Graphs and Combinatorics (2008) 24:1–9
Digital Object Identifier (DOI) 10.1007/s00373-008-0780-1

Graphs and Combinatorics

On Group Connectivity of Graphs

Hong-Jian Lai[1], Rui Xu[2], Ju Zhou[1]

[1] Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA. e-mail: hjlai@math.wvu.edu, zhouju2008@math.wvu.edu
[2] Department of Mathematics, University if West Georgia, Carrollton, GA 30118, USA. e-mail: xu@westga.edu

Abstract. Tutte conjectured that every 4-edge-connected graph admits a nowhere-zero Z_3-flow and Jaeger et al. [Group connectivity of graphs–a nonhomogeneous analogue of nowhere-zero flow properties, J. Combin. Theory Ser. B 56 (1992) 165-182] further conjectured that every 5-edge-connected graph is Z_3-connected. These two conjectures are in general open and few results are known so far. A weaker version of Tutte's conjecture states that every 4-edge-connected graph with each edge contained in a circuit of length at most 3 admits a nowhere-zero Z_3-flow. Devos proposed a stronger version problem by asking if every such graph is Z_3-connected. In this paper, we first answer this later question in negative and get an infinite family of such graphs which are not Z_3-connected. Moreover, motivated by these graphs, we prove that every 6-edge-connected graph whose edge set is an edge disjoint union of circuits of length at most 3 is Z_3-connected. It is a partial result to Jaeger's Z_3-connectivity conjecture.

Key words. Integer flow, group connectivity, edge connectivity.

1. Introduction

We follow the notations and terminology of [1] except otherwise stated.

Let G be a digraph, A be a nontrivial additive Abelian group and A^* be the set of nonzero elements in A. For any $v \in V(G)$, the set of all edges with tails at v is denoted by $E^+(v)$ and the set of all edges with heads at v is denoted by $E^-(v)$. We use $\delta^+(v)$ for $|E^+(v)|$ and $\delta^-(v)$ for $|E^-(v)|$.

We define

$$F(G, A) = \{f \mid f : E(G) \mapsto A\} \quad \text{and} \quad F^*(G, A) = \{f \mid f : E(G) \mapsto A^*\}.$$

For each $f \in F(G, A)$, the *boundary* of f is a function $\partial f : V(G) \mapsto A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. We define

$$Z(G, A) = \{b \mid b : V(G) \mapsto A \ \text{with} \sum_{v \in V(G)} b(v) = 0\}.$$

An A-*nowhere-zero-flow* (abbreviated as A-NZF) in G is a function $f \in F^*(G, A)$ such that $\partial f = 0$. For any $b \in Z(G, A)$, if there is a function $f \in F^*(G, A)$ such that $\partial f = b$, then we call f an (A, b)-NZF.

An undirected graph G is called A-*connected*, if G has some (and thus every) orientation G' such that for every function $b \in Z(G', A)$, there exists an (A, b)-NZF. Similarly, G is said to admit an A-NZF if G has some (and thus every) orientation G' such that G' admits an A-NZF.

The nowhere-zero-flow problems were introduced by Tutte [8], surveyed by Jaeger in [4] and by Zhang in [10]. The concept of A-connectivity was introduced by Jaeger et al. in [5], where A-NZF was successfully generalized to A-connectivity.

This paper is motivated by the following conjectures.

Conjecture 1. (Tutte, unsolved problem 48 in [1]) Every 4-edge-connected graph admits a Z_3-NZF.

Conjecture 2. (Jaeger et al.[5]) Every 5-edge-connected graph is Z_3-connected.

Kochol [7] proved that Conjecture 1 can be reduced to 5-edge-connected graphs, and then Conjecture 2 implies Conjecture 1. These two conjectures seem to be difficult and not many results (even partial results) are known. Then the following weaker conjectures are proposed.

Conjecture 3. (Jaeger [4]) There is an integer k such that every k-edge-connected graph admits a Z_3-NZF.

Conjecture 4. (Xu and Zhang [9]) Let G be a 4-edge-connected graph. If each edge of G is contained in a circuit of length at most 3, then G admits a Z_3-NZF.

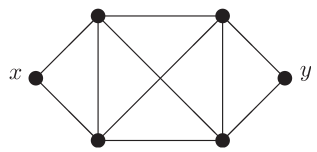
Devos proposed a stronger version problem of Conjecture 4 as follows.

Problem 1. (Devos [3]) Let G be a 4-edge-connected graph with each edge contained in a circuit of length at most 3. Must G be Z_3-connected?

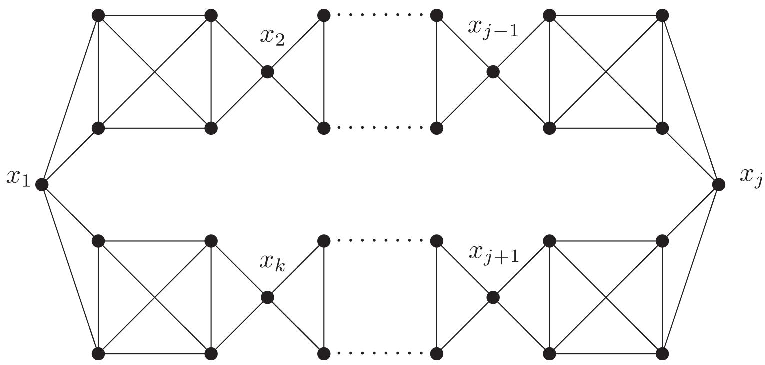
In this paper, we first answer Problem 1 in negative in Section 2 by constructing an infinite family of graphs. Then in Section 3, we prove that every 6-edge-connected graph whose edge set is an edge disjoint union of circuits of length at most 3 is Z_3-connected.

2. Triangles, 4-Edge Connectivity and Z_3-Connectivity

Let $L(x, y)$ be a graph as follows:



For $k \geq 3$, let $L_1, L_2, \ldots L_k$ be graphs such that for each i, $L_i(x_i, y_i) \cong L(x, y)$. Let $G(k)$ be a graph obtained from $L_1, L_2, \ldots, L_k$ by identifying y_i and x_{i+1}, where $x_{k+1} = x_1$ and $i = 1, 2, \ldots k$.



Theorem 1. *$G(k)$ is not Z_3-connected for $k \geq 3$.*

To prove this theorem, we need the following lemmas.

Let G be a 4-regular graph and $b \in Z(G, Z_3)$. Let $V_1 = b^{-1}(1) \subseteq V(G)$. Suppose that G admits a (Z_3, b)-NZF f. Then we may assume that $f(e) = 1$ for every edge $e \in E(G)$ by adjusting the orientation if needed. Let D denote such an orientation. Then we have the following lemmas:

Lemma 1. *For any $v \in b^{-1}(1)$, either $\delta^+(v) = 4$ or $\delta^+(v) = 1$ and $\delta^-(v) = 3$.*

Proof. Since f is a (Z_3, b)-NZF with $f(e) = 1$ for every edge of G, then for any $v \in b^{-1}(1)$, we have $\delta^+(v) - \delta^-(v) \equiv 1 \pmod 3$. Since G is 4-regular, then either $\delta^+(v) = 4$ ($4 \equiv 1 \pmod 3$) or $\delta^+(v) = 1$ and $\delta^-(v) = 3$ ($1 - 3 \equiv 1 \pmod 3$).

If $v \in b^{-1}(1)$ and $\delta^+(v) = 4$, we call v a *positive vertex*; if $v \in b^{-1}(1)$ and $\delta^-(v) = 3$ and $\delta^+(v) = 1$, we call v *a negative vertex*.

Lemma 2. *No two positive vertices can be adjacent.*

Proof. Suppose $u, v \in b^{-1}(1)$ and u, v are both positive vertices. If $uv \in E(G)$, then either (u, v) or (v, u) is in D, and so either $\delta^+(v) < 4$ or $\delta^+(u) < 4$, contradicting the definition of positive vertex.

Lemma 3. *If $x, y, z \in b^{-1}(1)$ are all negative vertices and $G[\{x, y, z\}] \cong K_3$, then $G[\{x, y, z\}]$ must be a dicircuit (and we call such a dicircuit a negative K_3).*

Proof. Without loss of generality, we assume that $(x, y) \in D$. Since $\delta^+(x) = 1$, we must have $(z, x) \in D$. Since $\delta^+(z) = 1$, we must have $(y, z) \in D$.

Lemma 4. *Any negative vertex u cannot be adjacent to two vertices in a negative K_3 which does not contain u.*

Proof. Suppose that u is a negative vertex and that $ux, uy \in E(G)$ such that x, y are in a negative K_3 which does not contain u. Since $\delta^+(x) = \delta^+(y) = 1$ and x, y are in the negative K_3, x, y have out degree 1, we must have $(u, x), (u, y) \in D$, and so $\delta^+(u) \geq 2$. This implies that u can't be a negative vertex, a contradiction.

Lemma 5. *Let G be a A-connected graph. Then for any $e \in E(G)$, G/e is A-connected. Therefore, group connectivity is closed under contraction.*

Proof. Suppose that $e = uu'$, $G' = G/e$ and the new resulting vertex in G' is denoted by u^*. For any $b' \in Z(G', A)$, let us define $b : V(G) \mapsto A$ as follows: $b(u) = b'(u^*)$, $b(u') = 0$ and $b(v) = b'(v)$ for any $v \notin \{u, u'\}$. Since $b' \in Z(G', A)$, then $b \in Z(G, A)$. Because G is A-connected, then there is an $f \in F^*(G, A)$ such that $\partial f = b$. Clearly f', the restriction of f to $E(G')$, is an (A, b')-NZF of G'.

Proof of Theorem 1. Since any $G(k)$ $(k \geq 4)$ can be contracted to a $G(3)$ with a series of edge contraction, by Lemma 5, we need only to prove that $G(3)$ is not Z_3-connected.

By contradiction. Suppose that $G(3)$ is Z_3-connected. Let $b \in Z(G(3), Z_3)$ with $b(v) = 1$ for each $v \in V(G(3))$. Then there is an $f \in F^*(G(3), Z_3)$ such that $\partial f = b$. We may assume that $G(3)$ has an orientation D such that $f \equiv 1$. Now $b^{-1}(1) = V(G)$.

By Lemma 2, Lemma 3 and Lemma 4, each K_4 must have exactly one positive vertex and 3 negative vertices (and so it contains a negative K_3).

Now we prove that x_1, x_2 can not be both positive or both negative. If both x_1 and x_2 are positive, then by Lemma 2, in subgraph $L(x_1, x_2)$ each vertex of the K_4 must be negative, contradicting Lemma 3 and Lemma 4. Suppose both x_1 and x_2 are negative. Then by Lemma 1, Lemma 2 and Lemma 3, there are exactly 3 negative vertices in the K_4 of $L(x_1, x_2)$ and the 3 negative vertices form a negative K_3 in $L(x_1, x_2)$. Then either x_1 or x_2 must be adjacent to two vertices in the negative K_3, contradicting Lemma 4.

Similarly, x_2, x_3 can not be both positive or both negative, x_1, x_3 can not be both positive or both negative. But this is impossible. □

Clearly, every graph $G(k)$ is 4-edge-connected and each edge is contained in a circuit of length 3 (triangle). Since group connectivity property is preserved under contraction (see Lemma 5), then we can get a larger family of such graphs as follows.

Corollary 1. *Every graph which can be contracted to a $G(k)$ for some integer $k \geq 3$ is not Z_3-connected.*

3. Triangles, 6-Edge Connectivity and Z_3-Connectivity

An edge cut X of G is *essential* if $G \setminus X$ has at least two nontrivial components. For any integer $k > 0$, a graph is *essentially k-edge-connected* if G has no essential edge cut X with $|X| < k$. Clearly, every k-edge-connected graph is essentially k-edge-connected.

For a $b \in Z(G, A)$, we say that G is *(A, b)-extendable from v* if, for any function $f : E(v) \to A^*$ with $\partial f(v) = b(v)$, f can be extended to a function $f \in F^*(G, A)$ such that $\partial f = b$. If for any $b \in Z(G, A)$, G is (A, b)-extendable from v, then G is A-extendable from v. Clearly, if G is *A-extendable from v*, then G is A-connected.

We use $D_k(G)$ to denote the set of vertices of G with degree k.

Let $\mathcal{F}$ be a set of graphs G which satisfy the following conditions:

(1) $E(G)$ is an edge disjoint union of circuits of length at most 3;
(2) G is essentially 6-edge-connected;
(3) $\delta(G) \geq 4$ and $|D_4(G)| \leq 1$.

Based on Theorem 1, we made the following conjecture which is a weak version of the Conjecture 2.

Conjecture 5. Let G be a 5-edge-connected graph with each edge contained in a circuit of length at most 3. Then G is Z_3-connected.

For partial results related to this conjecture, see [2, 6].

By the construction of $G(k)$, we can see that $G(k)$ is a 4-edge-connected graph such that $E(G)$ is an edge disjoint union of circuits of length at most 4. Motivated by this, we consider the family of essentially 6-edge-connected graphs whose edge set is an edge disjoint union of circuits of length at most 3 and get the following result.

Theorem 2. *Suppose $G \in \mathcal{F}$. Then G is Z_3-connected. Moreover, for any $v \in D_6(G)$, G is Z_3-extendable from v. In particular, every 6-edge-connected graph is Z_3-connected provided that its edge set is an edge disjoint union of circuits of length at most 3.*

Let $N(v) = \{v_1, v_2, \ldots, v_n\}$ denote the set of vertices adjacent to the vertex v, and let $Y = \{vv_1, vv_2\}$. Then $G_{[v,Y]}$ is the graph obtained from $G \setminus \{vv_1, vv_2\}$ by adding a new edge that joins v_1 and v_2.

Let us first introduce some useful lemmas.

Lemma 6. *For any abelian group A and any $b \in Z(G, A)$, if $G_{[v,Y]}$ has an (A, b)-NZF, then G has an (A, b)-NZF. Moreover, if $G_{[v,Y]}$ is A-extendable from a vertex u with $u \neq v$, so is G.*

Proof. Let e' be the new edge that joins v_1 and v_2. Since $b \in Z(G, A)$, then $b \in Z(G_{[v,Y]}, A)$. For an (A, b)-NZF f of $G_{[v,Y]}$, we define $f' : E(G) \mapsto A$ as follows: $f'(e) = f(e)$ if $e \notin \{vv_1, vv_2\}$ and $f'(vv_1) = f'(vv_2) = f(e')$. If e' is oriented as (v_1, v_2) in $G_{[v,Y]}$, then we orient vv_1, vv_2 as $(v_1, v), (v, v_2)$ in G. Clearly, f' is an (A, b)-NZF of G.

Now assume that $G_{[v,Y]}$ is A-extendable from a vertex u with $u \neq v$. For any $b \in Z(G, A)$, let $f^* : E(u) \mapsto A^*$ be a function such that $\partial f^*(u) = b(u)$. Clearly, $b \in Z(G_{[v,Y]}, A)$. Since $G_{[v,Y]}$ is A-extendable from u, f^* can be extended to a function $f \in F^*(G_{[v,Y]}, A)$ such that $\partial f = b$. By the same argument as above, we can get f' as an (A, b)-NZF of G. Since f is an extension of f^* and by the construction of f', f' is an extension of f^* as well. Then G is A-extendable from u.

Lemma 7. *Let G be a graph and H an A-connected subgraph of G. We define $G^* = G/H$ and denote the new resulting vertex in G^* by v_H. For any $b \in Z(G, A)$, we define $b' : V(G^*) \mapsto A$ as follows: $b'(v_H) = \sum_{v \in V(H)} b(v)$ and $b'(v) = b(v)$ for $v \neq v_H$. If G^* admits an (A, b')-NZF f^*, then f^* can be extended to an (A, b)-NZF of G.*

Proof. Let us define a new function $f' : E(G) \mapsto A$ such that $f'(e) = 0$ for any $e \in E(H)$ and $f'(e) = f^*(e)$ for any other edge of G. For $u \in V(H)$, let $b''(u) = b(u) - (\sum_{e \in E^+(u)} f'(e) - \sum_{e \in E^-(u)} f'(e))$. Since $b'(v_H) = \sum_{u \in V(H)} b(u) = \sum_{u \in V(H)} (\sum_{e \in E^+(u)} f'(e) - \sum_{e \in E^-(u)} f'(e))$, then $\sum_{u \in V(H)} b''(u) = 0$. Therefore $b'' \in Z(H, A)$. Since H is A-connected, there is $g \in F^*(H, A)$ such that $\partial g = b''$. Let $g' : E(G) \mapsto A$ be a function such that $g'(e) = g(e)$ for any $e \in E(H)$ and $g'(e) = 0$ for any other edge of G. Then $f = f' + g' \in F^*(G, A)$ satisfying $\partial f = b$ and f is an extension of f^*.

Lemma 8. *Suppose that G is a counterexample to Theorem 2 with $N(G) = |V(G)| + |E(G)|$ minimized. Then G does not have an essential 6-edge cut.*

Proof. Suppose that G has an essential 6-edge cut X such that $G \setminus X$ has two nontrivial components G_1 and G_2. Then $G/G_1 \in \mathcal{F}$ and $G/G_2 \in \mathcal{F}$. Let v_{G_1} be the new vertex in G/G_1 obtained from G by contracting G_1, v_{G_2} be the new vertex in G/G_2 obtained from G by contracting G_2.

Case 1. $D_6(G) \neq \emptyset$.

Without loss of generality, we may assume that $v \in D_6(G)$ and $v \in V(G/G_2)$. Then for any $b \in Z(G, Z_3)$ and any partial (Z_3, b)-NZF f_0 of $E(v)$, let us define $b_2 \in Z(G/G_2)$ by setting $b_2(u) = b(u)$ if $u \in V(G/G_2) \setminus \{v_{G_2}\}$ and $b_2(v_{G_2}) = \sum_{\omega \in V(G_2)} b(\omega)$. Since $N(G/G_2) < N(G)$, then by the selection of G, we can extend f_0 to a (Z_3, b_2)-NZF f_2 of G/G_2.

Now define $b_1 \in Z(G/G_1)$ by setting $b_1(u) = b(u)$ if $u \in V(G/G_1) \setminus \{v_{G_1}\}$ and $b_1(v_{G_1}) = \sum_{\omega \in V(G_1)} b(\omega)$. It is easy to see that $f_2|X$ is a partial (Z_3, b_1)-NZF of $E(v_{G_1})$ and $d_{G/G_1}(v_{G_1}) = 6$. Since $N(G/G_1) < N(G)$, then by the selection of G, we can extend $f_2|X$ to a (Z_3, b_1)-NZF f_1 of G/G_1.

Then $f = f_1 + f_2 - (f_2|X)$ is a (Z_3, b)-NZF of G extending from f_0. Therefore G is Z_3-connected, a contradiction.

Case 2. $D_6(G) = \emptyset$.

For any $b \in Z(G, Z_3)$, let us define b_2 and b_1 the same as in Case 1.

Since $N(G/G_2) < N(G)$, then by the selection of G, G/G_2 is Z_3-connected and there is a (Z_3, b_2)-NZF f_2 of G/G_2. It is easy to see that $f_2|X$ is a partial (Z_3, b_1)-NZF of $E(v_{G_1})$ and $d_{G/G_1}(v_{G_1}) = 6$. Since $N(G/G_1) < N(G)$, then by the selection of G, we can extend $f_2|X$ to a (Z_3, b_1)-NZF f_1 of G/G_1.

Then $f = f_1 + f_2 - (f_2|X)$ is a (Z_3, b)-NZF of G and G is Z_3-connected, a contradiction.

Lemma 9. *Suppose that G is a counterexample to Theorem 2 with $N(G) = |V(G)| + |E(G)|$ minimized. Then $|V(G)| \geq 4$ and $D_4(G) \neq \emptyset$.*

Proof. If $|V(G)| = 2$, since $G \in \mathcal{F}$, then $E(G)$ is the edge disjoint union of 2-circuits on the same 2 vertices. Clearly, G is Z_3-connected and Z_3-extendable from any degree 6 vertex, a contradiction.

If $|V(G)| = 3$, since $G \in \mathcal{F}$, then $d_G(u) \geq 4$ for any $u \in V(G)$ and G has at most one vertex of degree 4. So G has at least 8 edges and therefore, there are some 2-circuits. Then G is Z_3-connected. For any degree 6 vertex v of $V(G)$, $G \setminus \{v\}$ has at least 2 edges, therefore $G \setminus \{v\}$ contains a 2-circuit C. By the selection of G, G/C is Z_3-extendable from v, then by Lemma 7, G is Z_3-extendable from v, a contradiction. Then $|V(G)| \geq 4$.

Now suppose that $D_4(G) = \emptyset$. Then for any $v \in V(G)$, $d_G(v) \geq 6$. Since $G \in \mathcal{F}$, $E(G)$ is the edge disjoint union of circuits of length at most 3. We first show that at least one of such circuits is of length 3. Otherwise, if all such circuits are of length 2, then G is Z_3-connected (since 2-circuit is Z_3-connected). Since G has at least 4 vertices, if $v \in D_6(G)$, then there is a 2-circuit C such that $v \notin V(C)$. By the selection of G, G/C is Z_3-extendable from v. Then by Lemma 7, G is Z_3-extendable from v, this contradicts the choice of G.

So we may assume that uu_1u_2u is one such 3-circuit. For any $v \in D_6(G)$, there are at least two vertices in $\{u, u_1, u_2\}$ which are distinct from v. We may assume that $u \neq v$. Let $Y = \{uu_1, uu_2\}$. Then $G_{[u,Y]}$ has at most one vertex of degree 4 since $D_4(G) = \emptyset$ and $\delta(G) \geq 4$. By the construction of $G[u, Y]$, the edge set of $G[u, Y]$ is an edge disjoint union of circuits of length at most 3.

Case 1. $G_{[u,Y]}$ is essentially 6-edge-connected.

Then $G[u, Y] \in \mathcal{F}$ and $G[u, Y]$ has less edges than G. By the selection of G, $G_{[u,Y]}$ is Z_3-connected and $G_{[u,Y]}$ is Z_3-extendable from v. Then by Lemma 6, G is Z_3-connected and G is Z_3-extendable from v, a contradiction.

Case 2. $G_{[u,Y]}$ is not essentially 6-edge-connected.

If it has an essential 2-edge cut, then G has an essential 4-edge cut, a contradiction; if it has an essential 4-edge cut, then G has an essential 6-edge cut, contradicting Lemma 8.

Proof of Theorem 2. By contradiction. Let G be a counterexample with $N(G) = |V(G)| + |E(G)|$ minimized.

By Lemma 9, $D_4(G) \neq \emptyset$. Let $D_4(G) = \{u\}$. Let us consider the following 4 cases.

Case 1. $|N_G(u)| = 1$.

Let $N_G(u) = \{u_1\}$. By Lemma 9, $|V(G)| \geq 4$, then $d_G(u_1) \geq 12$. Otherwise, G has an essential k-edge cut with $k \leq 6$, contradicting Lemma 8. Contract all the circuits of length 2 which contain u and let G' be the resulting graph. For convenience, we still use u_1 for the new obtained vertex. Clearly, $d_{G'}(u_1) \geq 8$ and $G' \in \mathcal{F}$. By the selection of G, G' is Z_3-connected and for any $v \in D_6(G)$, G' is Z_3-extendable from v. By Lemma 7, G is Z_3-connected and Z_3-extendable from v, a contradiction.

Case 2. $|N_G(u)| = 2$.

Let $N_G(u) = \{u_1, u_2\}$. Since $d_G(u) = 4 > 2$, then there is at least one vertex in $N_G(u)$, say u_1, such that $C = uu_1u$ is a 2-circuit of G. We claim that $d_G(u_1) \neq 6$. Otherwise, let $A = \{u, u_1\}$, then $E_G(A, \bar{A})$ is an essential k-cut with $k \leq 6$. This contradicts Lemma 8 (if $k = 6$) or the fact that G is essential 6-edge-connected (if $k \leq 5$). Let v be any degree 6 vertex of G, then $v \neq u_1$. By the selection of G, G/C is Z_3-connected and G/C is Z_3-extendable from v. By Lemma 7, G is Z_3-connected and G is Z_3-extendable from v, a contradiction.

Case 3. $|N_G(u)| = 3$.

Let $N_G(u) = \{u_1, u_2, u_3\}$. Since $d_G(u) = 4 > 3$, then there is at least one vertex in $N_G(u)$, say u_1, such that $C = uu_1u$ is a 2-circuit of G. Similar to Case 2, we can get a contradiction.

Case 4. $|N_G(u)| = 4$.

Since $G \in \mathcal{F}$, there are four vertices u_1, u_2, u_3, u_4 such that uu_1u_2u and uu_3u_4u are two 3-circuits of G. For any degree 6 vertex v of $V(G)$, at least one of above two

3-circuits, say uu_1u_2u, contains no v. Let $Y = \{u_1u, u_1u_2\}$, then the graph $G_{[u_1,Y]}$ contains a 2-circuit uu_2u. We further contract this 2-circuit and get G', let u^* be the new vertex obtained by the contraction. Since $d_G(u_1) \geq 6$, $d_G(u_2) \geq 6$, we have $d_{G'}(u_1) \geq 4$ and $d_{G'}(u^*) = d_G(u_2) \geq 6$. Clearly, $E(G')$ is an edge disjoint union of circuits of length at most 3. If G' is not essential 6-edge-connected, then G' has an essential k-edge cut X with $k \leq 5$. But all the edge cuts must be of even size because G' is Eulerian. Then $k = 4$ or $k = 2$. Thus either X is an essential edge cut of G or $X \cup \{u_1u, u_1u_2\}$ is an essential edge cut of G. It follows that G has an essential k-edge cut with $k \leq 6$, contradicting Lemma 8 (if $k = 6$) or the fact that G is essential 6-edge-connected (if $k \leq 5$). Therefore $G' \in \mathcal{F}$. By the selection of G, G' is Z_3-connected and Z_3-extendable from v. By Lemma 7, then by Lemma 6, G is Z_3-connected and Z_3-extendable from v, a contradiction. And this case ends the proof of the theorem. □

References

1. Bondy, J.A., Murty, U.S.R.: *Graph Theory with Applications*, New York: American Elsevier, 1976
2. Devos, M., Xu, R., and Yu, G.-X.: Nowhere-zero Z_3-flows through Z_3-connectivity. Discrete Math. **306**, 26–30 (2006)
3. Devos, M. private communication.
4. Jaeger, F.: Nowhere-zero flow problems, In: *Selected topics in graph theory* (3, pp: 91–95), London/New York: Academic Press, 1988
5. Jaeger, F., Linial, N., Payan C., and Tarsi, M.: Group connectivity of graphs - a non-homogeneous analogue of nowhere-zero flow properties. J. Combin. Theory Ser. B **56**, 165–182 (1992)
6. Lai, H.-J.: Nowhere-zero 3-flows in locally connected graphs. J. Graph Theory **42**, 211–219 (2003)
7. Kochol, M.: An equivalent version of the 3-flow conjecture. J. Combin. Theory Ser. B **83**, 258–261 (2001)
8. Tutte, W.T.: A contribution to the theory of chromatical polynomials. Canad. J. Math. **6**, 80–91 (1954)
9. Xu, R., Zhang, C.-Q.: Nowhere-zero 3-flows in squares of graphs. Electron. J. Combin. **10**, R5 (2003)
10. Zhang, C.-Q.: *Integer flows and cycle covers of graphs*, New York: Marcel Dekker, 1997

Received: May 23, 2006
Final version received: January 13, 2008