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Note

Hamiltonian connected hourglass free line graphs

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Abstract

Thomassen [Reflections on graph theory, J. Graph Theory 10 (1986) 309–324] conjectured that every 4-connected line graph is hamiltonian. An hourglass is a graph isomorphic to $K_5 - E(C_4)$, where C_4 is a cycle of length 4 in K_5. In Broersma et al. [On factors of 4-connected claw-free graphs, J. Graph Theory 37 (2001) 125–136], it is shown that every 4-connected line graph without an induced subgraph isomorphic to the hourglass is hamiltonian connected. In this note, we prove that every 3-connected, essentially 4-connected hourglass free line graph, is hamiltonian connected.
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1. Introduction

Graphs considered in this section are finite and simple. Unless otherwise noted, we follow [1] for notations and terms. A graph G is *nontrivial* if $E(G) \neq \emptyset$. For a vertex v of a graph G, $d_G(v)$ denotes the degree of v in G and $E_G(v)$ denotes the set of edges incident with v in G. For an integer $i > 0$, $D_i(G) = \{v \in V(G) : d_G(v) = i\}$.

A graph G is *hamiltonian* if G has a cycle containing all vertices of G. A graph G is *hamiltonian connected* if for every pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-path (a path containing all vertices of G and starting from u and ending at v).

The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. Thomassen [8] conjectured that every 4-connected line graph is hamiltonian. This conjecture is still open.

An hourglass is a graph isomorphic to $K_5 - E(C_4)$, where C_4 is a cycle of length 4 in K_5. A graph is hourglass free if it does not have an induced subgraph isomorphic to the hourglass.

Theorem 1.1 (*Broersma et al. [2]*). *Every* 4-*connected hourglass free line graph is hamiltonian connected.*

A vertex (edge, respectively) cut X of a connected graph G is *essential* if $G - X$ has at least two nontrivial components. A graph G is essentially k-connected (essentially k-edge-connected, respectively) if G does not have an essential

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cut (essential edge cut, respectively) X with $|X| < k$. In this note, we shall improve Theorem 1.1 in the following form.

Theorem 1.2. *Every* 3-*connected*, *essentially* 4-*connected hourglass free line graph is hamiltonian connected.*

Since adding edges will not decrease the connectivity and the essential connectivity, applying the line graph closure of Ryjáček [6], Theorem 1.2 has the following corollary.

Corollary 1.3. *Every* 3-*connected*, *essentially* 4-*connected*, *claw free and hourglass free graph is hamiltonian.*

2. Mechanism

Graphs considered in Sections 2 and 3 are finite and loopless. Let G be a graph. An edge $e \in E(G)$ is *subdivided* when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. This process is called *subdividing* e. For a graph G with $e', e'' \in E(G)$, let $G(e')$ denote the graph obtained from G by subdividing e', and let $G(e', e'')$ denote the graph obtained from G by subdividing both e' and e''. Then,

$$V(G(e', e'')) - V(G) = \{v(e'), v(e'')\}.$$

For vertices $u, v \in V(G)$, an (u, v)-*trail* is a trail that starts with u and ends with v.

Theorem 2.1 (*Lai et al. [4]*). *If G is essentially* 4-*edge-connected such that for every vertex $v \in V(G)$ of degree* 3, *G has a cycle of length at most* 3 *containing v, then for every pair of edges $e', e'' \in E(G)$, $G(e', e'')$ has a spanning $(v(e'), v(e''))$-trail.*

Let G be a graph and let $X \subseteq E(G)$ be an edge subset. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. Note that contraction may generate multiple edges. Let G be a graph such that $\kappa(L(G)) \geqslant 3$ and $L(G)$ is not complete. The *core* of this graph G, denoted by G_0, is obtained from $G - D_1(G)$ by contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$.

Lemma 2.2 (*Lai et al. [5], Shao [7]*). *Let G be a connected nontrivial graph such that $\kappa(L(G)) \geqslant 3$, and let G_0 denote the core of G. If $\forall e', e'' \in E(G_0)$, $G(e', e'')$ has a spanning $(v(e'), v(e''))$-trail, then $L(G)$ is hamiltonian connected.*

3. Proof of Theorem 1.2

Let G_0 be the core of a simple graph G. Note that G_0 may have multiple edges. Then as $\kappa(L(G)) \geqslant 3$, by the definition of line graphs, G has no essential edge cuts of size less than 3. By the definition of core graphs, all vertices with degree 1 of G are deleted and degree 2 vertices of G disappear as two incident edges contract in G_0 and so $\delta(G_0) \geqslant 3$. Therefore $\kappa'(G_0) \geqslant 3$. Let X be an essential edge cut of G_0. Suppose that $|X| \leqslant 3$. If one side of $G_0 - X$ has only one edge, then by $\delta(G_0) \geqslant 3$, we must have $|X| \geqslant 4$, a contradiction. Therefore, both sides of $G - X$ must have a pair of adjacent edges, and so X corresponds to an essential vertex cut of $L(G)$. Since $L(G)$ is assumed to be essentially 4-connected, $|X| \geqslant 4$, a contradiction. Thus we have proved the claim:

(3.1) G_0 is essentially 4-edge-connected.

We shall prove the next claim:

(3.2) $\forall v \in D_3(G_0)$, G_0 has a cycle of length at most 3 intersecting $E_{G_0}(v)$.

For a contradiction, let $v \in D_3(G_0)$ with $E_{G_0}(v) = \{e_1, e_2, e_3\}$ such that no edge in $E_{G_0}(v)$ lies in a cycle of length at most 3. Let v_1, v_2, v_3 denote the vertices of G_0 adjacent to v such that $e_i = vv_i$ $(1 \leqslant i \leqslant 3)$. Since $\delta(G_0) \geqslant 3$, we can assume that $\{f_1, f_2\} \subseteq E_{G_0}(v_3) - \{e_3\}$ (see Fig. 1(a)). By the definition of a core, either $e_3 \in E(G)$ or e_3 is a new edge replacing a path of length 2 in G (see Fig. 2).

If $e_3 \in E(G)$, and if neither e_1 nor e_2 is adjacent to f_1 or f_2 (see Fig. 1(a) and (b)), then $L(G)$ would have an induced hourglass, contrary to the assumption that $L(G)$ is hourglass free. Hence we may assume that $e_1 f_1 \in E(L(G))$ (see Fig. 1(c)). Then $f_1 \in E_{G_0}(v_1)$, and so $G_0[\{e_1, e_3, f_1\}] \cong K_3$, contrary to the choice of v.

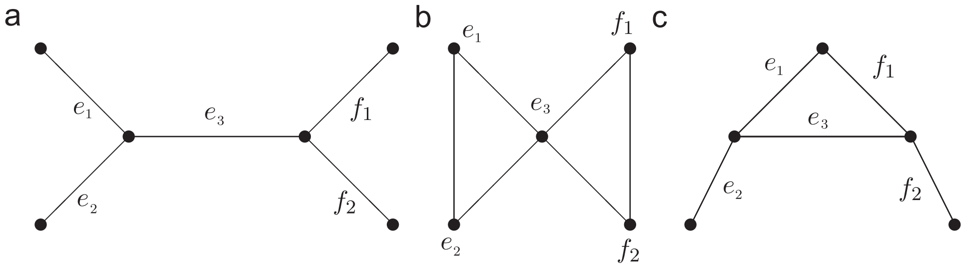

Fig. 1. $G[\{e_1, e_2, e_3, f_1, f_2\}]$ and $L(G)[\{e_1, e_2, e_3, f_1, f_2\}]$.

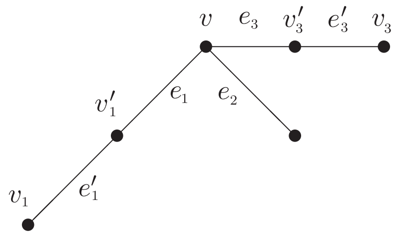

Fig. 2. The edge cut $\{e_1', e_2, e_3'\}$ in G.

By symmetry, we may assume that $\forall i \in \{1, 2, 3\}$, e_i is a new edge which replaces a path with edges $\{e_i, e_i'\}$ in G, where we also use e_i to denote the edge adjacent to v in G. Then $\{e_1', e_2, e_3'\}$ corresponds to an essential vertex cut of $L(G)$ (see Fig. 2), contrary to the assumption that $L(G)$ is essentially 4-connected. This proves Claim (3.2). □

By (3.1), (3.2) and by Theorem 2.1, $\forall e', e'' \in E(G_0)$, $G_0(e', e'')$ has a spanning $(v(e'), v(e''))$-trail, and so by Lemma 2.2, $L(G)$ is hamiltonian connected.

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