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The supereulerian graphs in the graph family $C(l, k)$[☆]

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ABSTRACT

For integers l and k with $l > 0$ and $k \geq 0$, let $C(l, k)$ denote the family of 2-edge-connected graphs G such that for every bond S with two or three edges, each component of $G - S$ has at least $(|V(G)| - k)/l$ vertices. In this note we get: (1) If $G \in C(6, 5)$ and $|V(G)| > 35$, then G is supereulerian if and only if G cannot be contracted to some well classified special graphs. (2) If $G \in C(6, 3)$, and $|V(G)| > 21$, then $L(G)$, the line graph of G, is Hamilton-connected if and only if $\kappa(L(G)) \geq 3$. Our results extend some earlier results in [P.A. Catlin, X.W. Li, Supereulerian graphs of minimum degree at least 4, J. Adv. Math. 28 (1999) 65–69], [H.J. Broersma, L.M. Xiong, A note on minimum degree conditions for supereulerian graphs, Discrete Appl. Math. 120 (2002) 35–43] and [D.X. Li, H.-J. Lai, M.Q. Zhan, Eulerian subgraphs and hamilton-connected line graphs, Discrete Appl. Math. 145 (2005) 422–428] by Catlin and Li, by Broersma and Xiong, and by Li, Lai and Zhan.

1. Introduction

Graphs in this paper are finite, undirected and loopless. Graphs may have multiple edges. A graph G is *nontrivial* if it contains at least one edge. We follow Bondy and Murty [1] for undefined notations and terminology. For a graph G, $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and the edge-connectivity of graph G, respectively, and $O(G)$ denotes the set of all odd degree vertices of G. For $X \subset E(G)$, the *contraction* G/X is obtained from G by contracting each edge of X and deleting the resulting loops. If $H \subset G$, we write G/H for $G/E(H)$. A graph G is *eulerian* if it is a connected graph with $O(G) = \emptyset$. A graph is *supereulerian* if it has a spanning eulerian subgraph. In particular, K_1 is both eulerian and supereulerian. For integers l and k with $l > 0$ and $k \geq 0$, let $C(l, k)$ denote the family of 2-edge-connected graphs G such that for every bond S with two or three edges, each component of $G - S$ has at least $(|V(G)| - k)/l$ vertices.

Catlin and Li, Broersma and Xiong, and D. Li et al. proved the following results concerning when a graph in a certain family $C(l, k)$ is supereulerian.

Theorem 1 (*Catlin and Li, of Theorem 6 [5]*). *If $G \in C(5, 0)$, then G is supereulerian if and only if G cannot be contracted to $K_{2,3}$.*

Theorem 2 (*Broersma and Xiong, of Theorem 7 [2]*). *If $G \in C(5, 2)$ and $n \geq 13$, then G is supereulerian if and only if G cannot be contracted to $K_{2,3}$ or $K_{2,5}$.*

Theorem 3 (*Li et al., of Theorem 1.3 [8]*). *If $G \in C(6, 0)$, then G is supereulerian if and only if G cannot be contracted to $K_{2,3}$, $K_{2,5}$ or $K_{2,3}(e)$, where $K_{2,3}(e)$ is the graph obtained from $K_{2,3}$ by replacing an edge $e \in E(K_{2,3})$ by a path of length 2.*

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Since $C(5,0) \subset C(5,2)$, and since when $n \geq 6k+1$, $C(5,k) \subset C(6,0)$, it is natural to consider the supereulerian problem for graph families $C(l,k)$ that properly contain $C(6,0)$. In this paper, we shall prove Theorem 14 below that if $G \in C(6,5)$ be a graph with $|V(G)| > 35$, then G is supereulerian if and only if G cannot be contracted to a well characterized collection of non supereulerian graphs. This result extends Theorems 1–3.

Let $L(G)$ denote the *line graph* of a graph G, which has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

A subgraph H of a graph G is *dominating* if $G - V(H)$ is edgeless. A (v_0, v_k)-*trail* of G is a vertex-edge alternating sequence $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ such that for all $i \neq j$, $e_i \neq e_j$, $i, j = 1, 2, \ldots, k$, where e_i is incident with both v_{i-1} and v_i. With the notation above, this (v_0, v_k)-trail is also called an (e_1, e_2)-*trail*, where all the vertices $v_1, v_2, \ldots, v_{k-1}$ are internal vertices of the trail. A *dominating* (e_1, e_k)-*trail* T of G is an (e_1, e_2)-trail such that every edge of G is incident with an internal vertex of T. A *spanning* (e_1, e_k)-*trail* T of G is a dominating (e_1, e_k)-trail such that $V(T) = V(G)$. Harary and Nash-Williams in [7] presented the following result:

Theorem 4 (*Harary and Nash-Williams [7]*). *Let G be a graph with $|E(G)| \geq 3$, then $L(G)$ is Hamiltonian if and only if G has a dominating eulerian subgraph.*

A graph G is *Hamilton-connected* if for any $u, v \in V(G)(u \neq v)$, there exists a (u, v)-path containing all vertices of G. Similar to Theorem 4, one can obtain the following theorem for Hamilton-connected line graphs.

Theorem 5. *Let G be a graph with $|E(G)| \geq 3$, then $L(G)$ is Hamilton-connected if and only if for any pair of edges $e_1, e_2 \in E(G)$, G has a dominating (e_1, e_2)-trail.*

Let G be a graph and let $X \subset E(G)$. The graph G_X is obtained from G by replacing each edge $e \in X$ with end u_e and v_e by a (u_e, v_e)-path P_e of length 2, where the internal vertex $w(e)$ of the path P_e is newly added. The graph G_X is called the subdivision graph of G to X. By the definition, the following result can be obtained immediately.

Theorem 6. *Let G be a graph and $X = \{e_1, e_2\} \subset E(G)$. If G_X has a spanning $(w(e_1), w(e_2))$-trail, then G has a spanning (e_1, e_2)-trail.*

Theorem 7 (*Li et al. [8]*). *If $G \in C(6,0)$ and $|V(G)| \geq 7$, then $L(G)$ is Hamilton-connected if and only if $\kappa(L(G)) \geq 3$.*

We get the following Hamilton-connectedness of line graphs of graphs in $C(l,k)$.

Theorem 8. *If $G \in C(6,3)$ and $|V(G)| \geq 21$, then $L(G)$ is Hamilton-connected if and only if $\kappa(L(G)) \geq 3$.*

In Section 2 we give a short description of Catlin's reduction method and some related results, which will be employed.

2. Catlin's reduction method and some useful results

A graph G is *collapsible* if for every set $R \subset V(G)$ with $|R|$ even, G has a spanning connected subgraph H_R with $O(H_R) = R$. Thus K_1 is both supereulerian and collapsible. Let G be a collapsible graph and let $R = \emptyset$. Then by definition G has a spanning connected subgraph H with $O(H) = \emptyset$, and so G is supereulerian. Thus every collapsible graph is also supereulerian.

In [3], Catlin showed that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The contraction of G obtained from G by contracting each H_i into a single vertex $(1 \leq i \leq c)$, is called the *reduction* of G. A graph is *reduced* if it is the reduction of itself. Thus a reduced graph does not have a nontrivial collapsible subgraph.

Theorem 9 (*Catlin, Theorem 3, Theorem 5 and Theorem 8 of [3]*). *Let G be a connected graph.*

(i) *Let H be a collapsible subgraph of a graph G, then G is supereulerian if and only if G/H is supereulerian.*
(ii) *G is reduced if and only if G has no nontrivial collapsible subgraphs.*
(iii) *Let G' be the reduction of G. Then G is supereulerian if and only if G' is supereulerian, and G is collapsible if and only if $G' = K_1$.*

Let $F(G)$ denote the minimum number of extra edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees.

Theorem 10 (*Catlin et al., Lemma 2.3 of [4]*). *If G is reduced with $|V(G)| \geq 3$, then $F(G) = 2|V(G)| - |E(G)| - 2$.*

Theorem 11 (*Catlin et al., Theorem 1.5 of [4]*). *Let G be a 2-edge-connected reduced graph. If $F(G) \leq 2$, then $G = K_1$ or $G = K_{2,t}$, where $t \geq 2$ is an integer.*

Let $D_i(G) = \{v \in V(G) | d(v) = i\}$, $N_G(v) = \{u | uv \in E(G),\ u \in V(G)\}$.

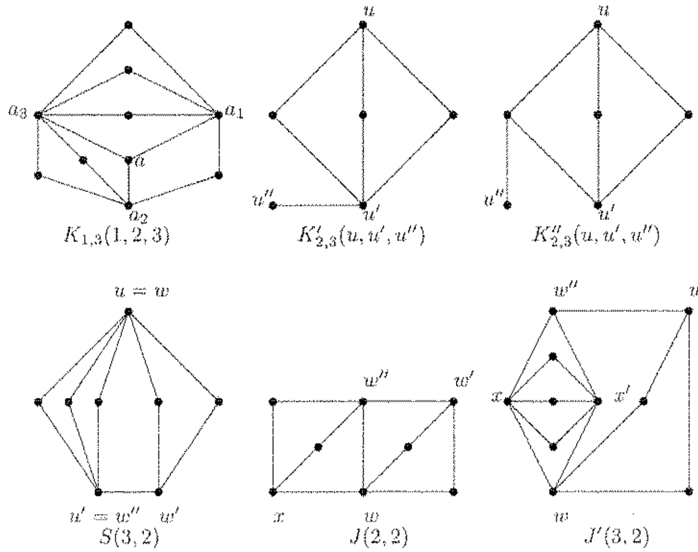

Fig. 1. Some graphs in $\mathcal{F}$ with small parameters.

Theorem 12 (*Catlin, Theorem 8 and Lemma 5 of [3]*). *If G is reduced, then G is simple and has no K_3. Moreover, if $\kappa'(G) \geq 2$, then $\sum_{i=2}^{3} |D_i(G)| \geq 4$, and when $\sum_{i=2}^{3} |D_i(G)| = 4$, G must be eulerian.*

Definition of $\mathcal{F}$: The Petersen graph is denoted by P. Let s_1, s_2, s_3, m, l, t be natural numbers with $t \geq 2$ and $m, l \geq 1$. Let $M \cong K_{1,3}$ with center a and ends a_1, a_2, a_3. Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from M by adding s_i vertices with neighbors $\{a_i, a_{i+1}\}$, where $i \equiv 1, 2, 3 \pmod 3$. Let $K_{2,t}(u, u')$ be a $K_{2,t}$ with u, u' being the nonadjacent vertices of degree t. Let $K'_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to u' only. Hence u'' has degree 1 and u has degree t in $K'_{2,t}(u, u', u'')$. Let $K''_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to a vertex of degree 2 of $K_{2,t}$. Hence u'' has degree 1 and both u and u' have degree t in $K''_{2,t}(u, u', u'')$. We shall use $K'_{2,t}$ and $K''_{2,t}$ for a $K'_{2,t}(u, u', u'')$ and a $K''_{2,t}(u, u', u'')$, respectively. Let $S(m, l)$ be the graph obtained from a $K_{2,m}(u, u')$ and a $K'_{2,l}(w, w', w'')$ by identifying u with w, and w'' with u'. Let $J(m, l)$ denote the graph obtained from a $K_{2,m+1}$ and a $K'_{2,l}(w, w', w'')$ by identifying w,w'' with the two ends of an edge in $K_{2,m+1}$,respectively. Let $J'(m, l)$ denote the graph obtained from a $K_{2,m+2}$ and a $K'_{2,l}(w, w', w'')$ by identifying w,w'' with two vertices of degree 2 in $K_{2,m+2}$, respectively. See Fig. 1 for examples of these graphs.

Let $\mathcal{F} = \{K_1, K_2, K_{2,t}, K'_{2,t}, K''_{2,t}, K_{1,3}(s, s', s''), S(m, l), J(m, l), J'(m, l), P\}$, where t, s, s', s'', m, l are nonnegative integers.

Theorem 13 (*Chen and Lai, Theorem 2.4 of [6]*). *If G is a connected reduced graph with $|V(G)| \leq 11$ and $F(G) \leq 3$, then $G \in \mathcal{F}$.*

3. Main results

We start with a lemma.

Lemma 1. *Let G be a 2-edge-connected reduced graph with $|V(G)| \leq 11$ and $F(G) \leq 3$, then*

(i) *If $|D_2(G)| = 4$, $|D_3(G)| = 2$, $|D_i(G)| = 0$, $i \geq 5$, then G is supereulerian or $G \in \{S(1, 2), J(2, 2)\}$.*
(ii) *If $|D_2(G)| = 5$, $|D_3(G)| = 1$, $|D_5(G)| = 1$, $|D_i(G)| = 0$, $i \geq 6$, then G is supereulerian or $G = S(3, 2)$.*
(iii) *If $|D_2(G)| = 6$, $|D_5(G)| = 2$, $|D_3(G)| = 0$, $|D_i(G)| = 0$, $i \geq 6$, then G is supereulerian or $G = S(4, 1)$.*

Proof. By Theorem 13, $G \in \mathcal{F}$. By the assumption of (i), (ii) and (iii), G cannot be $K_{2,t}$, P, $K'_{2,t}$, and $K''_{2,t}$. Hence we only consider G is one of $K_{1,3}(s_1, s_2, s_3)$, $S(s_1, s_2)$, $J(s_1, s_2)$ and $J'(s_1, s_2)$.

Case 1. For $K_{1,3}(s_1, s_2, s_3)$, as shown in Fig. 1 (in the figure only the case of $K_{1,3}(1, 2, 3)$ is shown).

By the definition of $K_{1,3}(s_1, s_2, s_3)$, we have $d(a) = 3$ and $\{a_1, a_2, a_3\} \subseteq N_G(a)$.

(i) In this case, G has only two odd degree vertices, since $d(a) = 3$, the other degree 3 vertex must be one of a_1, a_2 and a_3. Since $\{a_1, a_2, a_3\} \subseteq N_G(a)$, then G is supereulerian.
(ii) In this case, similarly, the other vertex of degree 5 must be one of a_1, a_2 and a_3. Since $\{a_1, a_2, a_3\} \subseteq N_G(a)$, then G is supereulerian.
(iii) Since $|D_3(G)| = 0$, G cannot be $K_{1,3}(s_1, s_2, s_3)$.

Case 2. For $S(s_1, s_2)$, as shown in the figure (In the figure is only shown the case of $S(3, 2)$). One can view $d(u) = s_1 + s_2$, $d(u') = s_1 + 1$ and $d(w') = s_2 + 1$.

(i) The two vertices of degree 3 must be two of u, u' and w'. If $d(u) \neq 3$ then $d(u') = d(w') = 3$. Since u' and w' are adjacent, then G is supereulerian. If $d(u) = 3$, w. l. o. g., we assume $d(w') = 3$, then $d(u) = s_1 + s_2 = d(w') = s_2 + 1 = 3$, thus $s_2 = 1, s_1 = 1$, hence $G = S(1, 2)(= K_{2,3}(e))$.
(ii) Since $s_1 \geq 1$ and $s_2 \geq 1$, we have $d(u) \geq d(u')$ and $d(u) \geq d(w')$, So $d(u) = s_1 + s_2 = 5$. W. l. o. g., let $d(w') = 3 = s_2 + 1$, then $s_2 = 2, s_1 = 3$. Hence $G = S(3, 2) = S(2, 3)$.
(iii) As shown in (ii), $d(u) = s_1 + s_2 = 5$. W. l. o. g., let $d(u') = 5 = s_1 + 1$, so $s_1 = 4, s_2 = 1$. Hence $G = S(4, 1)$.

Case 3. For $J(s_1, s_2)$, as shown in the figure (In the figure is only shown the case of $J(2, 2)$). One can view $d(w'') = s_1 + 2$, $d(w') = s_2 + 1$, $d(x) = s_1 + 1$ and $d(w) = s_2 + 2$.

(i) The two vertices of degree 3 must be that one is w'' or x, the other is w' or w. If the two vertices of degree 3 are adjacent then G is supereulerian. If the two vertices of degree 3 are not adjacent, then they must be x and w'. Therefore $d(x) = s_1 + 1 = d(w') = s_2 + 1 = 3$, thus $s_1 = s_2 = 2$, so $G = J(2, 2)$.
(ii) The vertex of degree 5 must be w'' or w. Suppose that $d(w'') = 5$. Then $s_1 + 2 = 5$, so $s_1 = 3$. Since $d(x) = s_1 + 1 = 4$, the vertex of degree 3 must be w or w'. As w and w' are both adjacent to w'', thus G is supereulerian. If $d(w) = 5$, similarly, G is supereulerian.
(iii) Considering $|D_i| = 0$ for $i \geq 6$, the two vertices of degree 5 should be w'' and w respectively, and $w''w \in E(G)$, then G is supereulerian.

Case 4. For $J'(s_1, s_2)$, as shown in the figure (In the figure is only shown the case of $J'(3, 2)$). One can view that $d(w'') = 3$, $d(w) = s_2 + 2$, $d(x) = d(x') = s_1 + 2$ and $d(w') = s_2 + 1$.

(i) As $d(w'') = 3$, the other vertex of degree 3 must be among x, x', w and w'. If the two vertices of degree 3 are adjacent, then G is supereulerian. If not, then w must has degree 3. Thus $d(w) = s_2 + 2 = 3$, so $s_2 = 1$. Since $|D_2| = 4$, then $s_1 + s_2 \leq 4$. Hence $s_1 \leq 3$.
If $s_1 = 1$, then $d(x) = d(x') = s_1 + 2 = 3$, contrary to the assumption that $|D_3| = 2$. If $s_1 = 2$, then $G = J'(2, 1) \in \mathcal{S}$. If $s_1 = 3$, then G has 5 vertices degree of 2, a contradiction.
(ii) Similar to (i), if the vertex degree of 5 is one of x, x' and w', then G is supereulerian. If $d(w) = 5$, then $s_2 + 2 = 5$ and $s_2 = 3$. As $s_1 \geq 1$, $d(x) = d(x') = s_1 + 2$ must be 4. Therefore $s_1 = 2$. Thus $G = J'(2, 3)$ is supereulerian.
(iii) As $|D_3| = 0$, G cannot be $J'(s_1, s_2)$. This completes the proof. □

Let $\mathcal{F}' = \{S(1, 2), S(3, 2), S(4, 1), J(2, 2), K_{2,3}, K_{2,5}\}$.

Theorem 14. *Let $G \in C(6, 5)$ be a graph with $n = |V(G)| > 35$. Then G is supereulerian if and only if G cannot be contracted to a member in $\mathcal{F}'$.*

Proof. Let G' is the reduction of G. By Theorem 9 (iii), it suffices to show if G' is not supereulerian then $G' \in \mathcal{F}'$. As $G' = K_1$ implies that G is supereulerian, we may assume that G' is 2-edge-connected and nontrivial. Let $d_i = |D_i(G')|$.

By Theorem 12, if $d_2 + d_3 = 4$, then G' is supereulerian. Therefore, we only consider the case when $d_2 + d_3 \geq 5$. We shall assume that

$$G' \text{ is not supereulerian,} \tag{1}$$

to find a contradiction or to get $G' \in \mathcal{F}'$.

Case 1. $d_2 + d_3 = 5$.

Subcase 1.1. $F(G') \leq 2$. By Theorem 11 and by (1), $G' = K_{2,t}$ with t being odd. Since $d_2 + d_3 = 5$, we have $t = 3$ or $t = 5$ and so $G' \in \{K_{2,3}, K_{2,5}\} \subset \mathcal{F}'$.

Subcase 1.2. $F(G') \geq 3$. By Theorem 10, we have

$$2F(G') = 4|V(G')| - 2|E(G')| - 4.$$

As $2|E(G')| = \sum_{j\geq 2} jd_j$ and $|V(G')| = \sum_{j\geq 2} d_j$, so

$$2F(G') + 4 = \sum_{j\geq 2} 4d_j - \sum_{j\geq 2} jd_j = \sum_{j\geq 2} (4-j)d_j = 2d_2 + d_3 + \sum_{j\geq 5} (4-j)d_j. \tag{2}$$

By $F(G') \geq 3$, by (2) and by $d_2 + d_3 = 5$, we have

$$10 + \sum_{j\geq 5} (j-4)d_j \leq 2d_2 + d_3 = d_2 + 5.$$

It follows that $d_2 = 5$, $d_3 = 0$, and $d_j = 0 (j \geq 5)$. Thus G' is eulerian, and so G' is supereulerian, contrary to (1).

Case 2. $d_2 + d_3 = 6$.

If $F(G') \leq 2$, then by (1) and by Theorem 11, $G = K_{2,t}$ with t odd. As $d_2 + d_3 = 6$, this is impossible. Therefore, we must have $F(G') \geq 3$.

Subcase 2.1. $F(G') = 3$. Let $c = d_2 + d_3$, and $H_1, H_2, \ldots, H_c$ denote the subgraphs of G whose contraction images in G' are the vertices of degree at most 3 in G'. Let $V'_4 = \{v \in G' | d_{G'}(v) \geq 4\}$. Since $G \in C(6, 5)$, we have $n = |V(G)| \geq \sum_{i=1}^{6} |V(H_i)| + |V'_4| \geq n - 5 + |V'_4|$, and so $|V'_4| \leq 5$ and $|V(G')| \leq 11$. By Theorem 13, $G' \in \mathcal{F}$. Thus by (2) and by $F(G') = 3$, we have

$$10 + \sum_{j \geq 5} (j - 4)d_j = 2d_2 + d_3 = d_2 + 6. \tag{3}$$

Thus by (3), $4 \leq d_2 \leq 6$. If $d_2 = 4$, then by (3), $d_3 = 2$ and $d_i = 0$ $(i \geq 5)$. By (1) and by Lemma 1(i), $G' \in \{S(1, 2), J(2, 2)\} \subset \mathcal{F}'$. If $d_2 = 5$, then by (3), $d_3 = 1$, $d_5 = 1$, $d_i = 0$ $(i \geq 6)$. By (1) and by Lemma 1(ii), $G' = S(3, 2) \in \mathcal{F}'$. Therefore, we may assume that $d_2 = 6$ and $d_3 = 0$. By (3) we have

$$\sum_{j \geq 5} (j - 4)d_j = 2. \tag{4}$$

By (4), either $d_5 = 2$ or $d_6 = 1$. If $d_5 = 2$, then by (4), $d_i = 0$ $(i \geq 6)$. By (1) and by Lemma 1(iii), $G' = S(4, 1) \in \mathcal{F}'$. If $d_6 = 1$, then $d_5 = 0$ and $d_i = 0$ $(i \geq 6)$, and so G' is eulerian, contrary to (1).

Subcase 2.2. $F(G') \geq 4$. By (2) and since $d_2 + d_3 = 6$,

$$12 + \sum_{j \geq 5} (j - 4)d_j \leq 2d_2 + d_3 = d_2 + 6.$$

It follows that $d_2 = 6$, $d_3 = 0$, and $d_i = 0$ $(i \geq 5)$, and so G' is eulerian, contrary to (1).

Case 3. $d_2 + d_3 \geq 7$.

Let $c = d_2 + d_3$, and $H_1, H_2, \ldots, H_c$ denote the subgraphs of G whose contraction images in G' are the vertices of degree at most 3 in G'. Since $G \in C(6, 5)$, for each i with $1 \leq i \leq c$, $|V(H_i)| \geq (n - 5)/6$. It follows any $c \geq 7$ that

$$n = |V(G)| \geq \sum_{i=1}^{7} |V(H_i)| \geq \frac{7(n - 5)}{6},$$

and so $6n \geq 7n - 35$, contrary to the assumption that $n > 35$.

This completes the proof of Theorem 14. □

The following lemma is useful in the Proof of Theorem 8.

By the definition of collapsible graph, we have:

Lemma 2. *If G is collapsible, then for any $u, v \in V(G)$, G has a spanning (u, v)-trail.*

Proof of Theorem 8. Clearly, if $L(G)$ is Hamilton-connected then $\kappa(L(G)) \geq 3$. Therefore we only need to show that when $\kappa(L(G)) \geq 3$, $L(G)$ is Hamilton-connected.

Let $X = \{e_1, e_2\} \in E(G)$, G' the reduction of G_X. By Theorems 5 and 6, Lemma 2 and Theorem 9 (iii), it suffices to show $G' = K_1$. Next we suppose $G' \neq K_1$ to find a contradiction. Let $d_i = |D_i(G')|$ $(i \geq 2)$.

Claim 1. $d_2 \leq 2$.

If $d_2 \geq 3$, then there exists $v \in D_2(G') - \{w(e_1), w(e_2)\}$ such that $d_{G'}(v) = 2$. Let H_v be the pre-image of v in G_X. Since $G \in C(6, 3)$ and $|V(G)| > 21$, $|V(H_v)| \geq (|V(G)| - 3)/6 > 1$. Thus $\kappa(L(G)) \leq 2$, contrary to $\kappa(L(G)) \geq 3$.

Claim 2. $d_2 + d_3 \leq 8$.

If $d_2 + d_3 \geq 9$, then $|D_2 \cup D_3 - \{w(e_1, w(e_2))\}| \geq 7$. Let $H_1, H_2, \ldots, H_7$ be the pre-images of 7 vertices in $D_2 \cup D_3 - \{w(e_1, w(e_2))\} \geq 7$ in G_X. By the hypothesis of Theorem 8, $|V(H_i)| \geq (|V(G)| - 3)/6$, for $i = 1, 2, \ldots, 7$. Hence

$$|V(G)| \geq \sum_{i=1}^{7} |V(H_i)| \geq \frac{7|V(G)| - 21}{6}$$

contrary to $|V(G)| > 21$.

Claim 3. $F(G') \leq 2$.

By Theorem 10 and $|V(G')| = \sum_{j \geq 2} d_j$, $2|E(G')| = \sum_{j \geq 2} jd_j$, we have

$$2d_2 + d_3 = 4 + 2F(G') + \sum_{j \geq 5} (j - 4)d_j. \tag{5}$$

Hence by (5), Claim 1 and Claim 2

$$2F(G') = 2d_2 + d_3 - 4 - \sum_{j \geq 5} (j - 4)d_j \leq 6.$$

Thus $F(G') \leq 3$. If $F(G') = 3$, by (5)

$$2d_2 + d_3 = 10 + \sum_{j\geq 5}(j-4)d_j.$$

Hence by Claim 1 and Claim 2, $d_2 = 2$ and $d_3 = 6$. From the proof of Claim 1, we have $D_2 = \{w(e_1), w(e_2)\}$. Let $H_1, H_2, \ldots, H_6$ be the pre-images of 3-vertex in G'. Let $V_4' = \{v|d(v) > 3, v \in G'\}$. We have

$$|V(G)| \geq \sum_{i=1}^{6} |V(H_i)| + |V_4'| \geq |V(G)| - 3 + |V_4'|.$$

So $|V_4'| \leq 3$ and $|V(G')| \leq 11$. Thus by Theorem 13 $G' \in \mathcal{F}$. As G' cannot be a Petersen graph and every 2-edge-connected graph in $\mathcal{F}$ has at least 3 vertices of degree 2 except Petersen graph, a contradiction obtains. Claim 3 holds.

As G' is 2-edge-connected and $d_2 \leq 2$, G cannot be $K_{2,t} (t \geq 2)$, so by Claim 3 and Theorem 11, $G' = K_1$, contrary to the assumption. This completes the proof. □

Corollary 1. *Theorem* 14 *implies Theorems* 1–3.

Proof. Let G' be the reduction of G.

(i) Theorem 14 implies Theorem 1. Since $C(5, 0) \subset C(6, 5)$, if $G \in C(5, 0)$, then $G \in C(6, 5)$. By Theorem 14, if G is not supereulerian then $G' \in \mathcal{F}'$. When $G' \in \mathcal{F}'$, since $G \in C(5, 0)$, G' cannot have more than 5 vertices degree at most 3, so $G' \notin \{S(1, 2), S(3, 2), S(4, 1), J(2, 2)\}$, and when G' has exactly 5 vertices degree at most 3, $\Delta(G') \leq 3$, so G' must be $K_{2,3}$.

(ii) Theorem 14 implies Theorem 2. If $G \in C(5, 2) \subset C(6, 5)$ and G is not supereulerian, then by Theorem 14, $G' \in \mathcal{F}'$. When $n = |V(G)| > 35 > 12$, G' cannot have more than 5 vertices degree at most 3, so $G' \notin \{S(1, 2), S(3, 2), S(4, 1), J(2, 2)\}$, so G' must be $K_{2,3}$ or $K_{2,5}$.

(iii) Theorem 14 implies Theorem 3. If $G \in C(6, 0) \subset C(6, 5)$ and G is not supereulerian, then by Theorem 14, $G' \in \mathcal{F}'$. Since $G \in C(6, 0)$, G' has at most 6 vertices degree no more than 3, and when G' has exactly 6 vertices degree at most 3, $\Delta(G') \leq 3$, so $G' \in \{S(1, 2)(= K_{2,3}(e)), K_{2,3}, K_{2,5}\}$.

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