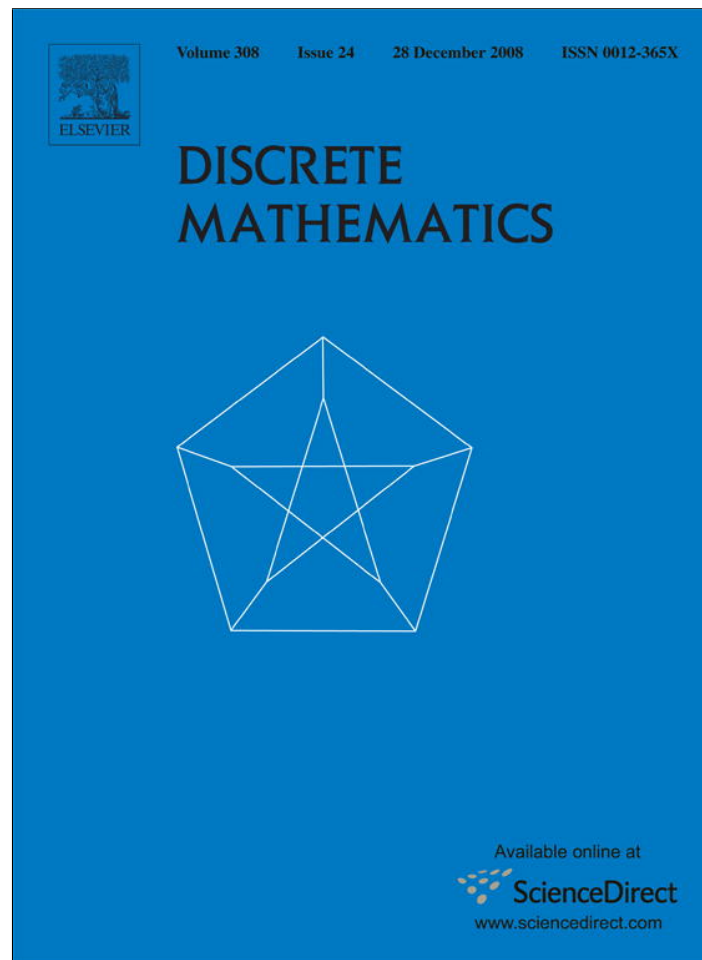


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Note

Degree sequence and supereulerian graphs

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Abstract

A sequence $d = (d_1, d_2, \dots, d_n)$ is graphic if there is a simple graph G with degree sequence d , and such a graph G is called a realization of d . A graphic sequence d is line-hamiltonian if d has a realization G such that $L(G)$ is hamiltonian, and is supereulerian if d has a realization G with a spanning eulerian subgraph. In this paper, it is proved that a nonincreasing graphic sequence $d = (d_1, d_2, \dots, d_n)$ has a supereulerian realization if and only if $d_n \geq 2$ and that d is line-hamiltonian if and only if either $d_1 = n - 1$, or $\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2} (d_j - 2)$.

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Keywords: Degree sequence; Collapsible graphs; Hamiltonian line graphs; Supereulerian graphs**1. Introduction**

We consider finite graphs in this note. Undefined terms can be found in [1]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A vertex $v \in V(G)$ is called a *pendent vertex* if $d(v) = 1$. Let $D_1(G)$ denote the set of all pendent vertices of G . An edge $e \in E(G)$ is called a *pendent edge* if one of its endpoints is a pendent vertex. If $v \in V(G)$, then $N_G(v) = \{u : uv \in E(G)\}$; and if $T \subseteq V(G)$, then $N_G(T) = \{u \in V(G) \setminus T : uv \in E(G) \text{ and } v \in T\}$. When the graph G is understood in the context, we may drop the subscript G .

A *circuit* is a connected 2-regular graph. A *cycle* is a graph such that the degree of each vertex is even. A cycle C of G is a *spanning eulerian subgraph* of G if C is connected and spanning. A graph G is *supereulerian* if G contains a spanning eulerian subgraph.

If G has vertices v_1, v_2, \dots, v_n , the sequence $(d(v_1), d(v_2), \dots, d(v_n))$ is called a *degree sequence* of G . A sequence $d = (d_1, d_2, \dots, d_n)$ is *nonincreasing* if $d_1 \geq d_2 \geq \dots \geq d_n$. A sequence $d = (d_1, d_2, \dots, d_n)$ is *graphic* if there is a simple graph G with degree sequence d . Furthermore, such a simple graph G is called a *realization* of d . Let \mathcal{G} denote the set of all graphic degree sequences. A sequence $d \in \mathcal{G}$ is *line-hamiltonian* if d has a realization G such that $L(G)$ is hamiltonian.

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The sequence S is called a bipartite graphic sequence if there is a bipartite graph G with bipartition $\{X, Y\}$ such that $\{d(x_1), \dots, d(x_m)\} = \{s_1, \dots, s_m\}$, and $\{d(y_1), \dots, d(y_n)\} = \{t_1, \dots, t_n\}$ where $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ and $d(v)$ is the degree of a vertex v ; the graph G is called a realization of S . In [9], Luo et al. proved the following theorem.

Theorem 1.1 (Luo, Zang, and Zhang [9]). *Every bipartite graphic sequence with the minimum degree $\delta \geq 2$ has a realization that admits a nowhere-zero 4-flow.*

In this paper, the following result is obtained.

Theorem 1.2. *Let $d = (d_1, d_2, \dots, d_n) \in \mathcal{G}$ be a nonincreasing sequence. Then d has a supereulerian realization if and only if either $n = 1$ and $d_1 = 0$, or $n \geq 3$ and $d_n \geq 2$.*

In [7], Jaeger proved the following result.

Theorem 1.3 (Jaeger [7]). *Every supereulerian graph admits a nowhere-zero 4-flow.*

Theorem 1.2, together with 1.3, implies a result analogous to Theorem 1.1.

Theorem 1.4 (Luo, Zang, and Zhang [9]). *Let $d = (d_1, d_2, \dots, d_n) \in \mathcal{G}$ be a nonincreasing sequence. Then d has a realization that admits a nowhere-zero 4-flow if and only if $d_n \geq 2$.*

The following characterization on line-hamiltonian graphic sequences is also obtained.

Theorem 1.5. *Let $d = (d_1, d_2, \dots, d_n) \in \mathcal{G}$ be a nonincreasing sequence with $n \geq 3$. The following are equivalent.*

- (i) d is line-hamiltonian.
- (ii) either $d_1 = n - 1$, or

$$\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2} (d_j - 2). \tag{1}$$

- (iii) d has a realization G such that $G - D_1(G)$ is supereulerian.

2. Collapsible sequences

Let $X \subseteq E(G)$. The contraction G/X is the graph obtained from G by identifying the endpoints of each edge in X and then deleting the resulting loops. Note that multiple edges may arise.

Let $O(G)$ denote the set of vertices of odd degree in G . A graph G is collapsible if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod{2}$, G has a connected spanning subgraph H_R such that $O(H_R) = R$. A sequence $d = (d_1, d_2, \dots, d_n)$ is collapsible if d has a simple collapsible realization.

Theorem 2.1. *Let G be a connected graph. Each of the following holds.*

- (i) (Catlin, Corollary of Lemma 3, [2]) *If H is a collapsible subgraph of G , then G is collapsible if and only if G/H is collapsible.*
- (ii) (Catlin, Corollary 1, [2]) *If G contains a spanning tree T such that each edge of T is contained in a collapsible subgraph of G , then G is collapsible.*
- (iii) (Caltin, Theorem 7, [2]) *C_2, K_3 (circuits of 2 or 3 edges) are collapsible.*
- (iv) (Caltin, Theorem 2, [2]) *If G is collapsible, then G is supereulerian.*

Theorem 2.1(ii) and (iii) imply Corollary 2.2(i); Theorem 2.1(i) and (iii) imply Corollary 2.2(ii).

Corollary 2.2. (i) *If every edge of a spanning tree of G lies in a K_3 , then G is collapsible.*

(ii) *If $G - v$ is collapsible and if v has degree at least 2 in G , then G is collapsible.*

Corollary 2.3. *If $d = (d_1, d_2, \dots, d_n)$ is a nonincreasing graphic sequence with $d_1 = n - 1$ and $d_n \geq 2$, then every realization of d is collapsible.*

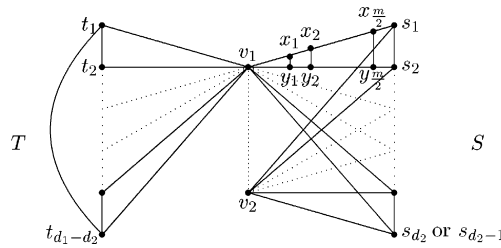


Fig. 1. G .

Proof. Let G be a realization of d with $N(v_1) = \{v_2, \dots, v_n\}$ and let T be the spanning tree with $E(T) = \{v_1v_k : 2 \leq k \leq n\}$. Since $d_n \geq 2$ and $N(v_1) = \{v_2, \dots, v_n\}$, for any $v_i \in \{v_k : 2 \leq k \leq n\}$, there is $v_j \in \{v_k : 2 \leq k \leq n\} \setminus \{v_i\}$ such that $v_iv_j \in E(G)$. It follows that every edge of T lies in a K_3 , and so by Corollary 2.2(i), G is collapsible. \square

Lemma 2.4. *If $d = (d_1, d_2, \dots, d_n)$ is a nonincreasing graphic sequence with $d_3 = \dots = d_n = 3$, then d is collapsible.*

Proof. Let v_1, v_2 be two vertices and let

$$S = \begin{cases} \{s_1, s_2, \dots, s_{d_2}\} & \text{if } d_2 \text{ is even} \\ \{s_1, s_2, \dots, s_{d_2-1}\} & \text{if } d_2 \text{ is odd} \end{cases}$$

be a set of vertices other than $\{v_1, v_2\}$ and let $T = \{t_1, t_2, \dots, t_{d_1-d_2}\}$ be a set of $d_1 - d_2$ vertices not in $S \cup \{v_1, v_2\}$. Let H denote the graph obtained from $\{v_1, v_2\} \cup S \cup T$ by joining v_2 to each vertex of S and joining v_1 to each vertex of $S \cup T$ (if d_2 is odd, then we also join v_1 and v_2). Note that $d_H(v_1) = d_2 + d_1 - d_2 = d_1$, $d_H(v_2) = d_2$, $d_H(s) = 2$ for $s \in S$ and $d_H(t) = 1$ for $t \in T$.

Case 1 $d_1 - d_2 \geq 3$. Let $C = t_1t_2 \dots t_{d_2-d_1}t_1$ be a circuit passing through all vertices of T and let $H' = H \cup E(C)$. As $|S|$ is even, we join all vertices of S in pairs (i.e., s_1s_2, s_3s_4, \dots) in H' and denote the resulting graph by H'' . Note that $d_{H''}(v_1) = d_1$, $d_{H''}(v_2) = d_2$ and $d_{H''}(v) = 3$ for $v \in S \cup T$.

Also note that

$$|V(H'')| = \begin{cases} 2 + d_1 & \text{if } d_2 \text{ is even} \\ 1 + d_1 & \text{if } d_2 \text{ is odd.} \end{cases}$$

Let $m = n - |V(H'')|$. Then

$$m = \begin{cases} n - (2 + d_1) & \text{if } d_2 \text{ is even} \\ n - (1 + d_1) & \text{if } d_2 \text{ is odd} \end{cases}$$

is even as n and d_1 have the same parity if d_2 is even while n and d_1 have different parity if d_2 is odd. By the construction of H'' , H'' contains a triangle $v_1s_1s_2$. We subdivide v_1s_1 and v_1s_2 $\frac{m}{2}$ times, respectively, and let $x_1, x_2, \dots, x_{\frac{m}{2}}$ and $y_1, y_2, \dots, y_{\frac{m}{2}}$ be the new vertices resulted in subdividing v_1s_1 and v_1s_2 , respectively. Then for $1 \leq j \leq \frac{m}{2}$, we join x_jy_j and denote the resulting graph by G (see Fig. 1). Hence, by the construction of G , G is a realization of d .

Case 2 $d_1 - d_2 = 2$. Let G be the construction as in Case 1 except that we join t_1 to s_1 , t_1 to t_2 , t_2 to s_2 , and delete s_1s_2 .

Case 3 $d_1 - d_2 = 1$. Let G be the construction as in Case 1 except that we join t_1 to both s_1 and s_2 , and delete s_1s_2 .

By Theorem 2.1(iii), K_3 is collapsible. If we contract $v_1x_1y_1$, then we get a triangle $v_1x_2y_2$ in the contraction, and if we contract $v_1x_2y_2$, then we get a triangle $v_1x_3y_3$ in the contraction. Repeat this process by contracting a triangle $v_1x_iy_i$ for each i with $1 \leq i \leq \frac{m}{2}$ in the subsequent contraction. In Case 2 and Case 3, this process results in a graph in which each edge lies in a triangle. In Case 1, this process eventually results in a triangle $v_1s_1s_2$. After contracting $v_1t_1t_2$ we obtained a graph in which each edge lies in a triangle. Since 2-circuit is collapsible, the contraction of a maximally collapsible graph will result in a simple graph. By Corollary 2.2(i) and (ii), G is collapsible in each case. \square

Theorem 2.5 (Havel [6], Hakimi [4]). *Let $d = (d_1, d_2, \dots, d_n)$ be a nonincreasing sequence. Then d is graphic if and only if $d' = (d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$ is graphic.*

Theorem 2.6 (Kleitman and Wang [8]). Let $d = (d_1, d_2, \dots, d_n)$ be a nonincreasing sequence. Then d is graphic if and only if $d' = (d_1 - 1, \dots, d_k - 1, d_{k+1}, \dots, d_{k-1}, d_{k+1}, \dots, d_n)$ is graphic.

Lemma 2.7. If $d = (d_1, d_2, \dots, d_n)$ is a nonincreasing sequence with $n \geq 4$ and $d_n = 3$, then d is graphic if and only if $d' = (d_1 - 1, d_2 - 1, d_3 - 1, d_4, \dots, d_{n-1})$ is graphic.

Proof. Theorem 2.6 implies Lemma 2.7 by letting $k = n$ and $d_k = 3$. \square

Theorem 2.8. If $d = (d_1, d_2, \dots, d_n)$ is a nonincreasing graphic sequence with $n \geq 4$ and $d_n \geq 3$, then d has a collapsible realization.

Proof. We argue by induction on n . If $n = 4$, then the assumption that $d_n \geq 3$ forces that the only realization of d is K_4 , and by Theorem 2.1(i), (iii), K_4 is collapsible.

Next we assume that $n \geq 5$. If $d_n \geq 4$, then $d_2 - 1 \geq d_3 - 1 \geq \dots \geq d_{d_1+1} - 1 \geq 3$ and $d_{d_1+2} \geq \dots \geq d_n \geq 3$. By Theorem 2.5 and the induction hypothesis, $(d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$ has a collapsible realization H . Assume that $V(H) = \{v_2, v_3, \dots, v_n\}$ such that $v_2, v_3, \dots, v_{d_1+1}$ have degrees $d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1$ in H , respectively, and such that v_{d_1+2}, \dots, v_n have degrees d_{d_1+2}, \dots, d_n in H , respectively. Then obtain a realization H' of d from H by adding a new vertex v_1 and joining v_1 to $v_2, v_3, \dots, v_{d_1+1}$, respectively. By Corollary 2.2(ii) H' is collapsible.

Therefore, we may assume that $d_n = 3$. If $d_3 = 3$, then by Lemma 2.4, $(d_1, d_2, 3, \dots, 3)$ is collapsible. Hence we assume further that $d_3 \geq 4$.

In this case, $d_1 - 1 \geq d_2 - 1 \geq d_3 - 1 \geq 3$ and $d_4 \geq \dots \geq d_n = 3$. By Lemma 2.7, $(d_1 - 1, d_2 - 1, d_3 - 1, d_4, \dots, d_{n-1})$ is graphic. By the induction hypothesis, $(d_1 - 1, d_2 - 1, d_3 - 1, d_4, \dots, d_{n-1})$ has a collapsible realization K with $V(K) = \{u_1, u_2, \dots, u_{n-1}\}$ such that u_1, u_2, u_3 have degrees $d_1 - 1, d_2 - 1, d_3 - 1$ in K , respectively, and such that u_4, u_5, \dots, u_{n-1} have degrees d_4, \dots, d_{n-1} in K , respectively. We obtain a realization K' of d from K by adding a new vertex u_n and joining u_n to u_1, u_2, u_3 , respectively. By Corollary 2.2(ii) K' is collapsible. \square

3. Supereulerian sequence and Hamiltonian line graph

Let X and Y be two sets. Then $X \Delta Y = (X \cup Y) - (X \cap Y)$ denotes the symmetric difference of X and Y . We start with the following observation (Lemma 3.1) and a few other lemmas. Throughout this section, we assume that $n \geq 3$.

Lemma 3.1 (Edmonds [3]). If $d = (d_1, d_2, \dots, d_n)$ is a nonincreasing graphic sequence with $d_n \geq 2$, then there exists a 2-edge-connected realization of d .

Lemma 3.2. Let $d = (d_1, d_2, \dots, d_n)$ be a nonincreasing sequence with $d_1 \leq n - 2$ and $d_n = 2$. Then d is graphic if and only if either of the following holds.

- (i) $d' = (d_1, d_2, \dots, d_{n-1})$ is graphic, or
- (ii) $d'' = (d_1, d_2, \dots, d_i - 1, \dots, d_j - 1, \dots, d_{n-1})$ is graphic for some $d_i \geq 3$ and $d_j \geq 3$, or
- (iii) both $d_{n-1} = d_n = 2$, and for some j with $1 \leq j < n - 1$ and with $d_j \geq 4$, $d''' = (d_1, d_2, \dots, d_{j-1}, d_j - 2, d_{j+1}, \dots, d_{n-2})$ is graphic, or
- (iv) $n = 3$ and $d = (2, 2, 2)$.

Proof. Suppose that $d = (d_1, d_2, \dots, d_n)$ is graphic. Then there exists a 2-edge-connected realization G of d with $d(v_i) = d_i$ for $1 \leq i \leq n$. Suppose that $N(v_n) = \{v_i, v_j\}$. If $v_i v_j \notin E(G)$, then $G - v_n + \{v_i v_j\}$ is a realization of $(d_1, d_2, \dots, d_{n-1})$, and so (i) holds. Thus we assume that $v_i v_j \in E(G)$.

If both v_i, v_j have degree at least 3 in G , then d'' is graphic and so (ii) must hold. Thus we may assume further that v_i has degree 2. If v_j also has degree 2 in G , then $n = 3$ and (iv) must hold. Therefore, we may assume that v_j has degree at least 3, and so v_j is a cut-vertex of G . Since G is 2-edge-connected and since v_j is a cut-vertex, $d_j = d(v_j) \geq 4$. In this case, d''' is the degree sequence of $G - \{v_n, v_i\}$, and so d''' is graphic. The sufficiency can be proved by reversing the arguments above. \square

Proof of Theorem 1.2. If a nonincreasing graphic sequence $d = (d_1, d_2, \dots, d_n)$ has a supereulerian realization, then we must have $d_n \geq 2$ as every supereulerian graph is 2-edge-connected.

We argue by induction on n to prove the sufficiency. If $n = 3$, then since $d_n \geq 2$, K_3 , a supereulerian graph, is the only realization of d .

Suppose that $n \geq 4$ and that the theorem holds for all such graphic sequences with fewer than n entries. Let $d = (d_1, d_2, \dots, d_n) \in \mathcal{G}$ be a nonincreasing sequence with $d_n \geq 2$. If $d_n \geq 3$, then by Theorem 2.8, d has a collapsible realization G . By Theorem 2.1(vi), G is supereulerian. If $d_1 = d_2 = \dots = d_n = 2$, then C_n is a supereulerian realization of d .

In the following, we assume that $d_1 > d_n = 2$. If $d_1 = n - 1$, then by Corollary 2.3, d has a realization G such that G is collapsible. By Theorem 2.1(iv), G is supereulerian. Thus d in this case must be supereulerian.

Thus we may assume that $2 < d_1 \leq n - 2$. By Lemma 3.2, one of the conclusions of Lemma 3.2 (except Lemma 3.2(iv)) must hold.

If Lemma 3.2(i) holds, then $d' = (d_1, d_2, \dots, d_{n-1})$ is graphic. By induction, there is a supereulerian realization G' of d' . Let C' be a spanning eulerian subgraph of G' and $e = uv$ be an edge of C' . Then by subdividing e of G' into uv_n, v_nv , we obtain a supereulerian realization of d as $d_n = 2$.

If Lemma 3.2(ii) holds, then for some i, j , $d'' = (d_1, d_2, \dots, d_i - 1, \dots, d_j - 1, \dots, d_{n-1})$ is graphic, with $d_i \geq 3$ and $d_j \geq 3$. By induction, there is a supereulerian realization G'' of d'' . Let C'' be a spanning eulerian subgraph of G'' . If $v_i v_j \in E(G'')$, then let $C_1 = v_i v_j v_n$ and so $G = G'' + \{v_i v_n, v_j v_n\}$ is a supereulerian realization of d . If $v_i v_j \notin E(G'')$, then we can get a realization G of d from $G'' + \{v_i v_j\}$ by subdividing an edge $e = uv$ of C' into uv_n and $v_n v$.

If Lemma 3.2(iii) holds, then both $d_{n-1} = d_n = 2$, and for some j with $1 \leq j < n - 1$ and with $d_i \geq 4$, $d''' = (d_1, d_2, \dots, d_{j-1}, d_j - 2, d_{j+1}, \dots, d_{n-2})$ is graphic. By induction, there is a supereulerian realization G''' of d''' . Let C''' be a spanning eulerian subgraph of G''' . Obtain G from G''' by adding two new vertices v_{n-1} and v_n and three new edges $v_j v_n, v_n v_{n-1}, v_{n-1} v_j$. Then G is a realization of d , and $E(C''') \cup \{v_j v_n, v_n v_{n-1}, v_{n-1} v_j\}$ is a spanning eulerian subgraph of G . \square

In order to prove Theorem 1.5, we need the following result which shows the relationship between hamiltonian circuits in the line graph $L(G)$ and eulerian subgraph in G . A subgraph H of G is dominating if $E(G - V(H)) = \emptyset$.

Theorem 3.3 (Harary and Nash-Williams, [5]). *Let $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

Proof of Theorem 1.5. (i) \Rightarrow (ii). Let G be a realization of d such that $L(G)$ is hamiltonian. By Theorem 3.3, G has a dominating eulerian subgraph H . If $d_1 = n - 1$, then we are done. Suppose that $d_1 \leq n - 2$. Then $|V(H)| \geq 2$. For any v_i with $d(v_i) = 1$, v_i must be adjacent to a vertex v_j in H and so $d_{G-E(H)}(v_j)$ is no less than the number of degree 1 vertices adjacent to v_j . Furthermore, since H is eulerian and nontrivial, $d_H(v_j) \geq 2$ and so (1) must hold.

(ii) \Rightarrow (iii) Suppose $d \in \mathcal{G}$ is a nonincreasing sequence such that $d_n \geq 1$ and $\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2} (d_j - 2)$. If $d_n \geq 2$, then by Theorem 1.2, d has a supereulerian realization. So we assume that $d_n = 1$.

Claim 3.4. *Any realization of d contains a nontrivial circuit.*

Suppose that there exists a realization G of d such that G is a tree. We may assume that $d_i \geq 2$ for $1 \leq i \leq k$ and $d_j = 1$ for $k + 1 \leq j \leq n$. Then

$$\sum_{i=1}^k d_i + (n - k) = \sum_{i=1}^k d_i + \sum_{i=k+1}^n d_i = \sum_{i=1}^n d_i = 2|E(G)| = 2(n - 1),$$

and so

$$\sum_{i=1}^k (d_i - 2) + (n - k) = 2(n - 1) - 2k.$$

Hence

$$\sum_{d_j \geq 2} (d_j - 2) = \sum_{i=1}^k (d_i - 2) = 2(n - 1) - 2k - (n - k) = n - k - 2 < n - k = \sum_{d_i=1} d_i,$$

contrary to (1). This completes the proof of the claim.

Thus we assume that G is a realization of d containing a nontrivial circuit C .

Claim 3.5. *There is a realization G of d such that $\delta(G - D_1(G)) \geq 2$.*

As G contains a nontrivial circuit C , $G - D_1(G)$ is not empty. Let $S = N(D_1(G))$. It suffices to show that for each $s \in S$, $N_{G-D_1(G)}(s) \geq 2$. Suppose, to the contrary, that there is $s \in S$ such that $N_{G-D_1(G)}(s) = 1$. Choose G to be a graph such that $P(G) = \{s : s \in S \text{ with } d_G(s) = d_t \geq 2 \text{ such that } N_{G-D_1(G)}(s) = 1\}$ is as small as possible. Let $x \in P(G)$. Then $x \notin C$. Choose $e \in E(C)$ and we subdivide e and let v_e denote the subdivision vertex. And we delete $d_t - 1$ pendent edges of x , add $d_t - 2$ pendent edges to v_e and denote the resulting graph G_x . (Note that if $d_t - 2 = 0$, then we subdivide e without adding any pendent edges.) Let $N_1(x)$ be the set of pendent vertices adjacent to x . So $d_{G_x}(v_e) = 2 + d_t - 2 = d_t$ and $|D_1(G_x)| = |(D_1(G) - N_1(x)) \cup \{x\}| + d_t - 2 = |D_1(G)| - (d_t - 1) + 1 + d_t - 2 = |D_1(G)|$ but $|P(G_x)| < |P(G)|$, contradicting the choice of G .

(iii) \Rightarrow (i) If G is a realization of d such that $G - D_1(G)$ is supereulerian, then by Theorem 3.3, $L(G)$ is hamiltonian. \square

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