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Hamilton-connected indices of graphs

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ABSTRACT

Let G be an undirected graph that is neither a path nor a cycle. Clark and Wormald [L.H. Clark, N.C. Wormald, Hamiltonian-like indices of graphs, ARS Combinatoria 15 (1983) 131–148] defined $hc(G)$ to be the least integer m such that the iterated line graph $L^m(G)$ is Hamilton-connected. Let diam(G) be the diameter of G and k be the length of a longest path whose internal vertices, if any, have degree 2 in G. In this paper, we show that $k-1 \le hc(G) \le \max\{\text{diam}(G), k-1\}$. We also show that $\kappa^3(G) \le hc(G) \le \kappa^3(G)+2$ where $\kappa^3(G)$ is the least integer m such that $L^m(G)$ is 3-connected. Finally we prove that $hc(G) \le |V(G)| - \Delta(G) + 1$. These bounds are all sharp.

1. Introduction

We use [1] for terminology and notation not defined here and we consider finite, undirected graphs. We allow graphs to have multiple edges but not loops. The multi-graph of order 2 with two edges will be called a 2-cycle and denoted by C_2. Let G be a graph. We use $\kappa(G)$ and $\kappa'(G)$ to denote the connectivity and the edge-connectivity of G, respectively. Denote by $O(G)$ the set of all odd vertices of G. For each $i = 0, 1, 2, \ldots$, let $D_i(G) = \{v \in V(G) | d_G(v) = i\}$, and $d_i(G) = |D_i(G)|$. A connected graph with at least two vertices is called a nontrivial graph. A **lane** in G is a nontrivial trail whose ends are not in $D_2(G)$ and whose internal vertices, if any, have degree 2 in G (and thus are in $D_2(G)$). Note that a lane may be a cycle. If the lane has length 1, then it has no internal vertices. The length of a lane is defined to be the number of its edges.

Let G be a connected graph. For any two vertices $v_1, v_2 \in V(G)$, the distance $d(v_1, v_2)$ between v_1 and v_2 is defined as the length of the shortest (v_1, v_2)-path in G. The diameter of G is $\text{diam}(G) = \max_{v \in V(G)}\{\max\{d(v, w) | w \in V(G)\}\}$. For $X \subseteq E(G)$, the **contraction** G/X is obtained from G by contracting each edge of X and deleting the resulting loops. If $H \subseteq G$, we write G/H for $G/E(H)$.

The **line graph** of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are incident. The iterated line graph is defined recursively by $L^0(G) = G$ and $L^{k+1}(G) = L(L^k(G))$ ($k \in \mathbf{N}$, where $\mathbf{N}$ stands for the set of all natural numbers). Chartrand [6] showed that if G is a connected graph that is not a path, then for some integer $k > 0$, $L^k(G)$ is hamiltonian.

A subgraph H of a graph G is **dominating** if $G - V(H)$ is edgeless. Let $v_0, v_k \in V(G)$. A (v_0, v_k) **-trail** of G is a vertex-edge alternating sequence

$$v_0, e_1, v_1, e_2, \ldots, e_k, v_k$$

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such that all the e_i's are distinct and for each $i = 1, 2, \ldots, k$, e_i joins v_{i-1} with v_i. With the notation above, this (v_0, v_k)-trail is also called an (e_1, e_k) **-trail**. All the vertices in $v_1, v_2, \ldots, v_{k-1}$ are internal vertices of the trail. A **dominating** (e_1, e_k) **-trail** T of G is an (e_1, e_k)-trail such that every edge of G is incident with an internal vertex of T. A **spanning** (e_1, e_k) **-trail** of G is an (e_1, e_k)-trail such that $V(T) = V(G)$. There is a close relationship between dominating eulerian subgraphs in graphs G and Hamilton cycles in $L(G)$. Xiong and Liu [13] extent this to the relationship between a certain even subgraph in G and Hamilton cycles in $L^m(G)$ for $m \geq 2$.

Theorem 1 (*Harary and Nash-Williams, [9]*). *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

A graph is **Hamilton-connected** if for any two vertices $u, v \in V(G)$, there exists a (u, v)-path containing all vertices of G. With an argument similar to that in the proof of Theorem 1, one can obtain the following theorem for Hamilton-connected line graphs.

Theorem 2. *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is Hamilton-connected if and only if for any pair of edges $e_1, e_2 \in E(G)$, G has a dominating (e_1, e_2)-trail.*

Corollary 3. *Let G be a graph that is not a cycle. For any $m \geq 0$ if $L^m(G)$ is Hamilton-connected, then $L^n(G)$ is Hamilton-connected for all $n \geq m$.*

We say that an edge $e \in E(G)$ is **subdivided** when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. The resulting two new edges are denoted by e' and e''. The process of taking an edge e and replacing it by the path of length 2 is called **subdividing** e. For a graph G and edges $e_1, e_2 \in E(G)$, let $G(e_1)$ denote the graph obtained from G by subdividing e_1, and let $G(e_1, e_2)$ denote the graph obtained from G by subdividing both e_1 and e_2. Thus

$$V(G(e_1, e_2)) - V(G) = \{v(e_1), v(e_2)\}.$$

From the definitions, one immediately has the following observation.

Proposition 4. *For a graph G and two edges $e_1, e_2 \in E(G)$, if $G(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail, then G has a spanning (e_1, e_2)-trail.*

In 1983, Clark and Wormald [8] introduced the concept of hamiltonian-connected index. Let G be an undirected graph that is neither a path nor a cycle. The hamiltonian index $h(G)$ (Hamilton-connected index $hc(G)$, respectively) is the least nonnegative integer k such that $L^k(G)$ is hamiltonian (Hamilton-connected, respectively).

Theorem 5 (*Combining Catlin, Janakiraman and Srinivasan, [5], and Lai, [10]*). *Let G be a connected graph that is neither a path nor C_2. Let k be the length of the longest lane in G. Then $h(G) \leq \min\{\mathrm{diam}(G), k + 1\}$.*

In this paper, we consider the Hamilton-connected index of a graph. In Section 2, we will describe Catlin's reduction method and state some relevant theorems. In Section 3, we get some results of Hamilton-connected index associated with diameter. In Section 4, we present the relations between the Hamilton-connected index and the connectivity of a graph. In Section 5, we give some relations between the Hamilton-connected index and the minimum and maximum degrees of a graph.

2. Catlin's reduction method

In [2] Catlin defined collapsible graphs. Let G be a graph. For $R \subseteq V(G)$, a subgraph Γ of G is called an **R -subgraph** if both $O(\Gamma) = R$ and $G - E(\Gamma)$ are connected. A graph is **collapsible** if G has an R-subgraph for every even set $R \subseteq V(G)$. In particular, K_1 is collapsible. For a graph G and its connected subgraph H, G/H denotes the graph obtained from G by contracting H, i.e. by replacing H by a vertex v_H such that the number of edges in G/H joining any $v \in V(G) - V(H)$ to v_H in G/H equals the number of edges joining v in G to H. A graph is contractible to a graph G' if G contains pairwise vertex-disjoint connected subgraphs $H_1, H_2, \ldots, H_k$ with $\bigcup_{i=1}^{k} V(H_i) = V(G)$ such that G' is obtained from G by successively contracting $H_1, H_2, \ldots, H_k$. The subgraph H_i of G is called the **preimage** of the vertex v_{H_i} of G', and v_{H_i} is called the image of H_i. For any vertex $v \in V(H_i)$, we also say that v_{H_i} is the image of the vertex v. Catlin [3] showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_k$ such that $\bigcup_{i=1}^{k} V(H_i) = V(G)$. The **reduction** of G is the graph obtained from G by successively contracting $H_1, H_2, \ldots, H_k$. A graph is **reduced** if it is the reduction of some graph. A nontrivial vertex in the reduction of G is a vertex which is the contraction image of a nontrivial connected subgraph of G.

Theorem 6 (*Catlin, [2]*). *Let G be a connected graph. Then each of the following holds.*

(i) *If G has a spanning tree T such that each edge of T is in a collapsible subgraph of G, then G is collapsible.*

(ii) *If G is reduced, then G is a simple graph and has no cycle of length less than four.*
(iii) *G is reduced if and only if G has no nontrivial collapsible subgraphs.*
(iv) *Let G' be the reduction of G. Then G is collapsible if and only if $G' = K_1$.*

Theorem 7 (*Catlin, Han and Lai, [4]*). *Let G be a connected reduced graph. If $2|V(G)| - |E(G)| \leq 4$, then G is a K_1, or a K_2 or a $K_{2,t}$ for some integer $t \geq 1$.*

Theorem 8 (*Lai, [11]*). *Let G be a 2-connected graph with $\delta(G) \geq 3$. If every edge of G is in a cycle of length at most 4, then G is collapsible.*

Lemma 9. *If G is collapsible, then for any pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-trail.*

Proof. Let $R = (O(G) \cup \{u, v\}) - (O(G) \cap \{u, v\})$. Then $|R|$ is even. Let Γ_R be an R-subgraph of G. Then $G - E(\Gamma_R)$ is a spanning (u, v)-trail of G. □

Lemma 10. *Let H be a collapsible subgraph of a graph G and $H' = G/H$. Let $u, v \in V(G)$ and $u', v' \in V(H')$ such that u', v' are the images of u, v respectively. Then G has a spanning (u, v)-trail if and only if H' has a spanning (u', v')-trail.*

Proof. It is clear that H' has a spanning (u', v')-trail if G has a spanning (u, v)-trail. So we only need to prove that G has a spanning (u, v)-trail if H' has a spanning (u', v')-trail.

Suppose that Γ' is a spanning (u', v')-trail in H'. Take one vertex $w_0 \notin V(G)$ and let Γ'' be a trail in H' with $V(\Gamma'') = V(\Gamma') \cup \{w_0\}$ and $E(\Gamma'') = E(\Gamma') \cup \{u'w_0, v'w_0\}$ if $u' \neq v'$, and let $\Gamma = \begin{cases} \Gamma', & \text{if } u' = v' \\ \Gamma'', & \text{if } u' \neq v' \end{cases}$. Then Γ is eulerian. Let $S = \{w \in V(H) : w \text{ is incident with an odd number of edges in } E(\Gamma)\}$. Then $|S|$ is even and $S \oplus O(H)$ is even too. Note that H is collapsible. Then there exists $L \subseteq H$ such that L is a connected, spanning subgraph in H such that $O(L) = S \oplus O(H)$. Thus $\Gamma \cup L$ is a spanning eulerian subgraph in $G + w_0$. Therefore G has a spanning (u, v)-trail. □

3. Hamilton-connected index and diameter

Let G be a graph. Denote $E' = E'(G) = \{e \in E(G) : e \text{ is in a cycle of } G \text{ of length at most 3}\}$ and $E'' = E(G) - E'(G)$.

Let H be an induced subgraph of G. The subgraph induced by the vertex set $E(H)$ in $L(G)$, denoted by $I_1(H)$, is called the 1-line-image of H, and H, denoted by $I_1^{-1}(I_1(H))$, is called the 1-line-preimage of $I_1(H)$. The subgraph induced by the vertex set $E(I_1(H))$ in $L^2(G)$, denoted by $I_2(H)$, is called the 2-line-image of H, and H, denoted by $I_2^{-1}(I_2(H))$, is called the 2-line-preimage of $I_2(H)$. Generally, the subgraph induced by the vertex set $E(I_k(H))$ in $L^{k+1}(G)$, denoted by $I_{k+1}(H)$, is called the $(k+1)$-line-image of H. Conversely, H, denoted by $I_{k+1}^{-1}(I_{k+1}(H))$, is called the $(k+1)$-line-preimage of $I_{k+1}(H)$. We adopt $I_{k+1}^{-1}(e)$ when $I_{k+1}(H)$ is a path induced by an edge e.

Lemma 11. *Let L be a lane in G with length d. Then $I_k(L)(k \leq d)$ is a lane in $L^k(G)$ with length $(d - k)$. Particularly, $I_{d-1}(L) \in E''(L^{d-1}(G))$.*

Lemma 12. *Let $e \in E''(I^{d-1}(G))$. Then $I_{d-1}^{-1}(e)$ is in a lane in G with length at least d.*

Theorem 13. *Let G be a connected graph that is neither a path nor C_n. If the length of a longest lane is k, then $k - 1 \leq hc(G) \leq \max\{\text{diam}(G), k - 1\}$.*

Proof. Since a longest lane of length k in G becomes a lane of length 2 in $L^{k-2}(G)$, and so $L^{k-2}(G)$ is not Hamilton-connected, $hc(G) \geq k - 1$.

The proof of the second inequality remains. If $\text{diam}(G) = 1$, then G is spanned by K_n. Thus $hc(G) = 0 \leq \max\{\text{diam}(G), k - 1\}$. Next we prove that the theorem holds for $d = \max\{\text{diam}(G), k - 1\} \geq 2$ by contradiction.

Let $f_1 = u_1v_1, f_2 = u_2v_2 \in E(L^{d-1}(G))$ and H be the reduction of $L^{d-1}(G)(f_1, f_2)$. By Lemma 9, Proposition 4 and Theorem 2, $H \neq K_1$. Note that

$$d_{L^{d-1}(G)(f_1,f_2)}(v(f_i)) = 2 (i = 1, 2).$$

Then either $\{f_i', f_i''\} \cap E(H) = \emptyset$ or $\{f_i', f_i''\} \subseteq E(H) (i = 1, 2)$. Let $H' = H/(E(H) \cap \{f_1'', f_2''\})$. Then $H' \neq K_2$.

Claim 1. $E(H') \subseteq E(L^{d-1}(G))$.

Proof. If $\{f_1', f_1'', f_2', f_2''\} \cap E(H) = \emptyset$, then $H' = H$. Thus $E(H') \subseteq E(L^{d-1}(G))$. If $\{f_1', f_1'', f_2', f_2''\} \cap E(H) \neq \emptyset$, without loss of generality, we assume $\{f_1', f_1''\} \subseteq E(H)$. Note that $H' = H/(E(H) \cap \{f_1'', f_2''\})$. Then $f_1 = f_1' \in E(H')$. Thus Claim 1 holds. □

By the definition of $E'(H')$ and Theorem 6(ii), we have the following claim.

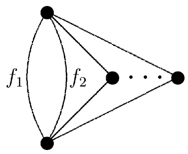

Fig. 1.

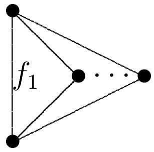

Fig. 2.

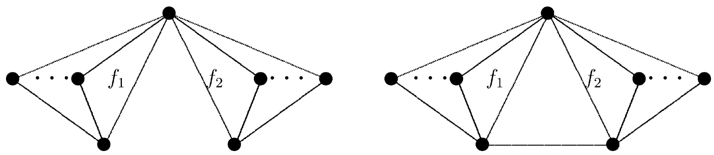

Fig. 3.

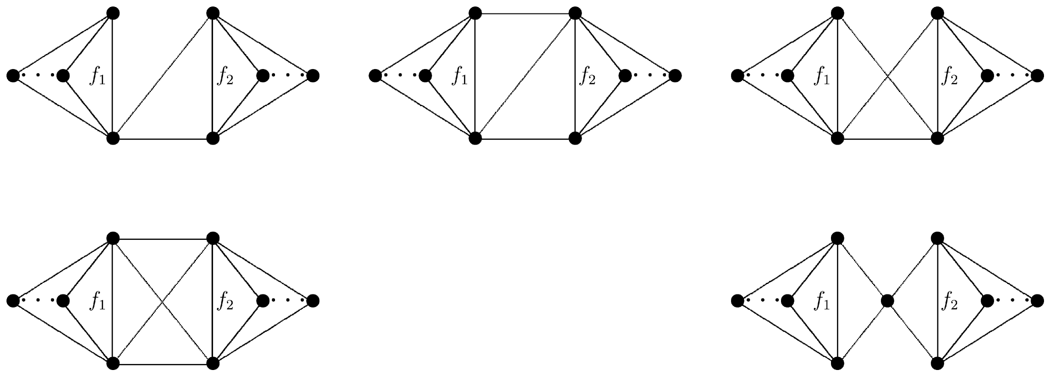

Fig. 4.

Claim 2. *Let $e \in E'(H')$. If e is in some 3-cycle of H', then this cycle contains f_1' or f_2'. If e is in a 2-cycle, then the two edges of the 2-cycle are f_1' and f_2'.*

Claim 3. $E(H') - E'(L^{d-1}(G)) \neq \emptyset$.

Proof. By contradiction. Suppose that $E(H') \subseteq E'(L^{d-1}(G))$. Let $e \in E(H')$. As $E''(H') \subseteq E''(H) \subseteq E''(L^{d-1}(G))$, we have $e \in E'(H')$. We consider three cases.

Case 1. e is in a 2-cycle of H'.

Note that an m-cycle with $m \leq 3$ is collapsible. By Claim 2 and the assumption that $E(H') \subseteq E'(L^{d-1}(G))$, H' must be the graph shown in Fig. 1. Thus, there exists a spanning $(v(f_1), v(f_2))$-trail in $L^{d-1}(G)(f_1, f_2)$ by Lemma 10, a contradiction.

Case 2. e is in a 3-cycle of H' containing exactly one of f_1' and f_2'.

Without loss of generality, we assume that this cycle contains f_1' only. Note again that an m-cycle with $m \leq 3$ is collapsible. By the assumption that $E(H') \subseteq E'(L^{d-1}(G))$, the graph H' must be the graph shown in Fig. 2, where $v(f_2)$ is in the preimage of some vertex. Thus, there exists a spanning $(v(f_1), v(f_2))$-trail in $L^{d-1}(G)(f_1, f_2)$ by Lemma 10, a contradiction.

Case 3. e is in a 3-cycle of H' containing both f_1' and f_2'.

Suppose that f_1 and f_2 are adjacent. Then the graph H' must be one of the graphs in Fig. 3. Thus, there exists a spanning $(v(f_1), v(f_2))$-trail in $L^{d-1}(G)(f_1, f_2)$ by Lemma 10, a contradiction. Now suppose that f_1 and f_2 are not adjacent. Then the graph H' must be one of the graphs in Fig. 4. Again, there exists a spanning $(v(f_1), v(f_2))$-trail in $L^{d-1}(G)(f_1, f_2)$ by Lemma 10, a contradiction. So Claim 3 holds. □

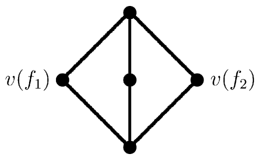

Fig. 5.

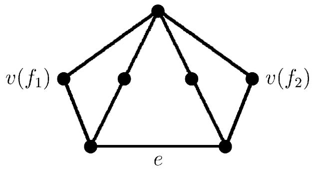

Fig. 6.

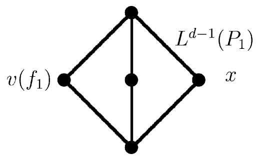

Fig. 7.

By Claim 3, let $e \in E(H') - E'(L^{d-1}(G))$. Then $I_{d-1}^{-1}(e)$ is in a lane in G with length at least d. Let P_1 be the maximal lane in G containing $I_{d-1}^{-1}(e)$. Suppose first P_1 is a (u, v)-path (for the case that P_1 is a cycle, we can argue similarly). Then $d_G(u), d_G(v) \geq 3$, and at least there exists another (v, u)-path P_2 in $G - E(P_1)$ (otherwise, let $u_1 \in N_G(u) - V(P_1)$ and $v_1 \in N_G(v) - V(P_1)$, then $\text{dist}(u_1, v_1) = d + 2$, a contradiction). Let $P_2, \ldots, P_m$ be all minimal (u, v)-paths in $G - E(P_1)$ with $E(P_i) \cap E(P_j) = \emptyset$ (when $i \neq j$).

Claim 4. *There exists a (u, v)-path P_i $(i \neq 1)$ that contains a lane with length at most $d - 1$.*

Proof. By contradiction. Suppose that each P_i contains a lane with length at least d. Then $1 \leq |L^{d-1}(P_i) \cap E''(L^{d-1}(G))| \leq 2$ and $\sum_{i=1}^{m} |L^{d-1}(P_i) \cap E''(L^{d-1}(G))| \leq m + 1$ since $d \geq \text{diam}(G)$. If $m = 2$, noting that $d_G(u) \geq 3, d_G(v) \geq 3$, so there always exist some $x \in N_G(u)$ and $y \in N_G(v)$ such that $\text{dist}_G(x, y) \geq d + 1 > \text{diam}(G)$, a contradiction. So $m \geq 3$. If $f_i \notin \bigcup_{i=1}^{m} E(L^{d-1}(P_i))$, then $e \notin E(H')$ since $m \geq 3$. So $\{f_1, f_2\} \subseteq \bigcup_{i=1}^{m} E(L^{d-1}(P_i))$, and $H = C_4$ or $K_{2,3}$. If $H = K_{2,3}$, then there is a spanning $(v(f_1), v(f_2))$-trail in H. If $H = C_4$, then one of vertices in H is trivial. Thus there is a dominating $(v(f_1), v(f_2))$-trail in H. In either case, $L^d(G)$ is Hamilton-connected, a contradiction. So Claim 4 holds. □

By Claim 4, suppose that P_2 contains a lane with length at most $d - 1$. Note that $P_1 \cup P_2$ is a cycle in G. Then $L^{d-1}(P_1 \cup P_2)$ is still a cycle in $L^{d-1}(G)$, $e \in L^{d-1}(P_1 \cup P_2)$ and at most two edges in $L^{d-1}(P_1 \cup P_2)$ are not in $E'(L^{d-1}(G))$. If $m \geq 3$, then $H = C_4$, $v(f_1), v(f_2) \in V(C_4)$ and one of the vertices of $V(H)$ is trivial. Thus $L^d(G)$ is Hamilton-connected. So we assume that $m = 2$.

Claim 5. $|E(L^{d-1}(P_1))| = 2$.

Proof. By contradiction. Suppose that $E(L^{d-1}(P_1)) = \{e\}$. If $e \in \{f_1, f_2\}$, without loss of generality, we assume that $e = f_1$. Then H is the graph $K_{2,3}$ (see Fig. 5). Thus $L^{d-1}(G)$ has a spanning $(v(f_1), v(f_2))$-trail by Lemma 10. Hence $L^d(G)$ is Hamilton-connected, a contradiction. Thus $e \notin \{f_1, f_2\}$.

Since each edge except e in $L^{d-1}(P_1 \cup P_2)$ is in $E'(L^{d-1}(G))$ and $e \notin E'(L^{d-1}(G))$, we have $f_1, f_2 \in E(L^{d-1}(P_1 \cup P_2))$. If f_1, f_2 are in the same triangle, then $L^{d-1}(P_1 \cup P_2)$ is collapsible, thus $e \notin E(H)$, a contradiction. Thus f_1, f_2 are not in the same triangle. Hence H is the graph shown in Fig. 6. Thus $L^{d-1}(G)$ has a spanning $(v(f_1), v(f_2))$-trail by Lemma 10. Hence $L^d(G)$ is Hamilton-connected, a contradiction. So Claim 5 holds. □

By Claim 5, $|E(L^{d-1}(P_1))| = 2$. Since each edge in $L^{d-1}(P_1 \cup P_2)$ except $L^{d-1}(P_1)$ is in $E'(L^{d-1}(G))$ and $L^{d-1}(P_1) \cap E'(L^{d-1}(G)) = \emptyset$, we have $|\{f_1, f_2\} \cap E(L^{d-1}(P_1 \cup P_2))| \geq 1$. If $|\{f_1, f_2\} \cap E(L^{d-1}(P_1 \cup P_2))| = 1$, without loss of generality, we assume that $f_1 \in E(L^{d-1}(P_1 \cup P_2))$. Then $v(f_2)$ is contracted. Thus $f_1 \notin E(L^{d-1}(P_1))$. (Otherwise H is collapsible and $H = K_1$.) Moreover $H = K_{2,3}$ and x is trivial in H (see Fig. 7). Thus the preimage of a vertex in $V(H) - \{x, v(f_1)\}$ contains $v(f_2)$. It is easy to check that $L^{d-1}(G)$ contains a dominating $(v(f_1), v(f_2))$-trail by Lemma 10. Hence $L^d(G)$ is Hamilton-connected. This contradicts the assumption. Thus $|\{f_1, f_2\} \cap E(L^{d-1}(P_1 \cup P_2))| = 2$.

We break this into three cases to finish the proof.

Case 1. $|\{f_1, f_2\} \cap E(L^{d-1}(P_1))| = 0$.

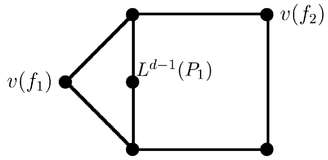

Fig. 8.

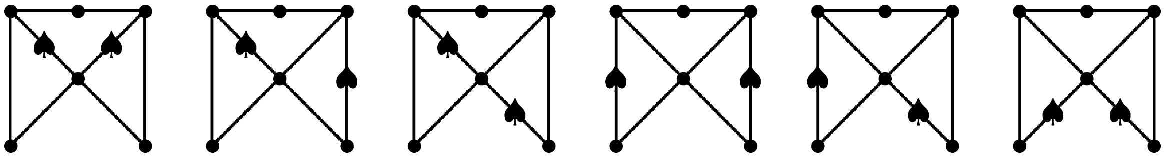

Fig. 9.

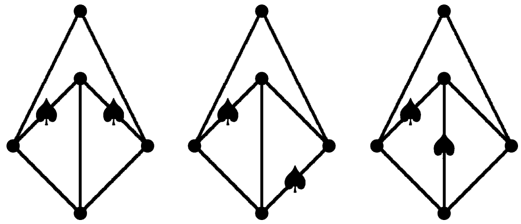

Fig. 10.

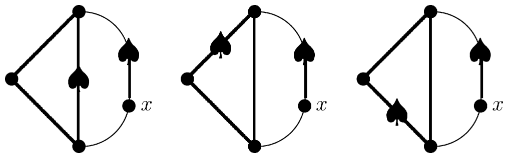

Fig. 11.

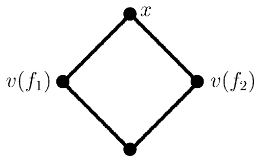

Fig. 12.

If f_1, f_2 are in the same triangle in H', then H is the graph shown in Fig. 8.

If f_1, f_2 are in two edge-disjoint triangles in H', then H is one of the graphs in Fig. 9. (In this and the following figures, the two ♠s stand for the vertices $v(f_1)$ and $v(f_2)$, respectively).

If f_1, f_2 are in two triangles sharing an edge in H', then H is one of the graphs shown in Fig. 10.

In either graph, H contains a spanning $(v(f_1), v(f_2))$-trail. Thus $L^d(G)$ is Hamilton-connected, a contradiction.

Case 2. $|\{f_1, f_2\} \cap E(L^{d-1}(P_1))| = 1$.

In this case, H is one of the graphs shown in Fig. 11. Note that $L^{d-1}(P_1)$ is a lane. Then the vertex x (see the graph) is trivial in H. Thus $L^{d-1}(G)$ contains a spanning $(v(f_1), v(f_2))$-trail by Lemma 10. Hence $L^d(G)$ is Hamilton-connected, a contradiction.

Case 3. $|\{f_1, f_2\} \cap E(L^{d-1}(P_1))| = 2$.

Then H is the graph shown in Fig. 12, and x is trivial in H. Thus $L^d(G)$ is Hamilton-connected, a contradiction.

Having exhausted the cases, we have completed the proof of Theorem 13. □

An obvious corollary is the following.

Corollary 14. *Let G be a connected graph that is neither a path nor C_n. If the length of a longest lane is k with $k \geq \text{diam}(G) + 1$, then $hc(G) = k - 1$.*

Noting that $k \leq 2\,\text{diam}(G) - 1$, we have the following corollary.

Corollary 15. *Let G be a connected graph that is neither a path nor C_n. Then $hc(G) \le 2(\mathrm{diam}(G) - 1)$.*

Let C be a cycle of length $2d(d > 1)$ and K be a complete graph of order $m > 2$. G is a graph obtained by combining C and K so that C and K share exactly one edge. Then $L^{2d-3}(G)$ has a 2-cut so that $L^{2d-3}(G)$ is not Hamilton-connected. On the other hand, $L^{2d-2}(G)$ is Hamilton-connected. Therefore Corollary 15 is best possible.

Theorem 16. *Let $d = \mathrm{diam}(G) \ge 3$. Then one of the following holds.*

(i) *$L^d(G)$ is Hamilton-connected;*
(ii) *$L^{d-1}(G)$ has a collapsible subgraph H such that $L^{d-1}(G)/H$ is a cycle of length at least 3.*

Proof. Suppose (i) does not hold. It suffices to show (ii) holds. Since $L^d(G)$ is not Hamilton-connected, there exists a lane L in G with length $k \ge d+2$ by Theorem 13. Suppose u and v are the two endvertices of L (possibly $u = v$ if L is a cycle), then $d(u) \ge 3$ and $d(v) \ge 3$ (otherwise $d > k$). Moreover, $G - L$ is connected. $L^{d-1}(L)$ is still a lane with length at least 3 and we assume $L' = L^{d-1}(L)$ with two endvertices u' and v' in $L^{d-1}(G)$. Let $H = (L^{d-1}(G) - L') \cup \{u', v'\}$. We are going to show that H is collapsible. Let H' be the reduction of H and we only need to show $H' = K_1$. For a contradiction, suppose there exists at least one edge xy in H'. Since H' is reduced, xy cannot be in a cycle of length at most 3. Correspondingly, there exists at least one edge $x'y'$ in the preimage of xy in H that is not contained in a cycle of length at most 3. By Lemma 12, the preimage of $x'y'$ in G must be a lane with length at least d and suppose the lane is Q. Take the midpoint w of P and the midpoint z of Q, then $\mathrm{dist}(w, z) \ge k/2 + d/2 \ge d + 1$, a contradiction. Note that $L^{d-1}(G)/H$ is a cycle obtained by identifying the two endvertices of P' and has length at least three. So we are done. □

Then we have the following corollary by the above theorem:

Corollary 17. *Let $d = \mathrm{diam}(G) \ge 3$. Then $L^d(G)$ is Hamilton-connected if and only if $\kappa(L^d(G)) \ge 3$.*

Proof. Necessity. This direction is trivial.

Sufficiency. For a contradiction, suppose that $\kappa(L^d(G)) \ge 3$ and $L^d(G)$ is not Hamilton-connected, then $L^{d-1}(G)$ is essentially 3-edge-connected. By (ii) of Theorem 16, $L^{d-1}(G)$ has an essential 2-edge-cut, a contradiction. □

4. Hamilton-connected index and connectivity

Let $\kappa^3(G) = \min\{m | L^m(G) \text{ is 3-connected}\}$. The following result shows that the Hamilton-connected index of G is not far from $\kappa^3(G)$.

Theorem 18. *Let G be a graph which is neither a path nor a cycle. Then $\kappa^3(G) \le hc(G) \le \kappa^3(G) + 2$.*

Proof. Let $c = \kappa^3(G)$. Noticing that a Hamilton-connected graph should be 3-connected, we have $hc(G) \ge c$.

It suffices to prove that $hc(G) \le c+2$. According to the definition of c, we know that $L^c(G)$ is 3-connected. So $\delta(L^c(G)) \ge 3$. Then $L^{c+1}(G)$ is the union of edge-disjoint complete subgraphs and $\delta(L^{c+1}(G)) \ge 4$, $\kappa(L^{c+1}(G)) \ge 3$. Hence each edge of $L^{c+1}(G)$ is in a triangle and $L^{c+1}(G)$ is collapsible by Theorem 8. Let $H = L^{c+1}(G)$. For any two edges e and f in H, we distinguish the following two cases.

Case 1. e and f are not in the same complete subgraph and, say, they are in two different complete subgraphs K_s and K_t of H respectively.

If both s and t are at least 4, then $H(e, f)$ is still collapsible since $K_s(e)$ and $K_t(f)$ have at least one triangle and exactly a C_4. Hence by Lemma 9, $H(e, f)$ has a spanning $(v(e), v(f))$-trail and so $L(H) = L^{c+2}(G)$ is Hamilton-connected. In the case that at least one of $\{s, t\}$ is 3, say $s = 3$, since $\kappa(L^{c+1}(G)) \ge 3$, the two endvertices of e are connected by a path P with $E(P) \bigcap E(K_s) = \emptyset$ in which each edge is in a complete subgraph of order at least 3, so the reduction of $H(e, f)$ is K_1, i.e., $H(e, f)$ is collapsible and so $L(H) = L^{c+2}(G)$ is Hamilton-connected.

Case 2. e and f are in the same complete subgraph K_t of H.

If $t \ge 5$, then $H(e, f)$ is still collapsible since $K_t(e, f)$ has at least a triangle. Hence by Lemma 9, $H(e, f)$ has a spanning $(v(e), v(f))$-trail and so $L(H) = L^{c+2}(G)$ is Hamilton-connected. If $t = 3$, since $\kappa(L^{c+1}(G)) \ge 3$, at least one of $\{e, f\}$, say e, has two endvertices that are connected by a path Q with $E(Q) \bigcap E(K_t) = \emptyset$ in which every edge is in a complete subgraph of order at least 3, so the reduction of $H(e, f)$ is K_1, i.e., $H(e, f)$ is collapsible and hence $L(H) = L^{c+2}(G)$ is Hamilton-connected. In the remaining case that $t = 4$, if e and f are incident, then the reduction of $H(e, f)$ is K_1 since $K_t(e, f)$ becomes a complete graph after contracting the unique triangle in it. Otherwise, if it is not collapsible, then the reduction of $H(e, f)$ must be the graph shown in Fig. 13. Then it has a spanning $(v(e), v(f))$-trail, so $L(H) = L^{c+2}(G)$ is Hamilton-connected.

In either case, there is a spanning $(v(e), v(f))$-trail in $H(e, f)$. Hence $hc(G) \le c + 2$. □

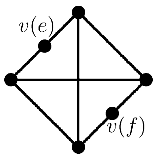

Fig. 13.

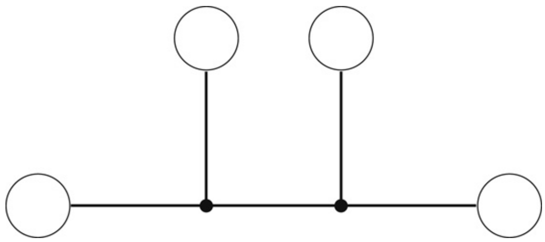

Fig. 14.

To show the sharpness of Theorem 18, we present an infinite family of graphs G with $hc(G) = \kappa^3(G) + 2$. Let P_{10} denote the Petersen graph and let $s \geq 1$ be an integer. Let $G(s)$ be obtained from P_{10} by first replacing every edge of P_{10} by a path of $s + 1$ edges, and then adding a pendent edge at each vertex of P_{10}.

Then $l(G(s)) = s + 1$ and $k^3(G(s)) = s$. However, $L^s(G(s))$ can be contracted to a P_{10}, each of whose vertices has a preimage with at least 4 edges. Let H_1 be the preimage of a vertex of P_{10} viewed as a contraction image of $L^s(G(s))$ (in fact, P_{10} is the reduction of $L^s(G(s))$ and we can just take H_1 as the preimage of any vertex in the reduction). Take two edges e_1, e_2 of $L^s(G(s))$ such that $e_1, e_2 \in V(H_1)$. Let T be an (e_1, e_2)-trail of $L^s(G(s))$. In the process when $L^s(G(s))$ is contracted to P_{10}, T is also contracted to an even subgraph T' (and so a cycle) of P_{10}, by the choices of e_1 and e_2. It follows that T' must miss at least one vertex of P_{10}, and so T cannot be an dominating (e_1, e_2)-trail of $L^s(G(s))$. This proves that $L^{s+1}(G(s))$ cannot be hamiltonian-connected, and so $hc(G(s)) > \kappa^3(G(s)) + 1$. By Theorem 18, $hc(G(s)) = \kappa^3(G(s)) + 2$.

5. Hamilton-connected index and degree

We start with some results on the hamiltonian index.

Theorem 19 (*Chartrand and Wall, [7]*). *Let G be a connected graph with minimum degree at least 3. Then $h(G) \leq 2$.*

Theorem 20 (*Saražin, [12]*). *Let G be a connected graph that is not a path. Then $h(G) \leq |V(G)| - \Delta(G)$.*

Accordingly, we have the following two theorems:

Theorem 21. *Let G be a connected graph with minimum degree at least 3. Then $hc(G) \leq 3$.*

Proof. Let $d, f \in E(L^2(G))$ and H be the reduction of $L^2(G)(e, f)$. Since $\delta(G) \geq 3$, $\delta(L(G)) \geq 4$. Thus every edge in $L^2(G)$ lies in some $K_t (t \geq 4)$. According to the proof of Theorem 18, $K_t(e)$ and $K_t(f)$ are collapsible. If H is not collapsible, then it must be as Fig. 13. Thus $L^3(G)$ is Hamilton-connected. This completes the proof of Theorem 21. □

In the graph shown in Fig. 14, each cycle stands for complete graphs $K_t (t \geq 4)$. Then $\delta(G) \geq 3$. On the other hand, $L^2(G)$ is not Hamilton-connected since $L^2(G)$ is not 3-connected, but $L^3(G)$ is Hamilton-connected. So Theorem 21 is best possible.

Theorem 22. *Let G be a connected graph that is neither a path nor a cycle. Then $hc(G) \leq |V(G)| - \Delta(G) + 1$.*

Proof. Since G is a connected graph that is neither a path nor a cycle, $\Delta(G) \geq 3$. Let $u \in V(G)$ with $d_G(u) = \Delta(G)$, and L be a longest lane with length k. Then

$$|(N_G(u) \cup \{u\}) \cap V(L)| \leq \begin{cases} 2, & \text{if } L \text{ is a path} \\ 3, & \text{if } L \text{ is a cycle} \end{cases}$$

$$|V(L)| = \begin{cases} k+1, & \text{if } L \text{ is a path} \\ k, & \text{if } L \text{ is a cycle} \end{cases}$$

and $|N_G(u) \cup \{u\}| = \Delta(G) + 1$. Thus

$$\begin{aligned} |(N_G(u) \cup \{u\}) \cup V(L)| &= |N_G(u) \cup \{u\}| + |V(L)| - |(N_G(u) \cup \{u\}) \cap V(L)| \\ &\geq \begin{cases} (\Delta + 1) + (k + 1) - 2, & \text{if } L \text{ is a path} \\ (\Delta + 1) + k - 3, & \text{if } L \text{ is a cycle} \end{cases} \\ &\geq \Delta + k - 2. \end{aligned}$$

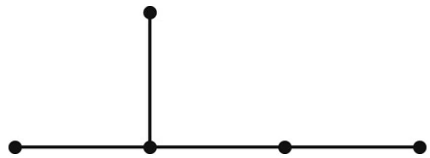

Fig. 15.

Therefore $n \geq \Delta(G) + k - 2$. If $\text{diam}(G) \leq k - 1$, then $hc(G) \leq k - 1 \leq n - \Delta(G) + 1$ by Theorem 13. Next we assume that $\text{diam}(G) \geq k$. Then $hc(G) \leq \text{diam}(G)$.

Let Q be a (v_1, v_2)-path satisfying $d_G(v_1, v_2) = \text{diam}(G)$. Let $|(N_G(u) \cup \{u\}) \cap V(Q)| = t$. Then $|(N_G(u) \cup \{u\}) \cup V(Q)| = |N_G(u) \cup \{u\}| + |V(Q)| - |(N_G(u) \cup \{u\}) \cap V(Q)| = (\Delta + 1) + (\text{diam}(G) + 1) - t = \Delta + \text{diam}(G) + 2 - t$. If $t \leq 2$, then $hc(G) \leq \text{diam}(G) \leq n - \Delta(G) - 2 + t \leq n - \Delta$. Thus we assume $t \geq 3$. Now we only need to discuss two cases.

Case 1. $u \notin V(Q)$.

Then $|N_G(u) \cap V(Q)| = t$, so we can assume $\{u_1, u_2, \ldots, u_t\} = N_G(u) \cap V(Q)$ such that $u_1, u_2, \ldots, u_t$ occur on $V(Q)$ in the order of the indices. Note that Q is a path satisfying $d_G(v_1, v_2) = \text{diam}(G)$. By the choice of Q, we have $u_i u_{i+1} \in E(Q)(1 \leq i \leq t - 1)$ and t must be 3. (Otherwise, if $t \geq 4$, then $d_G(v_1, v_2)$ can be shortened by discarding vertices $u_2, u_3, \ldots, u_{t-1}$ and adding u to Q.) Thus $Q' = Q - u_2 + u$ is still a (v_1, v_2)-path satisfying $d_G(v_1, v_2) = \text{diam}(G)$. Note that $n \geq |(N_G(u) \cup \{u\}) \cup V(Q')|$. Then $n \geq \Delta(G) + \text{diam}(G) + 2 - t = \Delta(G) + \text{diam}(G) - 1$. Thus $hc(G) \leq n - \Delta(G) + 1$.

Case 2. $u \in V(Q)$.

Then $|N_G(u) \cap V(Q)| = t - 1$, so we can assume $\{u_1, u_2, \ldots, u_{t-1}\} = N_G(u) \cap V(Q)$ such that $u_1, u_2, \ldots, u_{t-1}$ occur on $V(Q)$ in the order of the indices. Note that Q is a path satisfying $d_G(v_1, v_2) = \text{diam}(G)$ and u is also on the path. This forces t to be 3 and $u_1 u, uu_2 \in E(Q)$. Note that $n \geq |(N_G(u) \cup \{u\}) \cup V(Q)|$. Then $n \geq \Delta(G) + \text{diam}(G) + 2 - t = \Delta(G) + \text{diam}(G) - 1$. Thus $hc(G) \leq n - \Delta(G) + 1$. $\square$

Let G_0 be the (unique) tree on $n \geq 5$ vertices with exactly 3 leaves such that there are two leaves of G_0 having distances $n - 2$ from the third leaf. We have that $n = |V(G_0)|$, $\Delta(G) = 3$ and $L^{n-\Delta(G_0)}(G_0)$ is not 3-connected (hence not Hamilton-connected) and $L^{n-\Delta(G_0)+1}(G_0)$ is Hamilton-connected. For the case that $n = 5$, see Fig. 15. So Theorem 22 is best possible.

Acknowledgement

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References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London, Elsevier, New York, 1976.
[2] P.A. Catlin, A reduction method to find spanning eulerian subgraphs, Journal of Graph Theory 12 (1988) 29–44.
[3] P.A. Catlin, Supereulerian graph, collapsible graphs and 4-cycles, Congressus Numerantium 56 (1987) 223–246.
[4] P.A. Catlin, Z.Y. Han, H.-J. Lai, Graphs without spanning closed trails, Discrete Mathematics 160 (1996) 81–91.
[5] P.A. Catlin, T.N. Janakiraman, N. Srinivasan, Hamilton cycles and closed trails in iterated line graphs, Journal of Graph Theory 3 (1990) 347–364.
[6] G. Chartrand, On hamiltonian line-graphs, Transactions of the American Mathematical Society 134 (1968) 559–566.
[7] G. Chartrand, C.E. Wall, On the hamiltonian index of a graph, Studia Scientiarum Mathematicarum Hungarica 8 (1973) 43–48.
[8] L.H. Clark, N.C. Wormald, Hamiltonian-like indices of graphs, ARS Combinatoria 15 (1983) 131–148.
[9] F. Harary, C.St.J.A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs, Canadian Mathematical Bulletin 8 (1965) 701–710.
[10] H.-J. Lai, On the hamiltonian index, Discrete Mathematics 69 (1988) 43–53.
[11] H.-J. Lai, Graphs whose edges are in small cycles, Discrete Mathematics 94 (1991) 11–22.
[12] M.L. Saražin, A simple upper bound for the hamiltonian index of a graph, Discrete Mathematics 134 (1994) 85–91.
[13] L. Xiong, Z. Liu, Hamiltonian iterated line graphs, Discrete Mathematics 256 (2002) 407–422.