Available online at www.sciencedirect.com
ScienceDirect
European Journal of Combinatorics 29 (2008) 298–310
European Journal of Combinatorics
www.elsevier.com/locate/ejc

Spanning cycles in regular matroids without $M^*(K_5)$ minors

Hong-Jian Lai[a,b], Bolian Liu[c], Yan Liu[c], Yehong Shao[d]

[a] *Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA*
[b] *School of Mathematics, Physics and Software Enginneering, Lanzhou Jiaotong University, Lanzhou 730070, PR China*
[c] *Department of Mathematics, South China Normal University, Guangzhou, PR China*
[d] *Arts and Sciences, Ohio University Southern, Ironton, OH 45638, USA*

Received 20 April 2005; accepted 24 July 2006
Available online 7 November 2006

Abstract

Catlin and Jaeger proved that the cycle matroid of a 4-edge-connected graph has a spanning cycle. This result can not be generalized to regular matroids as there exist infinitely many connected cographic matroids, each of which contains a $M^*(K_5)$ minor and has arbitrarily large cogirth, that do not have spanning cycles. In this paper, we proved that if a connected regular matroid without a $M^*(K_5)$-minor has cogirth at least 4, then it has a spanning cycle.

1. Introduction

We shall assume familiarity with graph theory and matroid theory. For terms that are not defined in this note, see Bondy and Murty [5] for graphs, and Oxley [13] or Welsh [22] for matroids. To be consistent with the matroid terminology, a nontrivial 2-regular connected graph will be called a *circuit*, and a disjoint union of circuits will be called a *cycle*. For a subset X in a matroid M, $cl_M(X)$ is the closure of X in M.

For a graph G, let $O(G)$ denote the set of odd degree vertices of G. A graph G is *Eulerian* if G is connected with $O(G) = \emptyset$, and G is *supereulerian* if G has a spanning Eulerian subgraph. Boesch et al. [3] suggested that characterizing supereulerian graphs may be very difficult. Pulleyblank [14] showed that determining if a graph is supereulerian is a NP-complete problem.

E-mail address: hjlai@math.wvu.edu (H.-J. Lai).

0195-6698/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2006.07.011

Veblen [20] first showed that a connected graph G is Eulerian if and only if $E(G)$ is an edge-disjoint union of circuits. Welsh [23] defines a matroid M as Eulerian if $E(M)$ is a cycle of M. It is natural to define a matroid M to be a *supereulerian matroid* if M has a cycle L with $r(L) = r(M)$. Such a cycle L will be referred to as a *spanning cycle* of M.

For a graph G, if $V_1, V_2 \subseteq V(G)$ such that $V_1 \cap V_2 = \emptyset$, then denote $[V_1, V_2]_G = \{e = uv \in E(G) : u \in V_1, v \in V_2\}$. When G is understood from the context, we write $[V_1, V_2]$ for $[V_1, V_2]_G$.

To be consistent with the matroid contraction defined in [13] or in [22], for a graph G and a subset $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two end vertices of each edge in X. Note the new loops or new multiple edges may result from a contraction.

For a matroid $M, \mathcal{I}(M), \mathcal{C}(M), \mathcal{C}_0(M)$ and $\mathcal{B}(M)$ denote the set of all independent sets of M, the set of all circuits of M, the set of all cycles of M, and the set of all bases of M, respectively. Define

$$\tau(M) = \max\{k : \exists B_1, B_2 \ldots, B_k \in \mathcal{B}(M) \text{ such that } B_i \cap B_j = \emptyset \text{ whenever } i \neq j\},$$

and for a connected graph G, define $\tau(G) = \tau(M(G))$. The *girth* of a matroid M, is

$$g(M) = \begin{cases} \min\{|C| : C \in \mathcal{C}(M)\} & \text{if } \mathcal{C}(M) \neq \emptyset \\ \infty & \text{if } \mathcal{C}(M) = \emptyset. \end{cases}$$

The girth of the dual of M, $g(M^*)$, is often referred as the *cogirth* of a matroid M. When $M = M(G)$ is the cycle matroid of a connected graph G, $g(M^*)$ equals the edge-connectivity of G. The following is well known.

Theorem 1.1 (*Catlin [6], Jaeger [10]*). *If a graph G is 4-edge-connected, then G is supereulerian.*

A graph G is *collapsible* if for any subset $X \subseteq V(G)$ with $|X| \equiv 0 \pmod 2$, G has a spanning connected subgraph H_X such that $O(H_X) = X$. As examples, circuits of length at most 3 are collapsible. Catlin [6] showed that collapsible graphs are of particular importance in determining if a graph is supereulerian.

Theorem 1.2. *Each of the following holds.*

(i) *(Catlin, Theorem 3 of [6]). If L is a collapsible subgraph of G, and if G/L has a spanning eulerian subgraph H', then G has a spanning eulerian subgraph H with $E(H') \subseteq E(H)$. Thus G is supereulerian if and only if G/L is supereulerian.*
(ii) *(Nash-Williams [11] and Tutte [19]). If G is 4-edge-connected, then $\tau(G) \geq 2$.*
(iii) *(Catlin, Theorem 2 of [6]). If $\tau(G) \geq 2$, then G is collapsible.*

In this paper, we consider the question whether Theorems 1.1 and 1.2 can be extended to matroids. In Section 2, we present examples of connected cographic matroids which do not have spanning cycles even though the cogirth can be arbitrarily large, which indicate that Theorem 1.1 cannot be extended to cographic matroids in general. In Section 5, we will generalize Theorem 1.2(iii) to binary matroids.

Given matroids $N_1, N_2, \ldots, N_k$, let $EX(N_1, N_2, \ldots, N_k)$ denote the family of matroids that do not contain a minor isomorphic to any of the N_i's. The main purpose of this paper is to prove the following.

Theorem 1.3. *If a connected matroid $M \in EX(M^*(K_5), F_7, F_7^*, U_{2,4})$ has cogirth $g(M^*) \geq 4$, then M has a spanning cycle.*

Since $EX(M^*(K_5), F_7, F_7^*, U_{2,4})$ is a subset of the set of all regular matroids, we shall apply decomposition theorems of Seymour [16] and Wagner [21] to prove our result.

2. Examples

Let G be a graph and let $M = M^*(G)$ be the cographic matroid of G. Then it is easy to see that M is supereulerian if and only if $V(G)$ can be partitioned to sets V_1 and V_2 such that for both $i = 1, 2$, the induced subgraph $G[V_i]$ is acyclic. As a consequence, if M is supereulerian, then $\chi(G)$, the chromatic number of G, is at most 4. As an example, $M^*(K_5)$ cannot be supereulerian.

Theorem 2.1 (*Theorem 5 on page 128 of [4]*). *For any given integers $g \geq 4$ and $k \geq 4$, there exists a graph G with girth $g(G) \geq g$ and chromatic number $\chi(G) \geq k$.*

In particular, for arbitrarily large $g \geq 4$ and $k = 5$, there exists a graph G with $g(G) \geq g$ and $\chi(G) \geq 5$. This implies that there exists a cographic matroid M with cogirth $g(M^*) \geq g$ such that M is not supereulerian.

Note that each of such examples has chromatic number at least 5. Wagner [21] showed that the 4-Color-Theorem [1,2,15] is equivalent to that every 5 chromatic graph has a K_5-minor, a special case of the well known Hadwiger's coloring conjecture. Therefore, each of such examples suggested by Theorem 2.1 will have a K_5-minor.

3. Collapsible graphs

Catlin in [6] showed that for any graph G, G has a unique set of maximally collapsible subgraphs $L_1, L_2, \ldots, L_k$. The graph $G' = G/(L_1 \cup L_2 \cup \cdots \cup L_k)$ is the *reduction* of G. A graph G is *reduced* if G equals its own reduction. For a graph G, let $F(G)$ denote the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. Thus $\tau(G) \geq 2$ is equivalent to $F(G) = 0$. The following summarizes some of the useful facts about collapsible graphs and reductions.

Theorem 3.1. *Let G be a connected graph.*

(i) *(Catlin, Theorem 8 of [6]). If G is reduced, then G is simple, and G does not have a nontrivial subgraph which is collapsible.*
(ii) *(Catlin and Lai, Proposition in Section 3 of [8]). If G is reduced, then $F(G) = 2|V(G)| - |E(G)| - 2$.*
(iii) *(Catlin, Theorem 7 of [6]). If $F(G) \leq 1$, then G is collapsible if and only if its reduction is not isomorphic to K_2.*
(iv) *(Catlin et al. [9]). Let G be a connected graph with $F(G) \leq 2$. Then G is collapsible if and only if the reduction of G is not isomorphic to a member in $\{K_2, K_{2,t}, (t \geq 1)\}$.*

Theorem 3.2 (*Catlin [7], Zhan [24]*). *Let G be a graph. Then $\kappa'(G) \geq 4$ if and only if for any edges $e_1, e_2 \in E(G)$, $\tau(G - \{e_1, e_2\}) \geq 2$.*

Lemma 3.3. *Let G be a loopless graph, Z an edge subset and D an edge cut of G. Suppose that G, Z and D satisfy:*

(A) *Z does not contain any edge cut of G;*
(B) *if $|Z| = 3$, then Z is a circuit of G;*
(C) *if $D \cap Z = \emptyset$, then $|D| \geq 4$.*

Then each of the following holds.

(i) *If* $|Z| = 1$*, then either* $\tau(G - Z) \geq 2$*, or* G *has a 2-edge-cut* X *with* $Z \subset X$*, such that both components of* $G - X$ *are collapsible.*

(ii) *If* $|Z| = 3$*, then either* $G - Z \cong K_{2,1}$ *or* $G - Z$ *contains a nontrivial collapsible subgraph.*

Proof. (i) Let $Z = \{e\}$. By (A), Z is not an edge cut. Let X be a minimum edge cut containing e. Then $|X| \geq 2$.

Case 1: $|X| \geq 4$. By (C), $\kappa'(G) \geq 4$. By Theorem 3.2, $\tau(G - Z) = \tau(G_1 - e) \geq 2$.

Case 2: $|X| = 3$. Add to G a new edge e' parallel to e, and denote the resulting graph by $G_1 = G + e'$. Then by (C), $\kappa'(G_1) \geq 4$. By Theorem 3.2, $\tau(G - Z) = \tau(G_1 - e - e') \geq 2$.

Case 3: $|X| = 2$. Assume that $X = \{e, f\}$. Add to G two new edges e', e'' parallel to the edge e and denote the resulting graph by $G_1 = G + \{e', e''\}$. Then by (C), $\kappa'(G_1) \geq 4$. By Theorem 3.2, $\tau(G) = \tau(G_1 - \{e', e''\}) \geq 2$. Let T_1, T_2 be two edge disjoint spanning trees. Without loss of generality, we assume that $e \in E(T_1)$, $f \in E(T_2)$. Let H_1, H_2 be the two components of $G - X$. Then we can assume that $T_{11} \subseteq H_1, T_{12} \subseteq H_2$ are two components of $T_1 - e$ and $T_{21} \subseteq H_1, T_{22} \subseteq H_2$ are two components of $T_2 - f$. So T_{11} and T_{21} (T_{12} and T_{22}, respectively) are two edge disjoint spanning trees of H_1 (H_2, respectively). By Theorem 1.2(iii), both H_1 and H_2 are collapsible.

(ii) Let $Z = \{e_1, e_2, e_3\}$. By (A), $G - Z$ is connected.

Case 1: $\kappa'(G - Z) \geq 2$. Then by (A) and (C), $\kappa'(G) \geq 4$. Then $F(G - Z) \leq 1$ by Theorem 3.2. Since $\kappa'(G - Z) \geq 2$, the reduction of $G - Z$ cannot be a K_2. It follows by Theorem 3.1(iii) that $G - Z$ is collapsible.

Case 2: $\kappa'(G - Z) = 1$. Then we claim that $G - Z$ has at most two cut-edges. By contradiction, we assume that $G - Z$ has three cut-edges $\{f_1\}, \{f_2\}, \{f_3\}$. Then $G - (Z \cup \{f_1, f_2, f_3\})$ has at least 4 components. Since $|Z| = 3$, at least one of the components is vertex disjoint from the 3-circuit Z. We can assume that H is such a component that $V(H) \cap V(Z) = \emptyset$. Since $[V(H), V(G - H)]_G \subseteq \{f_1, f_2, f_3\}$, $D = [V(H), V(G - H)]_G$ is an edge cut with $|D| \leq 3$ and $D \cap Z = \emptyset$, contrary to (C).

Hence $G - Z$ has either one or two cut edges. We assume first that f_1, f_2 are the two cut-edges of $G - Z$. Note that every 3-edge-cut of G has either 0 or 2 edges in common with Z. By (C), every 3-edge-cut of $G - Z$ must contain 2 edges of Z. We can assume without loss of generality that $\{e_1, e_2, f_1\}, \{e_1, e_3, f_2\}$ are the only two 3-edge-cuts of G. We can add one edge e_1' to G parallel to the edge e_1 and denote the resulting graph by $G_1 = G + e_1'$. Then $\kappa'(G_1) \geq 4$ and $\tau(G_1 - \{e_1, e_1'\}) \geq 2$ by Theorem 3.2. Thus $F(G - Z) = F(G_1 - (Z \cup e_1')) \leq 2$. Since $G - Z$ is connected, by Theorem 3.1(iv), $G - Z$ is either collapsible or the reduction is isomorphic to a K_2 or a $K_{2,t}$ ($t \geq 1$). Since $G - Z$ has two cut edges, we must have $t = 1$. If $G - Z \cong K_{2,1}$, then done. Otherwise $G - Z \ncong K_{2,1}$, and so one of the vertex of $K_{2,1}$ must be contracted from a nontrivial collapsible subgraph of $G - Z$. It follows that $G - Z$ contains a nontrivial collapsible subgraph. The proof for the case when $G - Z$ has exactly one cut edge is similar, and so is omitted. □

For a subgraph H of a graph G, the *vertices of attachments* of H in G, denoted $A_G(H)$, is the set of vertices in $V(H)$ that are adjacent to a vertex not in H.

Definition 3.4. Let W_0 denote a graph isomorphic to a $K_{2,3}$ with $w_1, w_2 \in V(W_0)$ being the two vertices of degree 3 in W_0. Define W_1 to be the graph obtained from W_0 by contracting an edge, and W_2 to be the graph obtained from W_0 by contracting two edges incident with w_2.

Lemma 3.5. *Let G be a connected graph with $|E(G)| \geq 4$ and let $Z = \{e_1, e_2, e_3\}$ be a minimal edge cut of G such that $|Z| = 3$. If for every edge cut D of G such that $D \cap Z = \emptyset$, $|D| \geq 4$, then one of the following must hold.*

(i) $G \in \{W_1, W_2\}$.
(ii) *$G - Z$ has a nontrivial collapsible subgraph.*

Proof. If G has a cut vertex, then one of the end block H of G does not contain any edge in Z, and so H must be 4-edge-connected. By Theorem 1.2, H is a nontrivial collapsible subgraph of $G - Z$. Thus we assume that G is 2-connected.

Suppose first that $G - Z$ is a forest, and let H_1 and H_2 be the two components of $G - Z$ such that $|E(H_2)| \geq |E(H_1)|$. If $|E(H_2)| \geq 3$, then $|V(H_2)| \geq 4$, and so H_2 must have a vertex of degree v at most 3 which is not incident with any edge in Z. It follows that the edges in G incident with v form an edge cut D with $|D| \leq 3$ and $D \cap Z = \emptyset$, contrary to the assumption of the lemma. Therefore $|E(H_2)| \leq 2$. Suppose then $|E(H_2)| = 2$. Then $V(H_2) = \{v_0, v_1, v_2\}$ such that $E(H_2) = \{v_0v_1, v_0v_2\}$. For $i \in \{1, 2\}$, since H_i is a component of $G - Z$, every vertex in $A_G(H_i)$ must be incident with the edges in Z. If $|A_G(H_2)| \leq 2$, then either $v_0 \notin A_G(H_2)$, whence $\{v_0v_1, v_0v_2\}$ is an edge cut of G with two edges disjoint from Z; or $v_1 \notin A_G(H_2)$ (or $v_2 \notin A_G(H_2)$), whence v_0v_2 (or v_0v_1, respectively) is a cut edge disjoint from Z. In any case, a contradiction to the assumption of the lemma obtains. Therefore we may assume that v_0, v_1 and v_2 are incident with e_3, e_1 and e_2, respectively. Similarly, if $|E(H_1)| = 2$, then the three vertices of $V(H_1)$ must be incident with the three edges in Z. Thus $G - Z$ has two components each of which is a path of length 2. It follows that G has an edge cut D with $|D| = 2$ and with $D \cap Z = \emptyset$, contrary to the assumption of the lemma.

If $E(H_1) = \{e\}$, then since H_1 and H_2 are different components of $G - Z$ and since every vertex in $A_G(H_1)$ must be incident with edges in Z, we may assume that the two ends of e are incident either with e_1 and e_2, or with e_1 and e_3. In either case, $\{e, v_0v_1\}$ is an edge cut of G disjoint from Z, contrary to the assumption of the lemma. It follows that we must have either $|E(H_2)| = 2$ and $|E(H_1)| = 0$ whence $G \cong W_1$; or $|E(H_2)| = 1$ and $|E(H_1)| = 1$ whence $G \cong W_1$; or $|E(H_2)| = 1$ and $|E(H_1)| = 0$ whence $G \cong W_2$. Thus (i) must hold.

Now suppose that $G - Z$ has a component H which is not a tree. By the assumption of the lemma, H contains a subgraph H' of G such that $|A_G(H')| \leq |Z| = 3$ and such that for any $v \in A_G(H')$, $\deg_{H'}(v) \geq 2$; and for any $u \in V(H') - A_G(H')$, $\deg H'(u) \geq 4$. Thus counting the incidences of vertices in H', we have

$$2|E(H')| \geq 4(|V(H')| - 3) + 6 = 4|V(H')| - 6.$$

It follows by a result of Nash-Williams [12] that H' must contain a nontrivial subgraph H'' with $\tau(H'') \geq 2$. By Theorem 1.2, H'' is collapsible. This proves (ii). □

4. Decompositions

In this paper, we use $\triangle$ to denote both a set operator and a matroid operator. Given two sets X and Y, the *symmetric difference* of X and Y is defined as

$$X \triangle Y = (X \cup Y) - (X \cap Y).$$

Now suppose that M_1, M_2 are binary matroids on E_1 and E_2, respectively. We follow Seymour [16,17] to define the *binary sum* $M_1 \triangle M_2$ to be the matroid on the set $E_1 \triangle E_2$ such that the set of cycles of $M_1 \triangle M_2$ equals

$$\{C_1 \triangle C_2 \subseteq E_1 \triangle E_2 : C_i \text{ is a cycle of } M_i, i = 1, 2\}.$$

Three special cases of this operation are introduced by Seymour [16,17] as follows.

(i) If $E_1 \cap E_2 = \emptyset$ and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a *1-sum* of M_1 and M_2.
(ii) If $|E_1 \cap E_2| = 1$ and $E_1 \cap E_2 = \{z\}$, say, and z is not a loop or coloop of M_1 or M_2, and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a *2-sum* of M_1 and M_2.
(iii) If $|E_1 \cap E_2| = 3$ and $E_1 \cap E_2 = Z$, say, and Z is a circuit of M_1 and M_2, and Z includes no cocircuit of either M_1 or M_2, and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a *3-sum* of M_1 and M_2.

For $i = 1, 2, 3$, an i-sum of M_1, M_2 is denoted by $M_1 \oplus_i M_2$. The 1-sum $M_1 \oplus_1 M_2$ is also written as $M_1 \oplus M_2$. Seymour ([16], also see Exercise 6 in Section 12.4 of Oxley [13]) showed the following property of the dual of $M_1 \triangle M_2$ for binary matroids M_1 and M_2.

$$(M_1 \triangle M_2)^* = M_1^* \triangle M_2^*. \tag{1}$$

When $i = 1, 2$, the following is well known (Proposition 7.1.20 of [13]).

$$(M_1 \oplus_i M_2)^* = M_1^* \oplus_i M_2^*. \tag{2}$$

We use the notations in Definition 3.4, and let $G \in \{W_1, W_2\}$. Let Z be an edge subset of G separating w_1 and w_2. Then Z is a 3-cocircuit of $M(G)$. If a binary matroid $M = M_1 \oplus_3 M_2$ is a 3-sum of M_1 and M_2 such that $M_2 \cong M^*(G)$ with $E(M_1) \cap E(M_2) = Z$, then M_1 has a parallel extension M_1' such that $M = M_1' - Z$. In this case, we call $M_1 \oplus_3 M_2$ a *trivial 3-sum*.

Let R_{10} denote the vector matroid of the following matrix over $GF(2)$:

$$R_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

It is known that R_{10}^* is isomorphic to R_{10}. Based on the notion of matroid sums, Seymour proved the following decomposition theorem for regular matroids.

Theorem 4.1 (*Seymour [16]*). *Let M be a regular matroid. One of the following must hold.*

(i) *M is graphic.*
(ii) *M is cographic.*
(iii) $M \cong R_{10}$.
(iv) *For some $i \in \{1, 2, 3\}$, $M = M_1 \oplus_i M_2$ is the i-sum of two matroids M_1 and M_2, each of which is isomorphic to a proper minor of M.*

If a matroid M is isomorphic to the cycle matroid of a planar graph, then M is called a *planar matroid*. Thus a matroid M is planar if and only if M^* is planar. We can state Wagner's decomposition theorem as follows (see Seymour [16,21]).

Theorem 4.2 (*Wagner [21]*). *Let M be a graphic matroid that does not contain a minor isomorphic to $M(K_5)$. One of the following must hold.*

(i) *M is a planar matroid.*

(ii) $M \cong M(H_8)$.
(iii) $M \cong M(K_{3,3})$.
(iv) *For some* $i \in \{1, 2, 3\}$, $M = M_1 \oplus_i M_2$ *is the* i*-sum of two matroids* M_1 *and* M_2*, such that both* M_1 *and* M_2 *are proper minors of* M.

Theorem 4.3 (*Tutte [18]*). *A matroid* M *is regular if and only if* $M \in EX(F_7, F_7^*, U_{2,4})$.

Lemma 4.4. *Let* M *be a connected cographic matroid. If* $M \in EX(M^*(K_5))$*, then one of the following must hold.*

(i) M *is planar.*
(ii) $M \cong M^*(K_{3,3})$.
(iii) $M \cong M^*(H_8)$.
(iv) $M = M_1 \oplus_2 M_2$ *is a 2-sum of* M_1 *and* M_2*, such that each of* M_1 *and* M_2 *is isomorphic to a proper minor of* M*, and such that either* M_2 *is isomorphic to one of* $\{M^*(K_{3,3}), M^*(H_8)\}$ *or* M_2 *is planar.*
(v) $M^* = M_1^* \oplus_3 M_2^*$ *is a nontrivial 3-sum of* M_1^* *and* M_2^**, such that each of* M_1^* *and* M_2^* *is isomorphic to a proper minor of* M^**, and such that* M_2^* *is planar.*

Proof. Since M is cographic, there exists a connected graph H such that $M = M^*(H)$. As $M \in EX(M^*(K_5))$, $M^* = M(H) \in EX(M(K_5))$. By Theorem 4.2, one of the conclusions of Theorem 4.2 holds for M^*. If any of Theorem 4.2(i), (ii) or (iii) holds for M^*, then Lemma 4.4(i), (ii) or follows, respectively. Thus we may assume that Theorem 4.2(iv) holds. Since M^* is connected, M^* must be a 2-sum or a 3-sum of its proper minors. Hence M^* is obtained by taking a sequence of 2-sums and 3-sums of its minors isomorphic to planar matroids, copies of $M(H_8)$ or $M(K_{3,3})$. Pick such a decomposition of M so that the number of minors is minimized. Suppose the last one is denoted by M_2^*, then $M^* = M_1^* \oplus_i M_2^*$ and M_2^* is either in $\{M(K_{3,3}), M(H_8)\}$ or is planar. When $i = 2$, since by (2), $M = (M^*)^* = (M_1^* \oplus_2 M_2^*)^* = M_1 \oplus_2 M_2$, Lemma 4.4(iv) must hold. When $i = 3$, since the number of minors in this decomposition is minimized, the 3-sum $M^* = M_1^* \oplus_i M_2^*$ must be a nontrivial one. Since $K_{3,3}$ and H_8 are triangle free, M_2^* cannot be in $\{M(K_{3,3}), M(H_8)\}$, and so M_2^* must be planar. □

Theorem 4.5. *For every connected matroid* $M \in EX(M^*(K_5), F_7, F_7^*, U_{2,4})$*, one of the following must hold.*

(i) M *is graphic.*
(ii) $M \in \{R_{10}, M^*(K_{3,3}), M^*(H_8)\}$.
(iii) $M = M_1 \oplus_2 M_2$ *is a 2-sum of* M_1 *and* M_2*, such that each of* M_1 *and* M_2 *is isomorphic to a proper minor of* M*, and such that either* M_2 *is isomorphic to one of* $\{R_{10}, M^*(K_{3,3}), M^*(H_8)\}$ *or* M_2 *is graphic.*
(iv) $M = M_1 \oplus_3 M_2$ *is a nontrivial 3-sum of* M_1 *and* M_2*, such that each of* M_1 *and* M_2 *is isomorphic to a proper minor of* M*, and such that either* M_2 *is isomorphic to one of* $\{M^*(K_{3,3}), M^*(H_8)\}$ *or* M_2 *is graphic.*
(v) $M^* = M_1^* \oplus_3 M_2^*$ *is a nontrivial 3-sum of* M_1^* *and* M_2^**, such that each of* M_1^* *and* M_2^* *is isomorphic to a proper minor of* M*, and such that* M_2^* *is planar.*

Proof. Let $M \in EX(M^*(K_5), F_7, F_7^*, U_{2,4})$. By Theorem 4.3, M is regular. By Theorem 4.1, one of the conclusions of Theorem 4.1 must hold. If Theorem 4.1(i) or (iii) hold, then Theorem 4.5(i) or (ii) holds accordingly. Therefore we consider these cases.

Case 1: Theorem 4.1(ii) holds. Then M is a cographic matroid. By Lemma 4.4, Theorem 4.5(i), (ii), (iii) or (v) must hold.

Case 2: Theorem 4.1(iv) holds, then M is obtained from its minors each of which is isomorphic to a graphic matroid or a cographic matroid or an R_{10}, via 2-sums or 3-sums. Let the last one be denoted M_2'. Then $M = M_1' \oplus_i M_2'$, where $i \in \{2, 3\}$ and where M_2' is isomorphic to a graphic or R_{10} or a cographic matroid. If M_2' is isomorphic to a graphic matroid or R_{10}, then Theorem 4.5(iii) or (iv) must hold. Since R_{10} does not have a 3-circuit, when $M = M_1 \oplus_3 M_2$, M_2 cannot be an R_{10}. Suppose that M_2' is cographic. Then by Lemma 4.4, Theorem 4.5(iii) or (v) must hold. □

5. Reductions in binary matroids

All matroids considered in this section will be binary. In this section, we shall investigate binary matroids N with the property that whenever M is a binary matroid containing N as a restriction, it always holds that

$$M \text{ is supereulerian if and only if } M/N \text{ is superelerian.} \tag{3}$$

A binary matroid N with $|E(N)| \geq 1$ satisfying the property in (3) will be referred to as a *contractible* matroid. Our main goal in this section is to prove some useful facts on contractible matroids, including Theorem 5.4, which generalizes Theorem 1.2(iii) to binary matroids.

Lemma 5.1. *Suppose that a binary matroid $M = M_1 \oplus_i M_2$ with $E(M_1) \cap E(M_2) = Z$ such that either $i = 2$, and $Z = \{e_0\}$, or $i = 3$ and $Z = \{e_1, e_2, e_3\}$. Then*

$$M/(E(M_2) - Z) = M_1/Z.$$

Proof. Let $E_i = E(M_i) - Z$. In the definitions of 2-sums and 3-sums, we require that e_0 is not a loop nor a coloop in either M_1 or M_2, and Z does not contain a cocircuit in either M_1 or M_2. This means that M_2 has a basis disjoint from Z, and so $Z \subseteq cl_{M_2}(E_2)$. We shall show that both $M/(E(M_2) - Z)$ and M_1/Z have the same independent sets. Fix a basis $B_1 \in \mathcal{B}(M|E_2)$.

Pick $I \in \mathcal{I}(M/E_2)$. Then $I \cup B_1 \in \mathcal{I}(M)$. By contradiction, we assume that $I \cup e_0 \notin \mathcal{I}(M_1)$ (if $Z = \{e_0\}$) or $I \cup (Z - e_i) \notin \mathcal{I}(M_1)$ (if $|Z| = 3$ $e_i \in Z$, and $Z \in \mathcal{C}(M_1) \cap \mathcal{C}(M_2)$).

Suppose first that $Z = \{e_0\}$. Then $I \cup e_0$ has a circuit $C_1 \in \mathcal{C}(M_1)$ such that $e_0 \in C_1$. Since $Z \subseteq cl_{M_2}(E_2)$ and since $B_1 \in \mathcal{B}(M|E_2) \subseteq \mathcal{B}(M_2)$, $B_1 \cup e_0$ has a circuit $C_2 \in \mathcal{C}(M_2)$ with $e_0 \in C_2$. It follows that $C_1 \Delta C_2 \in \mathcal{C}(M)$. But $C_1 \Delta C_2 \subseteq I \cup B_1 \in \mathcal{I}(M)$, a contradiction. Thus $I \cup e_0 \in \mathcal{I}(M_1)$, and so $I \in \mathcal{I}(M_1/Z)$.

Suppose now that $Z = \{e_1, e_2, e_3\}$. Then for each $i \in \{1, 2, 3\}$, $I \cup (Z - e_i)$ has a circuit $C_i' \in \mathcal{C}(M_1)$. Since $Z \subseteq cl_{M_2}(E_2)$ and since $B_1 \in \mathcal{B}(M|E_2) \subseteq \mathcal{B}(M_2)$, $B_1 \cup e_i$ has a circuit $C_i'' \in \mathcal{C}(M_2)$ such that $e_i \in C_i''$. If $C_1' \cap Z = \{e_2, e_3\}$, then $C_1' \Delta C_2'' \Delta C_3''$ is a cycle of M, and $C_1' \Delta C_2'' \Delta C_3'' \subseteq I \cup B_1 \in \mathcal{I}(M)$, a contradiction; if $C_1' \cap Z = \{e_2\}$, then $C_1' \Delta C_2''$ is a circuit of M, and $C_1' \Delta C_2'' \subseteq I \cup B_1 \in \mathcal{I}(M)$, also a contradiction. Thus we must have $I \in \mathcal{I}(M_1/Z)$.

Conversely, assume that $I \in \mathcal{B}(M_1/Z)$. We prove first the case when $Z = \{e_1, e_2, e_3\}$. Then for any $i \in \{1, 2, 3\}$, $I \cup (Z - e_i) \in \mathcal{I}(M_1)$. To show that $I \in \mathcal{B}(M/E_2)$, we need to show $I \cup B_1 \in \mathcal{I}(M)$. Suppose not, then there exists a $C \in \mathcal{C}(M)$ and $C \subseteq I \cup B_1$. As $I \in \mathcal{B}(M_1/Z)$ and $B_1 \in \mathcal{B}(M|E_2)$, we must have both $C \cap I \neq \emptyset$ and $C \cap B_1 \neq \emptyset$. It follows that there exists a $C_1 \in \mathcal{C}(M_1)$ and $C_2 \in \mathcal{C}(M_2)$ such that $C = C_1 \Delta C_2$. Since $C_1, Z \in \mathcal{C}(M_1)$ and since $C_1 \neq Z$, we may assume that $C_1 \cap Z \subseteq Z - e_1$. Thus $C_1 \in \mathcal{C}(M_1)$ and $C_1 \subseteq I \cup (Z - e_1) \in \mathcal{I}(M_1)$, a contradiction.

When $Z = \{e_0\}$, the proof is similar. For $I \in \mathcal{B}(M_1/Z)$, $I \cup e_0 \in \mathcal{I}(M_1)$. If there exists a $C \in \mathcal{C}(M)$ and $C \subseteq I \cup B_1$, then there are $C_1 \in \mathcal{C}(M_1)$ and $C_2 \in \mathcal{C}(M_2)$ such that $C = C_1 \Delta C_2$. But then $C_1 \subseteq I \cup e_0 \in \mathcal{I}(M_1)$, a contradiction. □

Lemma 5.2. *Let M be a binary matroid.*

(i) *If $H \in \mathcal{C}_0(M)$ and $e \in E(M)$, then $H - e \in \mathcal{C}_0(M/e)$.*
(ii) *If $H \in \mathcal{C}_0(M)$ with $r(H) = r(M)$, then $H' = (H \cup e)/e \in \mathcal{C}_0(M/e)$ with $r(H') = r(M/e)$.*
(iii) *If M is a supereulerian matroid and $e \in E(M)$. Then M/e is also a supereulerian matroid.*

Proof. Clearly, (ii) implies (iii). Let $H \in \mathcal{C}_0(M)$ and let $H_1 = H - e$. We shall show that $H_1 \in \mathcal{C}_0(M/e)$. This certainly holds if e is a loop of M. Hence we assume that e is not a loop.

To see that H_1 is a cycle of M/e, it suffices to show that for any $D' \in \mathcal{C}((M/e)^*)$, $|D' \cap H_1| \equiv 0 \pmod 2$.

Pick $D' \in \mathcal{C}((M/e)^*)$. Since $e \notin D'$, we have $|D' \cap H_1| = |D' \cap H|$. Thus by the fact that $D' \in \mathcal{C}((M/e)^*) = \mathcal{C}(M^* - e) \subseteq \mathcal{C}(M^*)$, we have

$$|D' \cap H_1| = |D' \cap H| \equiv 0 \pmod 2,$$

where the last congruence follows from the fact that in a binary matroid, the cycle space and the cocycle space are orthogonal to each other. This proves that H_1 is a cycle of M/e, and so (i) follows.

To prove (ii), we now assume that H is a spanning cycle of M to show that $r(H') = r(M)$. This certainly holds if e is a loop, and so we assume that e is not a loop of M. If $e \in H$, then since $r(H) = r(M)$, H contains a basis $B_1 \in \mathcal{B}(M)$ with $e \in B_1$, and so $B_1 - e \subseteq H'$. It follows that $r(H') = r(M/e)$, by the definition of a contraction. Suppose that $e \notin H$. As $r(H) = r(M)$, there exists a $B_2 \in \mathcal{B}(M)$ such that $B_2 \subseteq H$. Then $B_2 \cup e$ has a unique circuit C_e. Since e is not a loop, there exists an $e' \in C_e - e \subseteq B_2$ such that $B_3 = B_2 \cup e - e' \in \mathcal{B}(M)$, and $B_3 - e \subseteq H'$. It also follows that $r(H') = r(M/e)$. □

Lemma 5.3. *Let M be a binary matroid and $X \subseteq E(M)$. Then*

$$\mathcal{C}_0(M/X) = \{C - X : C \in \mathcal{C}_0(M)\}.$$

Proof. Let $\mathcal{C}' = \{C - X : C \in \mathcal{C}_0(M)\}$. Then for any $H' \in \mathcal{C}_0(M/X)$, there exist $C_1', C_2', \ldots, C_t' \in \mathcal{C}(M/X)$ such that $H' = \Delta_{i=1}^t C_i'$. By the definition of a contraction, there exist $C_1, C_2, \ldots, C_t \in \mathcal{C}(M)$ such that for any $i \in \{1, 2, \ldots, t\}$, $C_i' = C_i - X$. Thus

$$H' = \Delta_{i=1}^t C_i' = \Delta_{i=1}^t (C_i - X) = \Delta_{i=1}^t C_i - X \in \mathcal{C}'.$$

Conversely, suppose that $H \in \mathcal{C}'$. Then for some circuits $C_1, C_2, \ldots, C_t \in \mathcal{C}(M)$, $H = \Delta_{i=1}^t C_i - X = \Delta_{i=1}^t (C_i - X)$. By Lemma 5.2, each $C_i - X \in \mathcal{C}_0(M/X)$, and so $H \in \mathcal{C}_0(M/X)$. □

Theorem 5.4. *Let M be a binary matroid and $X \subseteq E(M)$ such that $r(X) < r(M)$. If $\tau(M|X) \geq 2$, then the following are equivalent.*

(i) *M is supereulerian.*
(ii) *M/X is supereulerian.*
(iii) *$M/cl_M(X)$ is supereulerian.*

(Thus every binary matroid N with $\tau(N) \geq 2$ is contractible.)

Proof. By Lemma 5.2, (i) implies (ii), and (ii) implies (iii). Thus it suffices to show that (iii) implies (i). For notational convenience, we assume that $X = cl_M(X)$ is closed, and that M/X has a spanning cycle H', to prove that M has a spanning cycle. Since H' is a spanning cycle of M/X, there exist some mutually disjoint circuits $C'_1, C'_2, \ldots, C'_t \in \mathcal{C}(M/X)$ such that

$$H' = \Delta_{i=1}^{t} C'_i = \bigcup_{i=1}^{t} C'_i.$$

Let $B_1, B_2 \in \mathcal{B}(M|X)$ with $B_1 \cap B_2 = \emptyset$. Note that since $X = cl_M(X) = cl_M(B_1)$, we have $M/X = M/cl_M(B_1)$, and so for each i, there exists a $T_i \subset B_1$ (Proposition 3.1.11 of [13]) such that

$$C_i = C'_i \cup T_i \in \mathcal{C}(M).$$

Let $T = \Delta_{i=1}^{t} T_i \subseteq B_1$, and write $B_1 - T = \{e_1, e_2, \ldots, e_s\}$. For each $j \in \{1, 2, \ldots, s\}$, let $C_{M|X}(e_j, B_2)$ denote the fundamental circuit of e_j in $M|X$ with respect to B_2, and define

$$C_0 = \Delta_{j=1}^{s} C_{M|X}(e_j, B_2).$$

Now let $H = \Delta_{i=0}^{t} C_i$. Then $H \in \mathcal{C}_0(M)$ and $H = C_0 \cup (\Delta_{i=1}^{t} C'_i) \cup (\Delta_{i=1}^{t} T_i) = C_0 \cup H' \cup T$. Thus $H' \subseteq H$. Since $r(H') = r(M/X)$, there exists a $B' \in \mathcal{B}(M/X)$ such that $B' \subset H' \subseteq H$. Since $T \subseteq H$ and since $B_1 - T \subseteq C_0 \subseteq H$, we have $B_1 \subseteq H$, and so $B' \cup B_1 \subseteq H$. As $B' \cup B_1 \in \mathcal{B}(M)$, we have $r(H) = r(M)$. □

Proposition 5.5. *Let M, M_1 and M_2 be binary matroids such that $M = M_1 \triangle M_2$ with $Z = E(M_1) \cap E(M_2)$ and such that one of the following holds.*

(i) *$Z = \{e_0\}$ and $M = M_1 \oplus_2 M_2$ is a 2-sum, or*
(ii) *$Z = \{e_1, e_2, e_3\}$ and $M = M_1 \oplus_3 M_2$ is a 3-sum, or*
(iii) *$Z = \{e_1, e_2, e_3\}$ and $M^* = M_1^* \oplus_3 M_2^*$ is a 3-sum.*

Suppose that $M_2 = M(G)$ is graphic such that $G - Z$ contains a nontrivial collapsible subgraph L. If $M/E(L)$ is supereulerian, then M is also supereulerian.

Proof. Let $M' = M/E(L)$. Suppose first that $r(E(L)) < r(M_2)$. Then

$$M' = M_1 \triangle (M_2/E(L)) = M_1 \triangle M(G/L).$$

Let H' be a spanning cycle of M'. Then by the definition of binary sums, $H' = H_1 \triangle H'_2$, where $H_1 \in \mathcal{C}_0(M_1)$, $H'_2 \in \mathcal{C}_0(M_2/E(L))$ and $H_1 \cap Z = H'_2 \cap Z$. Note that H'_2 is an eulerian subgraph of G/L. Let $G' = G - (Z - H'_2)$. Since $Z \cap E(L) = \emptyset$, L is a subgraph of G' and H'_2 is an eulerian subgraph of G'/L. By Theorem 1.2 (i), G' has an eulerian subgraph H_2 with $E(H'_2) \subseteq E(H_2)$ and containing a spanning connected subgraph L_1 of L. Since G' is a spanning subgraph of G, H_2 is an eulerian subgraph of G with $E(H_2) \cap Z = E(H'_2) \cap Z = E(H_1) \cap Z$, and so $H = H_1 \triangle H_2$ is a cycle of M. Since $H' \subseteq H$, $L_1 \subseteq H$ and since $r(L_1) = r(M|E(L))$ and $r(H') = r(M/E(L))$, we have $r(H) = r(M)$, and so H is a spanning cycle of M.

Now we assume that $r(E(L)) = r(M_2)$. Note that if Z is a cocircuit of M_2 and $E(L) \cap Z = \emptyset$, we cannot have $r(E(L)) = r(M_2)$. Therefore, we only need to prove (i) and (ii). By the definition of collapsible graphs, if L is collapsible, then adding an edge with both ends in $V(L)$ also results a collapsible graph. Thus we may assume that $L = G - Z$.

For each $i \in \{1, 2\}$, let $E_i = E(M_i - Z)$. Note that $E_2 = E(L)$. Let $H' \subseteq E(M/E_2)$ be a spanning cycle of M/E_2. Then there exist $C'_1, C'_2, \ldots, C'_t \in \mathcal{C}(M/E_2)$ such that

$$H' = \Delta_{i=1}^{t} C'_i.$$

By Lemma 5.1, for all $i \in \{1, 2, \ldots, t\}$, there exists a $T_i \subseteq Z$ such that $C'_i \cup T_i \in \mathcal{C}(M_1)$.

Case 1: $|Z| = 1$ and $M = M_1 \oplus_2 M_2$.

We may assume that for some k with $0 \le k \le t$, $T_1 = T_2 = \cdots = T_k = \{e_0\}$, and $T_{k+1} = \cdots = T_t = \emptyset$, (we take the convention that $k = 0$ means $T_1 = T_2 = \cdots = T_t = \emptyset$). Thus $e_0 = uv$ is an edge in G such that $G - e_0$ is collapsible. Choosing $X = \{u, v\}$ and $X = \emptyset$, respectively, in the definition of a collapsible subgraph, we conclude that G has spanning connected subgraphs H_1 and H_2 such that $O(H_1) = \{u, v\}$ and $O(H_2) = \emptyset$.

Let $C''_i = H_1 \cup e_0$, for $i = 1, 2, \ldots, k$, and $C''_j = \emptyset$, for $j = k + 1, \ldots, t$. If k is odd, then $H = \Delta_{i=1}^t [(C'_i \cup T_i) \Delta C''_i] \in \mathcal{C}_0(M)$; if k is even (including the case when $T_i = \emptyset$ for all i), then let $H = (\Delta_{i=1}^t [(C'_i \cup T_i) \Delta C''_i]) \Delta H_2$. In either case, $H \in \mathcal{C}_0(M)$. Note that

$$H \cap E_1 = \Delta_{i=1}^t (C_i \cap E_1) = \Delta_{i=1}^t C'_i = H',$$

and $H \cap E_2 = H_1$ (if k is odd) or $H \cap E_2 = H_2$ (if k is even). Since $r(H') = r(M_1)$ and $r(H_i) = r(M_2)$, we have $r(H) = r(M)$, and so H is a spanning cycle of M.

Case 2: $Z = \{e_1, e_2, e_3\}$ and $M = M_1 \oplus_3 M_2$.

Denote the 3-circuit $Z = v_1 v_2 v_3 v_1$, where $e_1 = v_1 v_2$, $e_2 = v_2 v_3$ and $e_3 = v_3 v_1$. Let G' be the graph obtained from G by adding a new edge e'_i to G parallel to e_i, for each $i \in \{1, 2, 3\}$, and $Z' = \{e'_1, e'_2, e_3\}$. Let $M'_2 = M(G')$ and $M' = M_1 \oplus_3 M'_2$. Then M'_2 is obtained from M_2 by three parallel extensions, and $E(M_1) \cap E(M'_2) = Z$. Define the bijection

$$\phi : Z \mapsto Z' \quad \text{such that } \phi(e_i) = e'_i,\ 1 \le i \le 3.$$

For each $i \in \{1, 2, \ldots, t\}$, let

$$C''_i = T_i \cup (Z' - \phi(T_i)).$$

Then each $C''_i \in \mathcal{C}(M'_2)$, and so $(C'_i \cup T_i) \Delta C''_i \in \mathcal{C}_0(M')$. Thus $H_1 = \Delta_{i=1}^t (C'_i \cup T_i) \Delta C''_i \in \mathcal{C}_0(M')$. Let $T' = (\Delta_{i=1}^t (C'_i \cup T_i) \Delta C''_i) \cap Z'$. Then $0 \le |T'| \le 3$.

We will now find a spanning connected subgraph H_0 of G' according to the different cases of T'.

If $T' = \emptyset$, then since $G' - (Z \cup Z') = G - Z$ is collapsible, $G - Z$ has a spanning connected cycle L_0. Define $H_0 = L_0$.

If $|T'| = 1$, then without loss of generality, we assume that $T' = \{e'_1\}$. Note that with our notation, e'_1 is incident with v_1 and v_2 in $V(G') = V(G)$. Since $G - Z$ is collapsible, for $X = \{v_1, v_2\}$, we can find a spanning connected subgraph L_1 of $G - Z$ with $O(L_1) = \{v_1, v_2\}$. Define $H_0 = G'[E(L_1) \cup \{e'_1\}]$.

If $|T'| = 2$, then without loss of generality, we assume that $T' = \{e'_1, e'_2\}$. Note that with our notation, e'_1 is incident with v_1 and v_2, and e'_2 is incident with v_2 and v_3 in $V(G') = V(G)$. Since $G - Z$ is collapsible, for $X = \{v_1, v_3\}$, we can find a spanning connected subgraph L_2 of $G - Z$ with $O(L_2) = \{v_1, v_3\}$. Define $H_0 = G'[E(L_2) \cup \{e'_1, e'_2\}]$.

If $T' = Z'$, then define $H_0 = L_0 \cup Z'$, where L_0 is a spanning connected cycle of $G - Z$. Then in each case, $H_0 \in \mathcal{C}_0(M'_2)$. Recall that $H_1 = \Delta_{i=1}^t (C'_i \cup T_i) \Delta C''_i \in \mathcal{C}_0(M')$. Then $H = H_0 \Delta H_1 \in \mathcal{C}_0(M')$.

Since $H_1 \cap (Z \cup Z') = T' = H_0 \cap (Z \cup Z')$, $H \subseteq E(M') - Z' = E(M)$. It follows that $H \in \mathcal{C}_0(M' - Z') = \mathcal{C}_0(M)$. Moreover, as $H' \subseteq H$ and as H_0 contains a spanning connected subgraph of $G - Z$, $r(H) = r(M_1 \oplus_3 M_2) = r(M)$. □

Proposition 5.6. *Let M be a connected binary matroid such that $M = M_1 \oplus_2 R_{10}$. Let $N = R_{10} - E(M_1) \cap E(R_{10})$. If M/N is supereulerian, then M is supereulerian.*

Proof. Let e denote the only element in $E(M_1) \cap E(R_{10})$ and C' be a spanning cycle of $M/N \cong M_1/e$. Then for some disjoint circuits $C_1, C_2, \ldots, C_t, \ldots, C_{t+1}, \ldots, C_s \in \mathcal{C}(M_1)$ where $e \notin C_i$, $i = 1, 2, \ldots, t$ and $e \in C_j$, $j = t+1, \ldots, s$, such that $C' = (\bigcup_{i=1}^{t} C_i) \cup (\bigcup_{j=t+1}^{s} (C_j - e))$.

It is well known that the automorphism group of R_{10} acts transitively on $E(R_{10})$ and R_{10} is a disjoint union of a 4-circuit L_1 and a 6-circuit L_2. We may assume that $e \in L_1$. Thus $C = (\triangle_{i=1}^{t} C_i) \triangle (\triangle_{j=t+1}^{s} (C_j \triangle L_1)) \triangle L_2$ is a spanning cycle of M. □

Proposition 5.7. *Let M be a binary matroid and $T \in \mathcal{C}(M)$ with $|T| = 3$. Then T is contractible.*

Proof. By the definition of a contractible matroid, we need to show that M/T has a spanning cycle if and only if M has a spanning cycle. By Lemma 5.2, we only need to show the only if part.

Let H' be a spanning cycle of M/T. Since M/T is also binary, $H' = C'_1 \cup C'_2 \cdots \cup C'_k$ is a disjoint union of circuits of M/T. For each $i = 1, 2, \ldots, k$, by the definition of contractions, there exists a $C_i \in \mathcal{C}(M)$, such that $C'_i = C_i - T$, $i = 1, 2, \ldots, k$. Let $H_1 = \triangle_{i=1}^{k} C_i$. Then H_1 is a cycle of M. Since T is a 3-circuit of M, both H_1 and $H_1 \triangle T$ are cycles of M. Choose $H \in \{H_1, H_1 \triangle T\}$ so that $|H \cap T| \geq 2$. It remains to show that $r(H) = r(M)$.

Since $H' \subseteq H$ and since $r(H') = r(M/T)$,there exists a $B' \subseteq H' \subseteq H$ such that $B' \in \mathcal{B}(M/T)$. Since $|H \cap T| \geq 2$ and T is a 3-circuit, there exists a $B_T \subseteq H$ such that $\mathcal{B}_T \in \mathcal{B}(M|T)$. Thus $B = B' \cup B_T \in \mathcal{B}(M)$, and $B \subseteq H$, and so $r(H) = r(M)$. □

6. Proof of Theorem 1.3

Suppose that $M \in EX(M^*(K_5), F_7, F_7^*, U_{2,4})$ is a connected matroid such that $g^*(M) \geq 4$. We argue by contradiction and assume that

$$M \text{ is a counterexample to Theorem 1.3 such that } |E(M)| \text{ is minimized.} \tag{4}$$

If M contains a nonempty subset X such that $N = M|X$ is contractible, then M/X will also satisfy the hypothesis of Theorem 1.3 with $|E(M/X)| < |E(M)|$. Therefore, by (4), M/X is supereulerian. By (3), M is also supereulerian, contrary to the assumption that M is a counterexample. Therefore, we may assume that

$$M \text{ does not have a nonempty contractible restriction.} \tag{5}$$

If M is graphic, then by Theorem 1.1, M is supereulerian, contrary to (4); if $M = R_{10}$, then as R_{10} itself is a cycle, (4) is again violated. If $M = M^*(K_{3,3})$ or $M = M^*(H_8)$, then M has at least one 3-circuit T. By Proposition 5.7, T is contractible, contrary to (5).

Therefore by Theorem 4.5, we may assume that (iii), (iv) or (v) of Theorem 4.5 holds.

Case 1: Theorem 4.5(iii) holds and so $M = M_1 \oplus_2 M_2$ such that either M_2 is graphic or $M_2 \in \{R_{10}, M^*(K_{3,3}), M^*(H_8)\}$. Let e denote the element in $E(M_1) \cap E(M_2)$. Then e is neither a loop nor a coloop of M_i, $i \in \{1, 2\}$.

If $M_2 \cong R_{10}$, then by (4), $M/(M_2 - e)$ is supereulerian. By Proposition 5.6, M would be supereulerian, contrary to (4). If $M_2 \in \{M^*(K_{3,3}), M^*(H_8)\}$, then M contains a 3-circuit, by Proposition 5.7, M has a contractible restriction, contrary to (5). Hence M_2 must be a graphic matroid.

Let $M_2 = M(G)$, where G is a connected graph. Then as $g(M^*) \geq 4$, for any edge cut D of G such that $e \notin D$, we have $|D| \geq 4$. By Lemma 3.3(i), $G - e$ contains a nontrivial collapsible subgraph L. By Proposition 5.5, M has a contractible restriction, contrary to (4).

Case 2: Theorem 4.5(iv) holds, and so $M = M_1 \oplus_3 M_2$ is a nontrivial 3-sum of M_1 and M_2 such that either M_2 is isomorphic to one of $\{M^*(K_{3,3}), M^*(H_8)\}$ or M_2 is graphic. Let $Z = E(M_1) \cap E(M_2)$. Then $Z \in \mathcal{C}(M_1) \cap \mathcal{C}(M_2)$.

If $M_2 \in \{M^*(K_{3,3}), M^*(H_8)\}$, then by Proposition 5.7, M contains a contractible restriction, contrary to (5). Hence M_2 is a graphic, and so for some connected graph G, $M_2 = M(G)$. As $g(M^*) \geq 4$, for any edge cut D of G, if $D \cap Z = \emptyset$, then $|D| \geq 4$. By Lemma 3.3(ii), either $G - Z$ contains a nontrivial collapsible subgraph, whence by Proposition 5.5, M has a contractible restriction, contrary to (5); or $G - Z \cong K_{1,2}$, whence $M_1 \oplus_3 M_2$ is a trivial 3-sum, contrary to the assumption that M is a nontrivial 3-sum.

Case 3: Theorem 4.5(v) holds, and so $M^* = M_1^* \oplus_3 M_2^*$ is a nontrivial 3-sum of M_1^* and M_2^* such that M_2^* is planar. Let $Z = E(M_1^*) \cap E(M_2^*)$. Then $Z \in \mathcal{C}(M_1^*) \cap \mathcal{C}(M_2^*)$, and Z contains no circuits in M_1 or in M_2. By (1), we have $M = M_1 \triangle M_2$.

Since M_2^* is planar, $M_2 = M(G)$ for some connected planar graph G. As $g^*(M) \geq 4$, for any edge cut D of G, if $D \cap Z = \emptyset$, then $|D| \geq 4$. By Lemma 3.5, either $G - Z$ has a nontrivial collapsible subgraph, whence by Proposition 5.5, M has a contractible restriction, contrary to (5); or $G \in \{W_1, W_2\}$, whence M is a trivial binary sum, contrary to the assumption that $M^* = M_1^* \oplus_3 M_2^*$ is a nontrivial 3-sum.

These contradictions establish the theorem. □

Acknowledgements

Second author's research is supported in part by NFS of China (No. 10331020) and NFS of Guangdong Province (No. 04010389).

References

[1] K. Appel, W. Haken, Every planar map is four colorable, Part I: Discharging, Illinois J. Math. 21 (1977) 429–490.
[2] K. Appel, W. Haken, J. Koch, Every planar map is four colorable, Part II: Reducibility, Illinois J. Math. 21 (1977) 491–567.
[3] F.T. Boesch, C. Suffel, R. Tindell, The spanning subgraphs of Eulerian graphs, J. Graph Theory 1 (1977) 79–84.
[4] B. Bollobás, Graph Theory, Springer-Verlag, New York, 1979.
[5] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, American Elsevier, New York, 1976.
[6] P.A. Catlin, A reduction method to find spanning Eulerian subgraphs, J. Graph Theory 12 (1988) 29–44.
[7] P.A. Catlin, H.-J. Lai, Y. Shao, Edge-connectivity and edge-disjoint spanning trees (submitted for publication).
[8] P.A. Catlin, H.-J. Lai, Spanning trails joining two given edges, in: Y. Alavi, G. Chartrand, O. Oellermann, A. Schwenk, Kalamazoo (Eds.), Graph Theory, Combinatorics, and Applications, vol. 1, 1991, pp. 207–222.
[9] P.A. Catlin, Z.Y. Han, H.-J. Lai, Graphs without spanning closed trails, Discrete Math. 160 (1996) 81–91.
[10] F. Jaeger, A note on subeulerian graphs, J. Graph Theory 3 (1979) 91–93.
[11] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. London Math. Soc. 36 (1961) 445–450.
[12] C.St.J.A. Nash-Williams, Decomposition of finite graphs into forests, J. London Math. Soc. 39 (1964) 12.
[13] J.G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.
[14] W.R. Pulleyblank, A note on graphs spanned by Eulerian graphs, J. Graph Theory 3 (1979) 309–310.
[15] N. Robertson, D. Sanders, P. Seymour, R. Thomas, The four-color theorem, J. Combin. Theory Ser. B 70 (1997) 2–44.
[16] P.D. Seymour, Decomposition of regular matroids, J. Combin. Theory Ser. B 28 (1980) 305–359.
[17] P.D. Seymour, Matroids and multicommodity flows, European J. Combin. Theory Ser. B. 2 (1981) 257–290.
[18] W.T. Tutte, A homotopy theorem for matroids, I, II, Trans. Amer. Math. Soc. 88 (1958) 144–174.
[19] W.T. Tutte, On the problem of decomposing a graph into n connected factors, J. London Math. Soc. 36 (1961) 80–91.
[20] O. Veblen, An application of modular equations in analysis situs, Ann. of Math. 14 (1912–1913) 86–94.
[21] K. Wagner, Über eine eigneschaft der ebenen komplexe, Math. Ann. 144 (1937) 570–590.
[22] D.J.A. Welsh, Matroid Theory, Academic Press, London, 1976.
[23] D.J.A. Welsh, Euler and bipartite matroids, J. Combin. Theory 6 (1969) 313–316.
[24] S.M. Zhan, Hamiltonian connectedness of line graphs, Ars Combin. 22 (1986) 89–95.