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Note

On s-hamiltonian-connected line graphs

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Abstract

A graph G is hamiltonian-connected if any two of its vertices are connected by a Hamilton path (a path including every vertex of G); and G is s-hamiltonian-connected if the deletion of any vertex subset with at most s vertices results in a hamiltonian-connected graph. In this paper, we prove that the line graph of a $(t+4)$-edge-connected graph is $(t+2)$-hamiltonian-connected if and only if it is $(t+5)$-connected, and for $s \geqslant 2$ every $(s+5)$-connected line graph is s-hamiltonian-connected.
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1. Introduction

We consider finite loopless graphs. Undefined notation and terminology can be found in [1]. The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. It is well known that high connectivity does not assure the existence of a hamiltonian cycle, as evidenced by the complete bipartite graph $K_{m+1,m}$ for large m. However, for a line graph, Thomassen [9] made the following conjecture.

Conjecture 1 (*Thomassen [9]*). Every 4-connected line graph is hamiltonian.

An edge cut X of G is *essential* if each side of $G - X$ contains an edge. Note that the line graph $L(G)$ has a vertex cut of k vertices if and only if G has an essential edge cut of size k. A graph G is *hamiltonian-connected* if for any $u, v \in V(G)$, G has a hamiltonian (u, v)-path; and G is *s-hamiltonian-connected* if for any $X \subseteq V(G)$ with $|X| \leqslant s$, $G - X$ is hamiltonian-connected. By definitions, s-hamiltonian-connected graphs are hamiltonian-connected; and hamiltonian-connected graphs are hamiltonian. Zhan made the following progresses towards Thomassen's Conjecture.

Theorem 1.1 (*Zhan [10, Theorem 3]*). *If $\kappa'(G) \geqslant 4$, then $L(G)$ is hamiltonian-connected.*

Theorem 1.2 (*Zhan [11, Theorem 3]*). *If $\kappa(L(G)) \geqslant 7$, $L(G)$ is hamiltonian-connected.*

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The main purpose of this paper is to sharpen both theorems obtained by Zhan. In fact, we proved the following two theorems in this note, for integers $t \geqslant 0$ and $s \geqslant 2$.

Theorem 1.3. *The line graph of a* $(t+4)$*-edge-connected graph is* $(t+2)$*-hamiltonian-connected if and only if it is* $(t+5)$*-connected.*

Theorem 1.4. *Every* $(s+5)$*-connected line graph is* s*-hamiltonian-connected.*

In Section 2, we briefly review the needed tools in the proofs of the main results. The main results will be proved in Section 3. In the last section, we present an open problem to be further investigated.

2. Preliminaries

In this section, we review some of mechanisms needed in the arguments. A subgraph H of a graph G is *dominating* if $G - V(H)$ is edgeless. Harary and Nash-Williams proved a very useful connection between hamiltonian cycles in the line graph $L(G)$ and dominating eulerian subgraphs in G.

Theorem 2.1 (*Harary and Nash-Williams [6]*). *For a connected graph* G *with* $|E(G)| \geqslant 3$, $L(G)$ *is Hamiltonian if and only if* G *has a dominating eulerian subgraph.*

Let $e_1, e_2 \in E(G)$. A trail in G whose first edge is e_1 and last edge is e_2 is called an (e_1, e_2)*-trail*. Let T be an (e_1, e_2)*-trail*. Then T is *dominating* if every $e \in E(G)$ is incident with an internal vertex of T; and T is *spanning* if T is dominating and $V(T) = V(G)$. For $v_1, v_2 \in V(G)$, a trail in G whose origin is v_1 and terminus is v_2 is called a (v_1, v_2)*-trail*, and it is a *spanning* (v_1, v_2)*-trail* if it contains every vertex of G.

Theorem 2.2 (*Catlin and Lai [5, Theorem 4]*). *Let* G *be a graph and let* $e_1, e_2 \in E(G)$. *If* G *has two edge-disjoint spanning trees, then exactly one of the following holds*:

(a) *G has a spanning* (e_1, e_2)*-trail; and*
(b) $\{e_1, e_2\}$ *is an essential edge-cut of* G.

With similar arguments in [6], the following is obtained.

Theorem 2.3. *Let* G *be a connected graph with at least three edges. The line graph* $L(G)$ *is hamiltonian-connected if and only if for any* $e_1, e_2 \in E(G)$, G *has a dominating* (e_1, e_2)*-trail.*

A graph G is *collapsible* if for any even subset $R \subseteq V(G)$, G has a spanning connected subgraph H such that $O(H) = R$ where $O(H)$ denotes the set of odd vertices of H. The *reduction* of G is the graph obtained from G by contracting each maximal collapsible subgraph of G to a distinct vertex.

Theorem 2.4 (*Catlin [2, Theorem 2]*). *If* G *has two edge disjoint spanning trees, then* G *is collapsible.*

Theorem 2.5 (*Li et al. [8, Lemma 2.2]*). *If* G *is collapsible, then* $\forall x, y \in V(G)$, *there exists a* (x, y)*-trail* T *of* G *such that* $V(T) = V(G)$.

Define $F(G)$ to be the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees.

Theorem 2.6 (*Catlin [2, Theorem 7]*). *If* $F(G) \leqslant 1$, *then* G *is collapsible if and only if* $\kappa'(G) \geqslant 2$.

Theorem 2.7 (*Catlin et al. [4, Theorem 1.3]*). *If* G *is connected and if* $F(G) \leqslant 2$, *then* G *is collapsible or the reduction of* G *is either* K_2 *or a* $K_{2,t}$ *for some* $t \geqslant 1$.

Theorem 2.8 (*Zhan [11, Corollary 10]*). *Let G be a graph with $\kappa'(G)\geqslant 3$ and $\kappa(L(G))\geqslant 7$. Then for every pair x and y of edges of G, the subgraph $G-\{x,y\}$, or $G-\{x\}$ if x and y have an end-vertex of degree 3 in common, can be decomposed into two connected factors F_1 and F_2.*

The *core* of a graph G, denoted by G_0, is obtained by deleting all vertices of degree 1 and contracting exactly one edge of xy or yz for each path xyz in G with $d(y)=2$.

By the definition of the core graph G_o, all vertices of degree one or two are deleted or contracted and so $\delta(G_o)\geqslant 3$. Note that an essential edge cut of G corresponds to a vertex cut of $L(G)$ and vice versa. So if $\kappa(L(G))\geqslant 7$, then $\kappa'(G_o)\geqslant 3$ and $\kappa(L(G_o))\geqslant 7$.

For a graph G and an integer $i\geqslant 1$, $D_i(G)$ denotes the set of vertices of degree i in G and $\tau(G)$ denotes the number of edge-disjoint spanning trees of G. The following is useful.

Theorem 2.9 (*Catlin [3, Theorem 2]*). *Let G be a connected graph and let $k\geqslant 1$ be an integer, then $\kappa'(G)\geqslant 2k$ if and only if $\forall X\subseteq E(G)$ with $|X|\leqslant k$, $\tau(G-X)\geqslant k$.*

3. Main results

Throughout this section, we assume $t\geqslant 0$ and $s\geqslant 2$ are integers.

Theorem 3.1. *Let G be a graph with $\kappa'(G)\geqslant t+4$. Then $L(G)$ is $(t+2)$-hamiltonian-connected if and only if $\kappa(L(G))\geqslant t+5$.*

Proof. Note that K_4-e (where e is an edge of a complete graph K_4) is 2-connected, but not hamiltonian-connected. So a hamiltonian-connected graph is 3-connected and an s-hamiltonian-connected graph is $(s+3)$-connected. Thus if $L(G)$ is $(t+2)$-hamiltonian-connected, then $\kappa(L(G))\geqslant t+5$. It suffices to prove that if $\kappa(L(G))\geqslant t+5$, then $L(G)$ is $(t+2)$-hamiltonian-connected.

To show that $L(G)$ is $(t+2)$-hamiltonian connected, it suffices to show that $\forall Y\subseteq V(L(G))=E(G)$, with $|Y|\leqslant t+2$ and $\forall e_1,e_2\in E(G)-Y$, $L(G)-Y$ has a hamiltonian (e_1,e_2)-path. By Theorem 2.3, this amounts to show that $G-Y$ has a dominating (e_1,e_2)-trail.

Since $|Y|\leqslant t+2$, we can choose a subset $Y_1\subseteq Y$, and let $Y_2=Y-Y_1$, such that $|Y_1|\leqslant t$ and $|Y_2|\leqslant 2$. Since $\kappa'(G)\geqslant 4$, $\kappa'(G-Y_1)\geqslant t+4-t=4$. By Theorem 2.9, $\tau(G-Y)=\tau((G-Y_1)-Y_2)\geqslant 2$.

For any $e_1,e_2\in E(G)$, since G has no essential $(4+t)$-edge-cut, $G-Y$ has no essential 2-edge-cut. Therefore, $\{e_1,e_2\}$ is not an essential edge-cut of $G-Y$. By Theorem 2.2, $G-Y$ has a spanning (e_1,e_2)-trail. □

Let $t=0$ in Theorem 3.1, we obtain a result stronger than Theorem 1.1.

Corollary 3.2. *Let G be a graph with $\kappa'(G)\geqslant 4$. Then $L(G)$ is 2-hamiltonian-connected if and only if $\kappa(L(G))\geqslant 5$.*

Lemma 3.3. *If $\tau(G_o)\geqslant 2$ and $\kappa(L(G))\geqslant 3$, then $\forall e_1,e_2\in E(G)$, G has a dominating (e_1,e_2)-trail. Therefore, $L(G)$ is hamiltonian-connected.*

Proof. Let $e_1,e_2\in E(G)$ be given. Note that a spanning (e_1,e_2)-trail of G_o yields a dominating (e_1,e_2)-trail of G. For $i=1,2$, let $e_i=u_iv_i$, and suppose $d(u_1)\leqslant d(u_2)$, $d(u_i)\leqslant d(v_i)$. Since G does not have an essential 2-edge-cut, for each $i=1,2$, $d_G(v_i)\geqslant 3$ and so $v_i\in V(G_o)$.

We shall show that in each of the possible cases, G has a dominating (e_1,e_2)-trail.

Case 1: $e_1,e_2\notin E(G_o)$. By Theorem 2.4, G_o is collapsible since $\tau(G_o)\geqslant 2$. Let $v_1=x$, $v_2=y$. By Theorem 2.5,there exists (x,y)-trail T of G_o such that $V(T)=V(G_o)$. Therefore, $\{e_1\}\bigcup E(T)\bigcup\{e_2\}$ is a dominating (e_1,e_2)-trail of G.

Case 2: $e_1\notin E(G_o)$, $e_2\in E(G_o)$. Then subdividing e_2 by inserting a new vertex y, we get a new graph $G_o(e_2)$. Since $\tau(G_o)\geqslant 2$, $F(G_o(e_2))\leqslant 1$.

By Theorem 2.6, $G_o(e_2)$ is collapsible since $\kappa'(G_o(e_2))\geqslant 2$. By the notation $e_1=u_1v_1$ with $d(u_1)\leqslant d(v_1)$, we must have $v_1\in V(G_o)$. Let $v_1=x$. By Theorem 2.5, there exists a spanning (x,y)-trail T of $G_o(e_2)$. Since T is an (x,y)-trail, exact one of u_2y and yv_2 is in T. Assume $u_2y \in E(T)$. Then $\{e_1\} \bigcup (E(T) - \{u_2y\}) \bigcup \{e_2\}$ is a dominating (e_1, e_2)-trail of G.

Case 3: $e_1, e_2 \in E(G_o)$.Then subdividing e_1, e_2 by inserting new vertices x and y in e_1 and e_2, respectively, we get a new graph $G_o(e_1, e_2)$. Since $\tau(G_o) \geqslant 2$, $F(G_o(e_1, e_2)) \leqslant 2$.

By Theorem 2.7, $G_o(e_1, e_2)$ is either collapsible or its reduction is a $K_{2,t}$.

If $G_o(e_1, e_2)$ is collapsible, then there exists a spanning (x, y)-trail T of $G_o(e_1, e_2)$. Without loss of generality, assume $u_1x, u_2y \in E(T)$. Then $\{e_1\} \bigcup (E(T) - \{u_1x, u_2y\}) \bigcup \{e_2\}$ is a dominating (e_1, e_2)-trail of G.

If the reduction of $G_o(e_1, e_2)$ is isomorphic to a $K_{2,t}$, then denote $V(K_{2,t}) = \{x_1, x_2\} \bigcup \{y_1, \ldots, y_t\}$, where x_1, x_2 are the two nonadjacent vertices of degree t and where $\{y_1, \cdots, y_t\}$ are the vertices of degree 2 other than $\{x_1, x_2\}$. Since G_o is collapsible and $\kappa'(G_o) \geqslant 3$, then $t = 2$ and $\{y_1, y_2\} = \{x, y\}$. Therefore, $\{e_1, e_2\}$ is an essential 2-edge-cut, a contradiction. □

Lemma 3.4. *If $\kappa(L(G)) \geqslant 7$, then $\tau(G - Y)_o \geqslant 2$ for any $Y \subset V(L(G)) = E(G)$ with $|Y| \leqslant 2$.*

Proof. Note that $\kappa'(G_o) \geqslant 3$ and $\kappa(L(G_o)) \geqslant 7$. We mainly use Theorem 2.8 in each of the possible cases to prove $\tau(G - Y)_o \geqslant 2$.

Case 1: $Y = \phi$. By Theorem 2.8, $\tau(G_o - \{x\}) \geqslant 2$ for any $x \in E(G)$. So $\tau(G_o) \geqslant 2$.

Case 2: $Y = \{e\}$. Let $e = uv$ and suppose $d(u) \leqslant d(v)$.

Subcase 2.1: $u \in D_1$. Since $\kappa(L(G)) \geqslant 7$, G does not have an essential 6-edge-cut. And $E = \{e' | e'$ is incident with v and $e' \neq e\}$ is an essential edge-cut. So $|E| \geqslant 7$. Thus $d(v) \geqslant 7 + 1 = 8$. Therefore, $(G - e)_o = G_o$. By Theorem 2.8, $\tau(G - e)_o = \tau(G_o) \geqslant 2$.

Subcase 2.2: $u \in D_2$. Suppose $e' = uv'$ with $v' \neq v$. Since $E = \{e'' | e''$ is incident with v and $e'' \neq e\} \cup \{e'\}$ is an essential edge-cut and $\kappa(L(G)) \geqslant 7$, then $|E| \geqslant 7$. So $d(v) = |E \backslash \{e'\} \cup \{e\}| \geqslant 7$. Similarly, $d(v') \geqslant 7$. Contract e' such that $e \in G_o$ when we obtain G_o from G, then $(G - e)_o = G_o - e$. By Theorem 2.8, $\tau(G - e)_o = \tau(G_o - e) \geqslant 2$.

Subcase 2.3: $u \in D_3$. Then $d(v) \geqslant 6$. Let $e' = uv'$, $e'' = uv''$ with $e' \neq e''$ and $v', v'' \neq v$. Note that by Theorem 2.8, $G_o - e$ has two edge disjoint spanning trees T' and T''. Since $d_{G_o - e}(u) = 2$, exactly one of e' and e'' is in T' and the other is in T''. Assume $e' \in T'$ and $e'' \in T''$. Then $T' - e'$ and $T'' - e''$ are two edge disjoint spanning trees of $(G_o - e)/e'$. And $(G - e)_o = (G_o - e)/e'$. Thus $\tau(G - e)_o = \tau((G_o - e)/e') \geqslant 2$.

Subcase 2.4: $d(u) \geqslant 4$. Since $d_{G-e}(u) \geqslant 3$, $(G - e)_o = G_o - e$. By Theorem 2.8, $\tau(G - e)_o = \tau(G_o - e) \geqslant 2$.

Case 3: $Y = \{e, e'\}$. Let $e = u_1v_1$, $e' = u_2v_2$ and suppose $d(u_1) \leqslant d(u_2)$, $d(u_i) \leqslant d(v_i)$, for each $i = 1, 2$.

Subcase 3.1: $u_1, u_2 \in D_1$. Then $d(v_1), d(v_2) \geqslant 8$. So $(G - \{e, e'\})_o = G_o$. By Theorem 2.8, $\tau(G - \{e, e'\})_o = \tau(G_o) \geqslant 2$.

Subcase 3.2: $u_1 \in D_1$, $u_2 \notin D_1$. Then $(G - \{e, e'\})_o = ((G - e) - e')_o$. Apply the same argument as in Subcases 2.2–2.4 to $G - e$. We conclude that $\tau(G - \{e, e'\})_o \geqslant 2$.

Subcase 3.3: $u_1, u_2 \notin D_1$.

Subcase 3.3.1: $u_1, u_2 \in D_2$. Then $d(v_i) \geqslant 7$, for each $i = 1, 2$. If $u_1 = u_2$, contract e' such that $e \in G_o$ when we obtain G_o from G. Then $(G - \{e, e'\})_o = G_o - e$. Thus $\tau(G - \{e, e'\})_o = \tau(G_o - e) \geqslant 2$. If $u_1 \neq u_2$, we obtain G_o by contracting other edges such that $e, e' \in G_o$. Then $(G - \{e, e'\})_o = G_o - \{e, e'\}$. By Theorem 2.8, $\tau(G - \{e, e'\})_o = \tau(G_o - \{e, e'\}) \geqslant 2$.

Subcase 3.3.2: $u_1 \in D_2$, $d(u_2) \geqslant 3$. We obtain G_o by contracting the other edge such that $e \in G_o$. If $d(u_2) = 3$, $(G - \{e, e'\})_o = (G_o - \{e, e'\})/e''$ where $e'' = u_2v''$ with $v'' \neq v_2$. Similar to Subcases 2.2 and 2.3,$\tau(G - \{e, e'\})_o \geqslant 2$. If $d(u_2) \geqslant 4$, then $(G - \{e, e'\})_o = G_o - \{e, e'\}$. Thus $\tau(G - \{e, e'\})_o = \tau(G_o - \{e, e'\}) \geqslant 2$.

Subcase 3.3.3: $d(u_1), d(u_2) \geqslant 3$. If $u_1, u_2 \in D_3$ and $u_1 = u_2$, then suppose $e'' = u_1v''$ with $v'' \neq v_1, v_2$. Therefore, $(G - \{e, e'\})_o = (G_o - \{e, e'\})/e''$. Since $\tau(G_o - e) \geqslant 2$ and $d_{G_o - e}(u_1) = 2$, $G_o - e$ has two edge disjoint spanning trees T' and T'' which contain e' and e'', respectively. Therefore, $T' - e'$ and $T'' - e''$ are two edge disjoint spanning trees of $(G_o - \{e, e'\})/e''$. So $\tau(G - \{e, e'\})_o = \tau(G_o - \{e, e'\})/e'' \geqslant 2$. If $u_1, u_2 \in D_3$ and $u_1 \neq u_2$, then suppose $e_3 = u_1v_3$, $e_4 = u_2v_4$ with $v_3 \neq v_1$, $v_4 \neq v_2$. Then $(G - \{e, e'\})_o = (G_o - \{e, e'\})/\{e_3, e_4\}$. Similar to Subcase 2.3, $\tau(G - \{e, e'\})_o = \tau((G_o - \{e, e'\})/\{e_3, e_4\}) \geqslant 2$.

If $u_1 \in D_3$ and $d(u_2) \geqslant 4$, let $e_3 = u_1v_3$ with $v_3 \neq v_1$. Then $(G - \{e, e'\})_o = (G_o - \{e, e'\})/e_3$. Similar to Subcases 2.3 and 2.4, $\tau(G - \{e, e'\})_o \geqslant 2$.

If $d(u_1), d(u_2) = 4$ and $u_1 = u_2$, suppose $e_3 = u_1v_3$ with $v_3 \neq v_1, v_2$. Then $(G - \{e, e'\})_o = (G_o - \{e, e'\})/e_3$. Since $\tau(G_o - \{e, e'\}) \geqslant 2$, similar to Subcase 2.3 $\tau(G_o - \{e, e'\})/e_3 \geqslant 2$. If $d(u_1), d(u_2) = 4$ and $u_1 \neq u_2$ or $d(u_2) > 4$, then $(G - \{e, e'\})_o = G_o - \{e, e'\}$. Thus $\tau(G - \{e, e'\})_o \geqslant 2$. □

Theorem 3.5. *If $\kappa(L(G)) \geqslant s+5$, then $L(G)$ is s-hamiltonian-connected.*

Proof. Let $Y \subset V(L(G)) = E(G)$ with $|Y| \leqslant s$. If $|Y| \leqslant 2$, let $Y_1 = Y$ and if $|Y| \geqslant 3$, let $Y_1 \subset Y$ with $|Y_1| = 2$ and $Y_2 = Y - Y_1$, $|Y_2| \leqslant s-2$. Since $\kappa(L(G)) \geqslant s+5 \geqslant 7$, $\kappa(L(G)-Y_2) \geqslant 7$. By Lemma 3.4, we have $\tau(G-Y)_o = \tau((G-Y_2)-Y_1)_o \geqslant 2$. By Lemma 3.3, $L(G)-Y$ is hamiltonian-connected. Thus $L(G)$ is s-hamiltonian-connected. □

When $s = 2$, the corollary below extends Theorem 1.2.

Corollary 3.6. *Every 7-connected line graph is 2-hamiltonian-connected.*

4. A remark

We conclude this paper with the following remark.

Theorem 3.5 suggests that for any $s \geqslant 2$, there exists a minimum number $f(s)$ such that if $\kappa(L(G)) \geqslant f(s)$, then $L(G)$ is s-hamiltonian-connected. What is the exact value of $f(s)$? Theorem 3.5 showed that for $s \geqslant 2$, $f(s) \leqslant s+5$. As any s-hamiltonian-connected graph must be $(s+3)$-connected, we conjecture that for large values of s, $f(s) = s+3$.

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