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Discrete Mathematics 308 (2008) 4779–4785
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

The s-Hamiltonian index

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Received 28 July 2006; received in revised form 15 August 2007; accepted 16 August 2007
Available online 24 September 2007

Abstract

For integers k, s with $0 \leqslant k \leqslant s \leqslant |V(G)| - 3$, a graph G is called *s-Hamiltonian* if the removal of any k vertices results in a Hamiltonian graph. For a simple connected graph that is not a path, a cycle or a $K_{1,3}$ and an integer $s \geqslant 0$, we define $h_s(G) = \min\{m : L^m(G)$ is s-Hamiltonian$\}$ and $l(G) = \max\{m : G$ has a divalent path of length m that is not both of length 2 and in a $K_3\}$, where a divalent path in G is a non-closed path in G whose internal vertices have degree 2 in G. We prove that $h_s(G) \leqslant l(G) + s + 1$.
Published by Elsevier B.V.

Keywords: s-Hamiltonian graph; s-Hamiltonian index; Divalent path; Spanning Eulerian subgraph; Dominating Eulerian subgraph

1. Introduction

We use [1] for terminology and notations not defined here, and consider finite connected graphs only. For a graph G and a vertex $v \in V(G)$, define

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

For integers k, s with $0 \leqslant k \leqslant s \leqslant |V(G)| - 3$, G is called *s-Hamiltonian* if the removal of any k vertices results in a Hamiltonian graph. The *line graph* of G, denoted by $L(G)$ or $L^1(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have a common vertex. For an integer $m \geqslant 1$, we define $L^m(G) = L(L^{m-1}(G))$ with $L^0(G) = G$.

In 1973, Chartrand and Wall [2] introduced the *Hamiltonian index* of a connected graph G that is not a path to be the minimum number of applications of the line graph operator so that the resulting graph is Hamiltonian. He showed that the Hamiltonian index exists as a finite number. In 1983, Clark and Wormald [3] extended this idea of Chartrand and introduced the Hamiltonian-like indices. The *s-Hamiltonian index* $h_s(G)$ of G is the least nonnegative integer m such that $L^m(G)$ is s-Hamiltonian. Note that a 0-Hamiltonian graph is a Hamiltonian graph and $h_0(G) = h(G)$ is the Hamiltonian index of the graph G.

A non-closed path P of G is called a *divalent* in G if all the internal vertices of P have degree 2 in G. We define $l(G) = \max\{m : G$ has a divalent path of length m that is not both of length 2 and in a $K_3\}$.

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0012-365X/$ - see front matter Published by Elsevier B.V.
doi:10.1016/j.disc.2007.08.073

Theorem 1.1. *Let G be a simple connected graph that is not a path, a cycle or a* $K_{1,3}$. *Then* $h_s(G) \leqslant l(G) + s + 1$.

The case that $s = 0$ in Theorem 1.1 yields the theorem below, extending a former result by Lai.

Theorem 1.2 (*Lai* [5]). *Let G be a simple connected graph that is not a path, a cycle or a* $K_{1,3}$. *Then* $h(G) \leqslant l(G) + 1$.

2. Spanning Eulerian subgraph

Let $O(G)$ denote the set of all vertices in G with odd degree. A graph G is *Eulerian* if $O(G) = \emptyset$ and G is connected. A spanning closed trail of G is called a *spanning Eulerian subgraph* of G. A subgraph H of G is *dominating* if $G - V(H)$ is edgeless. If a closed trail C of G satisfies $E(G - V(C)) = \emptyset$, then C is called a *dominating Eulerian subgraph*. An edge cut X of G is *essential* if each side of $G - X$ has at least one edge. For a graph G and an edge subset $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by contracting each edge in X. Note that loops and/or multiple edges may be resulted from a contraction.

Lemma 2.1. *Let G be a connected graph and H an edge subset of G.*

(i) *If H is an edge set consisting of loops of G and* $G - H = G/H$ *has a spanning Eulerian subgraph, then G has a spanning Eulerian subgraph.*
(ii) *If H is a pair of parallel edges or the edge set of a* C_3 *and* G/H *has a spanning Eulerian subgraph, then G has a spanning Eulerian subgraph.*

Proof. (i) Let T be a spanning Eulerian subgraph of $G - H$. Since H is an edge set consisting of loops of G, T or $T \cup H$ is a spanning Eulerian subgraph of G.

(ii) *Case* 1: Let T be a spanning Eulerian subgraph of G/H. $H = \{e_1, e_2\}$ is an edge set of parallel edges of G. Let v_1, v_2 be the two endpoints of H and v_H the vertex in G/H onto which H is contracted. Since $d_{(G/H)[T]}(v_H)$ is even, $d_{G[T]}(v_1) + d_{G[T]}(v_2)$ is even. If $d_{G[T]}(v_1)$ and $d_{G[T]}(v_2)$ are both even, then $T \cup H$ is a spanning Eulerian subgraph of G; if $d_{G[T]}(v_1)$ and $d_{G[T]}(v_2)$ are both odd, then $T \cup e_1$ is a spanning Eulerian subgraph of G.

Case 2: Let T' be a spanning Eulerian subgraph of G/H. $H = \{e_1, e_2, e_3\}$ is an edge set of C_3 in G. Let v_1, v_2, v_3 be the three endpoints of the edges in H (see Case 2 in Fig. 1) and v_H the vertex in G/H onto which H is contracted. Since $d_{(G/H)[T']}(v_H)$ is even, $d_{G[T']}(v_1) + d_{G[T']}(v_2) + d_{G[T']}(v_3)$ is even. If $d_{G[T']}(v_1)$, $d_{G[T']}(v_2)$ and $d_{G[T']}(v_3)$ are all even, then $T' \cup H'$ is a spanning Eulerian subgraph of G; if two of them are odd and we assume without loss of generality that $d_{G[T']}(v_1)$ and $d_{G[T']}(v_2)$ are odd, then $T' \cup \{e_1, e_2\}$ is a spanning Eulerian subgraph of G. □

Theorem 2.2 (*Mantel* [7]). *Let G be a simple graph on n vertices and m edges, where* $m > n^2/4$. *Then G contains a triangle.*

Lemma 2.3. *Let G be a connected graph without essential cut edges of size* 1 *and* G_1 *the graph obtained by contracting all the triangles, loops and multiple edges repeatedly from G. If* $|V(G_1)| \leqslant 4$, *then G has a dominating Eulerian subgraph.*

Proof. Since G_1 is simple, connected and has no 3-cycles, $|E(G_1)| \leqslant |V(G_1)|^2/4$ by Theorem 2.2. So $|E(G_1)| = 2$ when $|V(G_1)| = 3$ and $|E(G_1)| = 3, 4$ when $|V(G_1)| = 4$.

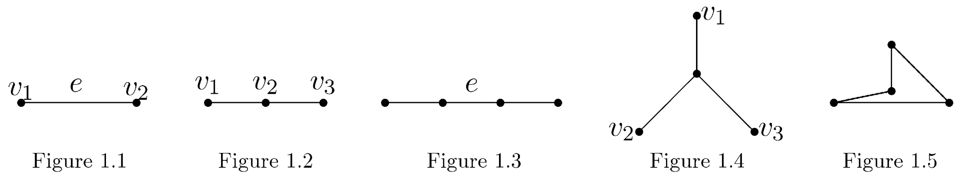

Fig. 1. G_1.

Case 1: $|V(G_1)|=1$. By Lemma 2.1, G has a spanning Eulerian subgraph.

Case 2: $|V(G_1)|=2$ (See Fig. 1(1)). Since G has no essential cut edges of size 1, one of the vertices is a vertex of G. We assume that $v_1 \in V(G)$. Then $G_1 - v_1 = \{v_2\} \cong K_1$ is a spanning Eulerian subgraph. As $G_1 - v_1 = \{v_2\}$ is obtained from the contractions described in Lemma 2.1, by Lemma 2.1, $G - v_1$ has a spanning Eulerian subgraph T, and so T is a dominating Eulerian subgraph of G.

Case 3: $|V(G_1)|=3$ (See Fig. 1(2)). Then by the same argument as in Case 2, $v_1, v_3 \in V(G)$. By Lemma 2.1, $G - \{v_1, v_3\}$ has a spanning Eulerian subgraph T, and so T is a dominating Eulerian subgraph of G.

Case 4: $|V(G_1)|=4$ (see Figs. 1(3–5)).

In Fig. 1(3), G has an essential cut edge e, a contradiction; In Fig. 1(5), by Lemma 2.1, G has a spanning Eulerian subgraph. For Fig. 1(4), by the same argument as in Case 2, we have that $v_1, v_2, v_3 \in V(G)$. By Lemma 2.1, $G - \{v_1, v_2, v_3\}$ has a spanning Eulerian subgraph T, and so T is a dominating Eulerian subgraph of G. □

3. Line graphs

We summarize some of the properties of line graphs as follows:

Proposition 3.1. *Let G be a simple graph with $|V(G)|=n$.*

(i) *The line graph $L(G)$ is the edge-disjoint union of n complete graphs, each of which has $d_G(v)$ vertices where $v \in V(G)$.*
(ii) *Let $e = xy \in E(G)$. Then the corresponding vertex v_e is of degree $d_G(x) + d_G(y) - 2$ in $L(G)$. In particular, if $\delta(G) \geqslant 2$, then $\delta(L(G)) \geqslant \delta(G)$.*
(iii) *Let $S' \subseteq E(G)$ and $S \subseteq V(L(G))$ the corresponding vertex set of the edge set S'. Then $L(G) - S = L(G - S')$.*

Proof. (i) By the definition of a line graph, for each $v \in V(G)$, the vertex set in $L(G)$ corresponding to $E_G(v)$ induces a complete graph of $d_G(v)$ vertices.

(ii) Follows directly from the definition of line graphs. If $\delta(G) \geqslant 2$, then $d_{L(G)}(v_e) = d_G(x) + d_G(y) - 2 \geqslant \delta(G) + \delta(G) - 2 \geqslant \delta(G)$.

(iii) By the definition of a line graph, $|V(L(G-S'))| = |E(G-S')| = |E(G)| - |S'| = |V(L(G))| - |S| = |V(L(G)-S)|$. For any $u, v \in V(L(G - S'))$, let e_u, e_v be the corresponding edges in $G - S'$. Then $uv \in E(L(G - S')) \Leftrightarrow e_u$ and e_v share a common vertex in $G - S' \Leftrightarrow e_u$ and e_v share a common vertex in $G \Leftrightarrow uv \in E(L(G)) \Leftrightarrow uv \in E(L(G) - S)$. □

A graph G is *k-triangular* if each edge of G lies in at least k triangles of G. In particular, G is *triangular* if it is 1-triangular.

Lemma 3.2. *Let G be a simple connected graph that is not a path, a cycle or a $K_{1,3}$ with $l(G) = l$. Then each of the following holds:*

(i) *For integers $l \geqslant 1$ and $m \geqslant 0$,*

$$l(L^m(G)) = \begin{cases} l(G) - m & \text{if } 0 \leqslant m < l(G), \\ 1 & \text{if } m \geqslant l(G). \end{cases}$$

(ii) *For an integer $s \geqslant 0$,*

$$\delta(L^{l+s}(G)) \geqslant \begin{cases} 2 & \text{if } s = 0 \text{ or } s = 1, \\ 2^{s-2} + 2 & \text{if } s \geqslant 2. \end{cases}$$

(iii) *$L^l(G)$, $L^{l+1}(G)$ and $L^{l+2}(G)$ are triangular and $L^{l+s}(G)$ is 2^{s-3}-triangular when $s \geqslant 3$.*
(iv) *For an integer $s \geqslant 0$, $\kappa(L^{l+s}(G)) \geqslant s + 1$.*

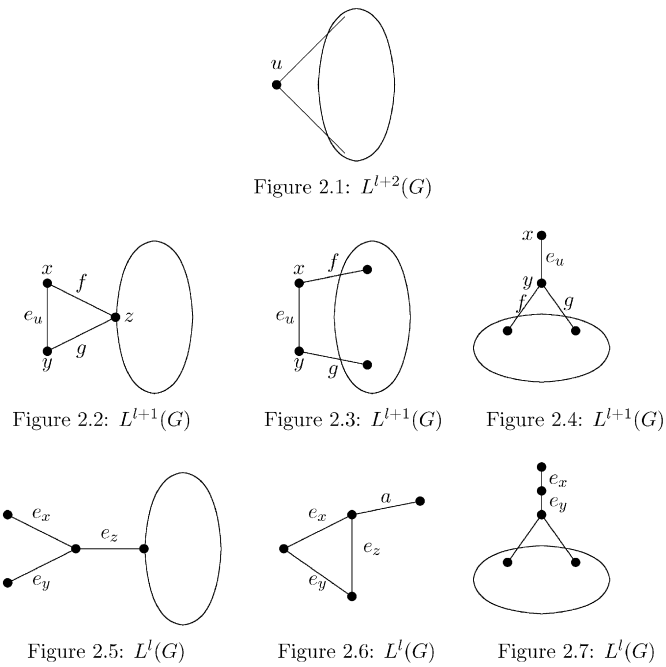

Figure 2.1: $L^{l+2}(G)$

Figure 2.2: $L^{l+1}(G)$ Figure 2.3: $L^{l+1}(G)$ Figure 2.4: $L^{l+1}(G)$

Figure 2.5: $L^{l}(G)$ Figure 2.6: $L^{l}(G)$ Figure 2.7: $L^{l}(G)$

Fig. 2.

Proof. (i). We consider each of the following cases.

Case 1: $l(G) = 1$.

We need to show that $l(L^m(G)) = 1$ for any $m \geqslant 0$. By the definition of a divalent path, $l(G) = 1$ if and only if one of the following holds

(A) $\delta(G) \geqslant 3$;

(B) $\delta(G) \leqslant 2$ and every vertex of degree 2 is contained in a triangle.

If (A) holds, then $\delta(L^m(G)) \geqslant \delta(G) \geqslant 3$ and so $l(L^m(G)) = 1$ for any $m \geqslant 0$. Hence, we assume that (B) holds. By way of contradiction we assume that $l(L(G)) \geqslant 2$. Let $P_0 = v_0 v_1 \cdots v_l$ be a divalent path in $L(G)$ of length $l(L(G)) \geqslant 2$. So $d_{L(G)}(v_1) = d_{L(G)}(v_2) = \cdots = d_{L(G)}(v_{l-1}) = 2$. By the definition of a divalent path, $v_0 v_1 v_2$ is an induced path of length 2, i.e., $v_0 v_1, v_1 v_2 \in E(L(G))$, but $v_0 v_2 \notin E(L(G))$. Assume that $e_{v_0}, e_{v_1}, e_{v_2}$ are edges in $E(G)$ corresponding to v_0, v_1, v_2 in $L(G)$. So $e_{v_0} e_{v_1} e_{v_2}$ is a path of length 3 in G with two internal vertices of degree 2 and e_{v_0}, e_{v_2} do not share a common vertex. Hence $l(G) \geqslant 3$, contrary to $l(G) = 1$. Hence $l(L(G)) = 1$. Inductively, $l(L^m(G)) = 1$ for any $m \geqslant 0$ when $l(G) = 1$.

Case 2: $l(G) \geqslant 2$.

Let $P_0 = v_0 v_1 \cdots v_l$ be a divalent path of length $l(G) \geqslant 2$. By the definition of a divalent path, $d_G(v_i) = 2$ for $i = 1, 2, \ldots, l-1$. Let $u_1, u_2, \ldots, u_l$ be the vertices in $L(G)$ corresponding to the edges $v_0 v_1, v_1 v_2, \ldots, v_{l-1} v_l$ in G, respectively. Then $u_1 u_2 \cdots u_l$ is a divalent path in $L(G)$ with length $l-1$. So $l(L(G)) = l-1$. Inductively, we have $l(L^m(G)) = l - m$ when $0 \leqslant m < l$. So $l(L^{l-1}(G)) = 1$. By Case 1 $l(L^m(G)) = 1$ when $m \geqslant l$.

(ii). First we show that $\delta(L^l(G)) \geqslant 2$. We assume by way of contradiction that there exists a vertex v of degree 1 in $L^l(G)$. Then the corresponding edge of v in $L^{l-1}(G)$ must have only one adjacent edge and these two edges in $L^{l-1}(G)$ induce a path of length 2 with the internal vertex of degree 2 in $L^{l-1}(G)$. So this path is a divalent path of length 2, contrary to the fact that $l(L^{l-1}(G)) = 1$. Hence $\delta(L^l(G)) \geqslant 2$ and by Proposition 3.1(ii) $\delta(L^{l+1}(G)) \geqslant 2$.

Next we show that $\delta(L^{l+2}(G)) \geqslant 3$. We assume that there exists a vertex u in $L^{l+2}(G)$ of degree 2 (see Fig. 2(1)). Then the corresponding edge $e_u = xy$ in $L^{l+1}(G)$ of u is incident with 2 edges. Hence if $d_{L^{l+1}(G)}(x) = 1$, then $d_{L^{l+1}(G)}(y) = 3$ (see Fig. 2(4)) and if $d_{L^{l+1}(G)}(x) = 2$, then $d_{L^{l+1}(G)}(y) = 2$ (see Fig. 2(2) and 2(3)). The graph $L^{l+1}(G)$ in Fig. 2(4) is the line graph of $L^l(G)$ in Fig. 2(7), contrary to (i) that $l(L^l(G)) = 1$. In Fig. 2(3), $\{f, e_u, g\}$ forms a divalent path of length 3 where f and g do not share any common vertex, which contradicts $l(L^{l+1}(G)) = 1$ by (i). So we preclude

Figs. 2(3), 2.4 and consider Fig. 2(2). In Fig. 2(2), $\{f, e_u, g\}$ forms a triangle in $L^{l+1}(G)$. Assume that e_x, e_y, e_z are edges in $L^l(G)$ corresponding to the vertices x, y, z in $L^{l+1}(G)$. The induced graph of $\{e_x, e_y, e_z\}$ corresponds to either a $K_{1,3}$ (see Fig. 2(5)or a C_3 (see Fig. 2(6)) in $L^l(G)$. Since $l \geqslant 1$, $L^l(G)$ is a line graph and so it is $K_{1,3}$-free, which excludes the graph in Fig. 2(5).

Since G is not a C_3 or $K_{1,3}$, $L^l(G)$ is not a graph isomorphic to a C_3. So $E(L^l(G)) - \{e_x, e_y, e_z\} \neq \emptyset$. Without loss of generality we assume that e_x and e_z are adjacent to an edge $a \in E(L^l(G)) - \{e_x, e_y, e_z\}$ (see Fig. 2(6)), contrary to the fact that $d_{L^{l+1}(G)}(x) = 2$. So we have $\delta(L^{l+2}(G)) \geqslant 3$.

Since $\delta(L^{l+2}(G)) \geqslant 3 = a_1$, by Proposition 3.1(ii), every edge in $L^{l+2}(G)$ is adjacent to at least $4 = 2 \cdot 3 - 2 = a_2$ edges and so $\delta(L^{l+3}(G)) \geqslant 4 = a_2$. Inductively, every edge in $L^{l+s-1}(G)$ is adjacent to at least $a_{s-1} = 2a_{s-2} - 2$ edges and so $\delta(L^{l+s}(G)) \geqslant 2^{s-2} + 2$ for $s \geqslant 2$.

(iii) First we prove that $L^l(G)$ is triangular. Since $xy \in E(L^l(G))$, the corresponding edges e_x and e_y in $L^{l-1}(G)$ share a common vertex v in $L^{l-1}(G)$. If $d_{L^{l-1}(G)}(v) = 2$, then e_x and e_y lie in a triangle by $l(L^{l-1}(G)) = 1$. So, $d_{L^{l-1}(G)}(v) \geqslant 3$. By Proposition 3.1(i), e lies in a triangle of $L^l(G)$.

And by definition of a line graph, $L^{l+1}(G)$ and $L^{l+2}(G)$ are also triangular. If $s \geqslant 3$, by (ii), $\delta(L^{l+s-1}(G)) \geqslant 2^{s-3} + 2$, so by Proposition 3.1(i) the vertex set corresponding to the incident edges of each vertex form a complete graph with order at least $2^{s-3} + 2$ in $L^{l+s}(G)$. Then each edge of $L^{l+s}(G)$ lies in at least 2^{s-3} triangles, that is, $L^{l+s}(G)$ is 2^{s-3}-triangular. (iv) Since G is connected, $L^l(G)$ is connected, i.e., $\kappa(L^l(G)) \geqslant 1$.

Next we prove that $\kappa(L^{l+s}(G)) \geqslant s + 1$ for $s \geqslant 1$ by induction. Assume that $\kappa(L^{l+s-1}(G)) \geqslant s$. Let X be an essential edge cut of $L^{l+s-1}(G)$ and H_1, H_2 are the two nontrivial components of $L^{l+s-1}(G) - X$ (a nontrivial component is a component containing at least one edge). By the induction hypothesis $|X| \geqslant \kappa(L^{l+s-1}(G)) \geqslant s$.

We now show that $|X| \geqslant s + 1$. By way of contradiction we assume that $|X| = s$. Since $L^{l+s-1}(G)$ is triangular when $s \geqslant 1$ by (iii), we may assume that there are $t(\geqslant 2)$ edges of X incident with a same vertex (say y) in one of $\{H_1, H_2\}$ (say H_1). Let $Y \subseteq V(H_1)$ denote the set of endpoints of X in H_1, then

Fact 1. $|Y| \leqslant s - t + 1 \leqslant s - 1$.

Fact 2. $\delta(L^{l+s-1}(G)) \geqslant s$ for $s \geqslant 1$.

By Lemma 3.2(ii), $\delta(L^{l+s-1}(G)) \geqslant 2^{s-3} + 2 \geqslant s$ when $s \geqslant 3$ and $\delta(L^{l+s-1}(G)) \geqslant 2 \geqslant s$ when $s = 1, 2$.

Fact 3. $V(H_1) - Y \neq \emptyset$.

By way of contradiction we assume that $Y = V(H_1)$. By Fact 1, if $|Y| \leqslant s - t$, then $d_{L^{l+s-1}(G)}(y) \leqslant |Y - \{y\}| + t = (s - t - 1) + t = s - 1$ where t is the number of incident edges in X of y, contrary to Fact 2. So by Fact 1 we must have $|Y| = s - t + 1$. Since $|X| = s$ and $|Y - \{y\}| = s - t$, $\forall z \in Y - \{y\}$, z is incident with exactly one edge of X. Let $z \in Y - \{y\}$, then $d_{L^{l+s-1}(G)}(z) \leqslant |Y - \{z\}| \leqslant (s - t) + 1 = s - t + 1 \leqslant s - 1$, contrary to Fact 2. Hence $V(H_1) - Y \neq \emptyset$.

By Facts 1 and 3, Y is a $(s - 1)$-cut of $L^{l+s-1}(G)$, contrary to the induction hypothesis. Hence every essential edge cut of $L^{l+s-1}(G)$ has size at least $s + 1$. Notice that for an integer $k \geqslant 0$, a non-complete line graph $L(H)$ has no vertex-cut of size less than k if and only if H has no essential edge-cut of size less than k. So $\kappa(L^{l+s}(G)) \geqslant s + 1$. □

Lemma 3.3. *Let G be a simple connected graph that is not a path, a cycle or $K_{1,3}$ with $l(G) = l$ and s be a nonnegative integer. Then for any $S' \subseteq E(L^{l+s}(G))$ with $|S'| \leqslant s$, $L^{l+s}(G) - S'$ has a dominating Eulerian subgraph.*

Proof. By Lemma 3.2(iii), $L^{l+s}(G)$ is 2^{s-3}-triangular. Since $2^{s-3} > s + 1$ when $s \geqslant 6$, we have that $L^{l+s}(G)$ is $(s+1)$-triangular when $s \geqslant 6$. In this case, every edge of $L^{l+s}(G) - S'$ lies in at least one triangle since $|S'| \leqslant s$. By Lemma 2.3, $L^{l+s}(G) - S'$ has a spanning Eulerian subgraph since we can contract all the triangles of $L^{l+s}(G) - S'$ to get a K_1. It suffices to prove this lemma for $s = 0, 1, 2, 3, 4, 5$.

By Proposition 3.1(i), $L^{l+s}(G)$ is an edge-disjoint union of complete graphs, each of which is induced by $E_{L^{l+s-1}(G)}(v)$ with $v \in V(L^{l+s-1}(G))$. We can assume that $\{E_1, E_2, \ldots, E_r\}$ is such an edge partition of $L^{l+s}(G)$, where each induced graph of E_i in $L^{l+s}(G)$ is a complete graph in $L^{l+s}(G)$ corresponding to $E_{L^{l+s-1}(G)}(v)$ with $v \in V(L^{l+s-1}(G))$, for $1 \leqslant i \leqslant r$. Consider $L^{l+s}(G) - S'$ and notice that deleting some edges in $L^{l+s}(G)$ may result in some of the edges in $L^{l+s}(G) - S'$ not lying in any triangles. For simplification let $H = L^{l+s}(G)$. Since any complete graph $H[E_i]$ with an order t is $(t - 2)$-triangular, $H[E_i] - S'$ is still triangular when $0 \leqslant |E_i \cap S'| \leqslant t - 3$. Thus if $H[E_i] - S'$ is not triangular, we must have

$$|E_i \cap S'| \geqslant t - 2. \tag{1}$$

Table 1

Case 3	e_1	e_2		
Case 3.1	K_4	K_4		
Case 3.2	K_3	K_3		
Case 3.3	K_3	K_3'		
Case 4	e_1	e_2	e_3	
Case 4.1	K_5	K_5	K_5	
Case 4.2	K_4	K_4	K_4	
Case 4.3	K_4	K_4	K_3	
Case 4.4	K_3	K_3	K_3'	
Case 4.5	K_3	K_3'	K_3''	
Case 5	e_1	e_2	e_3	e_4
Case 5.1	K_6	K_6	K_6	K_6
Case 5.2	K_5	K_5	K_5	
Case 5.3	K_4	K_4	K_4	
Case 5.4	K_4	K_4	K_4'	K_4'

By Lemma 3.2(iv), $\kappa(L^{l+s+1}(G) \geqslant s+2$. Then every essential edge cut of $L^{l+s}(G)$ has size at least $s+2$. So every essential edge cut of $L^{l+s}(G) - S'$ has size at least 2. Let G_1 be the graph obtained by contracting all the triangles, multiple edges and loops repeatedly from $L^{l+s}(G) - S'$. By Lemma 2.3, if $|V(G_1)| \leqslant 4$ for each case, then $L^{l+s}(G) - S'$ has a dominating Eulerian subgraph.

Case 1: $s = 0$. By Lemma 3.2(iii) $L^l(G)$ is triangular, then $G_1 = K_1$.

Case 2: $s = 1$. Consider $L^{l+1}(G) - S'$.

By Lemma 3.2(iii) $L^l(G)$ is triangular. From Proposition 3.1(i) $L^{l+1}(G)$ can be viewed as an edge-disjoint union of complete graphs each of which corresponds to the set of incident edges of a vertex in $L^l(G)$. If each edge of a triangle in $L^{l+1}(G)$ lies in a complete graph K_2 which corresponds to the incident edges of a vertex of degree 2 in $L^l(G)$, then $L^l(G)$ must be a C_3 or $K_{1,3}$, contrary to our assumption that $G \neq C_3$ and $G \neq K_{1,3}$. Thus for each triangle of $L^{l+1}(G)$, there are at most two edges (say z_1, z_2) each of which lies in a complete graph with order less than 3. We assume $z_1 = H[E_i] \cong K_2$, $z_2 = H[E_j] \cong K_2$ in $L^{l+1}(G)$ respectively. If $S' = \{z_1\} = E_i = E(K_2)$, then the only possible edge of $L^{l+s}(G) - S'$ not lying in a 3-cycle is z_2. So $|V(G_1)| \leqslant 2$. Next we assume that each $H[E_i]$ has at least 3 vertices.

By (1), the only possibility of making some $H[E_i] - S'$ not triangular is that $S' \subseteq E(K_3) \cong H[E_i]$. We can assume that $H[E_1] = K_3$ and by (1) and Proposition 3.1(i), for any $e \in E_2 \cup E_3 \cdots \cup E_n$, e lies in at least one triangle of $L^{l+1}(G) - S'$. That means the induced graph of $E_2 \cup E_3 \cdots \cup E_n$ in $L^{l+1}(G) - S'$ is triangular which will be contracted to a single vertex in G_1. By Lemma 3.2(iv) $\kappa(L^{l+1}(G)) \geqslant 2$, so $H[E_1] = K_3$ shares at least two vertices with $H[E_2 \cup E_3 \cdots \cup E_n]$ in $L^{l+1}(G)$, and so $|V(G_1)| \leqslant 2$.

Case 3: $s = 2$. Let $S' = \{e_1, e_2\}$ and consider $L^{l+2}(G) - S'$.

By Lemma 3.2(iii) $L^l(G)$ is triangular. Since $G \neq C_3$, $L^l(G) \neq C_3$. So any triangle of $L^l(G)$ has at most 2 vertices both of degree 2. Thus by the definition of a line graph, any triangle of $L^{l+1}(G)$ has at most one vertex of degree two and any triangle of $L^{l+2}(G)$ has at most one edge lying in a complete graph $H[E_i] \cong K_2$ of $L^{l+2}(G)$. If there is an $e \in S'$ such that $e \in E_i = E(K_2)$, then $L^{l+s}(G) - e$ is still triangular. Next we assume that each $H[E_i]$ has at least 3 vertices. By (1), the possibilities of making some $H[E_i] - S'$ not triangular are illustrated in Case 3 of Table 1.

In Table 1, Case 3.1 is the case when e_1, e_2 are contained in some K_4; Case 3.2 is the case when they are contained in some K_3; Case 3.3 is the case when one of them is contained in a K_3 and the other is contained in a different K_3. Let Z be the union of the complete graphs each of which contains some edges of S'. So we have that $Z = E(K_4)$ (Case 3.1), $Z = E(K_3)$ (Case 3.2) or $Z = E(K_3) \cup E(K_3')$ (Case 3.3) and by Proposition 3.1(i) for any $e \in E(H) - Z$, e lies in at least one triangle of $L^{l+2}(G) - S'$. By Lemma 3.2(iv) $\kappa(L^{l+2}(G)) \geqslant 3$. Then $H[Z]$ shares at least three vertices with $H[E(H) - Z]$. If $Z = E(K_4)$, then $|V(G_1)| \leqslant 2$; if $Z = E(K_3)$, then $|V(G_1)| \leqslant 1$; if $Z = E(K_3) \cup E(K_3')$, then $|V(G_1)| \leqslant 4$.

Case 4: $s = 3$. Let $S' = \{e_1, e_2, e_3\}$ and consider $L^{l+3}(G) - S'$. By Lemma 3.2(ii), $\delta(L^{l+2}(G)) \geqslant 3$. So each $H[E_i]$ is a complete graph with an order at least 3. By (1), the possibilities of making some $H[E_i] - S'$ not triangular are illustrated in Case 4 of Table 1.

For Case 4.1 through Case 4.4, let $Z = K_5$, $Z = K_4$, $Z = K_4 \cup K_3$, $Z = K_3 \cup K_3'$, respectively. By Proposition 3.1(i) for any $e \in E(H) - Z$, e lies in at least one triangle of $L^{l+3}(G) - S'$. By Lemma 3.2(ii) $\kappa(L^{l+3}(G)) \geqslant 4$. Then $H[Z]$ shares at least four vertices with $H[E(H) - Z]$. So $|V(G_1)| \leqslant 4$.

For Case 4.5, assume that $L^{l+3}(G)$ has distinct cliques $L_i \cong K_3$ such that $e_i \in E(L_i)$ where each L_i is isomorphic to some $H[E_i]$ described in Proposition 3.1(i), $i = 1, 2, 3$.

Since $\kappa'(L^{l+3}(G)) \geqslant \kappa(L^{l+3}(G)) \geqslant 4$, $L^{l+3}(G) - S$ is connected, and it has a cut edge if and only if for some e', $\{e', e_1, e_2, e_3\}$ is an edge cut of $L^{l+3}(G)$. Since each $e_i \in E(L_i)$(each L_i is a 3-cycle) and since for every edge cut D, $|D \cap E(L_i)| \equiv 0 \pmod 2$, $\{e_1, e_2, e_3\}$ cannot be in an edge cut of size 4. Thus $\kappa'(L^{l+3}(G) - S') \geqslant 2$.

Since $e_i \in E(L_i)$, $i = 1, 2, 3$, $L^{l+3}(G) - S'$ has at most 6 edges not lying in a 3-cycle. It follows that G_1 is 2-edge-connected with at most 6 edges (and so at most 6 vertices). Since every 2-edge-connected graph with at most 6 vertices either has a spanning Eulerian subgraph, or is isomorphic to $K_{2,3}$, we may assume that $G_1 \cong K_{2,3}$. Since $L^{l+3}(G)$ is an edge-disjoint union of cliques each of which has order at least 3, this can be true only when each of e_1, e_2, e_3 is incident with both preimages of the two vertices of degree 3 in G_1, and so $L^{l+3}(G)$ has at least one edge cut of size 3, contrary to the fact that $\kappa'(L^{l+3}(G)) \geqslant 4$. This contradiction indicates that G_1 has a spanning Eulerian subgraph, and by Lemma 2.1 $L^{l+3}(G) - S'$ has a dominating Eulerian subgraph.

Case 5: $s = 4$. Consider $L^{l+4}(G) - S'$. By Lemma 3.2(ii), $\delta(L^{l+3}(G)) \geqslant 4$. So each $H[E_i]$ is a complete graph with an order at least 4. By (1), the possibilities of making some $H[E_i] - S'$ not triangular are illustrated in Case 5 of Table 1.

For Case 5.1 through Case 5.4, let $Z = K_6, K_5$, $Z = K_4$, $Z = K_4 \cup K_4'$, respectively. By Proposition 3.1(i) for any $e \in E(H) - Z$, e lies in at least one triangle of $L^{l+4}(G) - S'$. By Lemma 3.2(iv) $\kappa(L^{l+4}(G)) \geqslant 5$. Then $H[Z]$ shares at least five vertices with $H[E(H) - Z]$. So $|V(G_1)| \leqslant 4$.

Case 6: $s = 5$. Consider $L^{l+5}(G) - S'$. By Lemma 3.2(ii), $\delta(L^{l+4}(G)) \geqslant 6$. So each $H[E_i]$ is a complete graph with order at least 6. By (1), the possibilities of making some $H[E_i] - S'$ not triangular are that S' is contained in some K_7 or at least 4 edges of S' are contained in some K_6. By Lemma 3.2(iv) $\kappa(L^{l+5}(G)) \geqslant 6$, so K_7 or K_6 shares at least six vertices with other E_is. So $|V(G_1)| \leqslant 2$. □

4. Proof of the main theorem

Theorem 4.1 (*Harary and Nash-Williams [4]*). *Let H be a graph with $E(H) \geqslant 3$. The line graph $L(H)$ of a graph H is Hamiltonian if and only if H has a dominating Eulerian subgraph.*

Theorem 4.1 reveals the relationship between a dominating Eulerian subgraph in H and a Hamiltonian cycle in $L(H)$. We use this theorem and Lemma 3.3 to prove our main result.

Proof of Theorem 1.1. Let S be a vertex set of $L^{l+s+1}(G)$ with $|S| \leqslant s$. Let S' be the edge set of $L^{l+s}(G)$ corresponding to s. By Proposition 3.1(iii), $L^{l+s+1}(G) - S = L(L^{l+s}(G) - S')$. By Lemma 3.2(iv), $\kappa(L^{l+s+1}(G)) \geqslant s + 2$. Then $\kappa(L^{l+s+1}(G) - S) \geqslant 2$ and so $L^{l+s}(G) - S'$ has no essential 1-edge-cuts. By Lemma 3.3, $L^{l+s}(G) - S'$ has a dominating Eulerian subgraph. Then by Theorem 4.1, $L^{l+s+1}(G) - S$ is Hamiltonian. And so $L^{l+s+1}(G)$ is s-Hamiltonian. □

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London, Elsevier, New York, 1976.
[2] G. Chartrand, C.E. Wall, On the Hamiltonian index of a graph, Studia Sci. Math. Hungar. 8 (1973) 43–48.
[3] L.H. Clark, N.C. Wormald, Hamilton-like indices of graphs, Ars Combin. 15 (1983) 131–148.
[4] F. Harary, C.St.J.A. Nash-Williams, On Eulerian and Hamiltonian graphs and line graphs, Canad. Math. Bull. 8 (1965) 701–709.
[5] H.-J. Lai, On the Hamiltonian index, Discrete Math. 69 (1988) 43–53.
[7] W. Mantel, Problem 28 (solution by H. Gouwentak, W. Mantel, J. Texeira de Mattes, F. Schuh and W. A. Wythoff), Wiskundige Opgaven 10 (1907) 60–61.