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An s-Hamiltonian Line Graph Problem

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Abstract. For an integer $k > 0$, a graph G is k-triangular if every edge of G lies in at least k distinct 3-cycles of G. In (J Graph Theory 11:399–407 (1987)), Broersma and Veldman proposed an open problem: for a given positive integer k, determine the value s for which the statement "Let G be a k-triangular graph. Then $L(G)$, the line graph of G, is s-hamiltonian if and only $L(G)$ is $(s+2)$-connected" is valid. Broersma and Veldman proved in 1987 that the statement above holds for $0 \le s \le k$ and asked, specifically, if the statement holds when $s = 2k$. In this paper, we prove that the statement above holds for $0 \le s \le \max\{2k, 6k-16\}$.

1. Introduction

Graphs considered in this paper are simple and finite. Undefined terms and notation can be found in [1]. The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. An edge cut X of G is essential if each side of $G - X$ contains an edge. Note that G has an essential edge cut of size k if and only if $L(G)$ has a vertex cut of k vertices. For an integer $k \ge 0$, a graph G is k-triangular if every edge of G lies in at least k triangles of G. A graph G is k-hamiltonian if for every subset $U \subseteq V(G)$ such that $|U| \le k$, $G - U$ is hamiltonian. Throughout this paper, for a graph G and an integer $i \ge 1$, $D_i(G)$ denotes the set of vertices of degree i in G.

Let G be a k-triangular graph. In [1], Broersma and Veldman asked for which values of s (as a function of k) is $L(G)$ s-hamiltonian if and only if $L(G)$ is $(s+2)$-connected. They proved the following theorem.

Theorem 1.1 (Broersma and Veldman [2]). *Let $k \ge s \ge 0$ be integers and let G be a k-triangular simple graph. Then $L(G)$ is s-hamiltonian if and only $L(G)$ is $(s+2)$-connected.*

In particular, they asked if Theorem 1.1 is still valid when $s = 2k$. In this paper, we investigate this problem and prove that s can be much larger than k, and prove the following main theorem.

Theorem 1.2. *Let k and s be positive integers such that $0 \leq s \leq \max\{2k, 6k-16\}$, and let G be a k-triangular simple graph. Then $L(G)$ is s-hamiltonian if and only $L(G)$ is $(s+2)$-connected.*

As noted in [2], when $k = \delta(G) - 2$ and $s = 2k = 2\delta(G) - 4$, Theorem 1.2 implies the following former result.

Corollary 1.3 (Lesniak-Foster [10]). *If G is a 2-connected simple graph with $\delta(G) \geq 4$, then $L(L(G))$ is $(2\delta(G) - 4)$-hamiltonian.*

The problem is still opened for larger value of s. One can even asked the question whether every $(s+2)$-connected line graph $L(G)$ is s-hamiltonian for sufficiently large values of s, without knowing if G is triangulated. This is certainly not true if $s = 1$ and $s = 0$, as there exist 3-connected line graphs that are not hamiltonian.

The technique employed in this paper is a modified version of Catlin's reduction method, different from that used in [2]. Section 2 provides certain backgrounds of the reduction method and their connection to the current problem. The proof for the main result is in Section 3.

2. Catlin's Reduction Method

For a graph G, $O(G)$ denotes the set of vertices of G with odd degree in G. A connected graph G is eulerian if $O(G) = \emptyset$. A subgraph H of a graph G is dominating if $G - V(H)$ is edgeless. A dominating eulerian subgraph is also called a DES, and a spanning eulerian subgraph is also called an SES. Clearly, every SES of a graph G is a DES of G. A graph with an SES is also called a supereulerian graph. See Catlin's survey [4] and its update [7] for an overview of supereulerian graphs.

There is a close relationship between dominating eulerian subgraphs in graphs and hamilton cycles in $L(G)$.

Theorem 2.1 (Harary and Nash-Williams [9]). *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a DES.*

To search for eulerian subgraphs with certain properties, Catlin in [3] invented the collapsible graphs. Let G be a graph and let $R \subseteq V(G)$ be a subset with $|R|$ even. A subgraph Γ of G is called an R-subgraph if $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph G is collapsible if for any even subset R of $V(G)$, G has an R-subgraph. Catlin showed in [3] that every vertex of G lies in a unique maximal collapsible subgraph of G. The reduction of G is obtained from G by contracting all maximal collapsible subgraphs. A graph G is reduced if G has no nontrivial collapsible subgraphs. A nontrivial vertex in a contraction of G is a vertex whose contraction preimage is a nontrivial connected subgraph of G. Note that if G has an $O(G)$-subgraph Γ, then $G - E(\Gamma)$ is a spanning eulerian subgraph of G. Therefore, every collapsible graph is supereulerian. We summerize some results on Catlin's reduction method and other related facts as follows.

Theorem 2.2. *Let G be a graph and let H be a collapsible subgraph of G. Let v_H denote the vertex onto which H is contracted in G/H. Each of the following holds.*

(i) (*Catlin, Theorem 3 of* [3]) *G is collapsible (supereulerian, respectively) if and only if G/H is collapsible (supereulerian, respectively). In particular, G is supereulerian if and only if the reduction of G is supereulerian; and G is collapsible if and only if the reduction of G is K_1.*
(ii) *Let G' be the reduction of G. Then G has a DES if and only if G' has a DES that contains all the nontrivial vertices of G'.*
(iii) *2-cycles and 3-cycles are collapsible.*
(iv) (*Catlin et al.* [5]) *Let G be a 2-edge-connected reduced graph with $n > 1$ vertices. Then either $|D_2(G)| = n - 2$ or $2|D_2(G)| + |D_3(G)| \geq 10$.*
(v) *A subgraph of a reduced graph is reduced.*

Theorem 2.3 (Chen [6]). *Let M be a maximum matching of a connected reduced graph G, with $|V(G)| = n$, $\delta(G) \geq 2$ and $|D_2(G)| = l$. Then*

$$|M| \geq \min\left\{\frac{n-1}{2}, \frac{n+4-l}{3}\right\}.$$

Theorem 2.4 (Chen and Lai [8]). *Let G be a 3-edge-connected reduced graph with $|V(G)| \leq 13$. Then either G is supereulerian or G is the Petersen graph.*

3. The Proof of Main Result

Since any hamiltonian graph must be 2-connected, it is necessary that any s-hamiltonian graph be $(s + 2)$-connected. Therefore, it suffices to show that if $L(G)$ is $(s + 2)$-connected, then $L(G)$ is s-hamiltonian.

Throughout the rest of this section, k denotes a positive integer, G denotes a simple k-triangular graph and s denotes an integer with $0 \leq s \leq \max\{2k, 6k - 16\}$.

We argue by contradiction and assume that $L(G)$ is $(s+2)$-connected but $L(G)$ is not s-Hamiltonian. Therefore, there exists an edge subset $U \subset E(G)$ with $|U| = s$ such that $L(G)-U$ is not hamiltonian. Let $G_0 = (G-U)-(D_0(G-U)\cup D_1(G-U))$, and let G_0' denote the reduction of G_0. Note that each edge in G_0' is also an edge in G. By Theorem 2.2 (iii), every edge in $G - U$ lying in a cycle of length at most 3 is in a collapsible subgraph of $G - U$. Since G is k-triangulated, every edge in G_0' is adjacent to an edge of U in G. Then by Theorem 2.1 and by Theorem 2.2 (i), we have

Lemma 3.1. *Each of the following holds.*

(i) *G_0' does not have an eulerian subgraph H such that H contains all nontrivial vertices as well as those vertices that are adjacent to a vertex in $D_1(G - U)$.*
(ii) *G_0' is not supereulerian and not collapsible.*

Proof. By Theorem 2.2 (ii), the subgraph H described in Lemma 3.1(i), if it exists, would correspond to a DES of G, and so by Theorem 2.1, $L(G) - U$ would be hamiltonian, contrary to the assumption that $L(G) - U$ is not hamiltonian. As collapsible graphs are supereulerian, and as an SES of G_0' satisfies the description for H, G_0' cannot be supereulerian nor collapsible. □

Lemma 3.2. *G_0' is 2-edge-connected.*

Proof. If G_0' has more than one components, then U will contain an edge cut of G separating two edges of G, contrary to the assumption that $L(G)$ is $(s+2)$-connected. If G_0' has a cut edge e, then e cannot be incident with a vertex in $D_1(G-U)$, for otherwise the degree one vertex would have been deleted in obtaining G_0. Therefore, $U \cup \{e\}$ contains an edge cut of G separating two edges of G, contrary to the assumption that $L(G)$ is $(s+2)$-connected. □

Let $e \in E(G)$. If e is incident with vertices u and v, then write $V(e) = \{u, v\}$. Let $\mathcal{C}(e)$ denote the collection of 3-cycles in G that contains e and let

$$E(e) = \cup_{C \in \mathcal{C}(e)} E(C) - \{e\}.$$

Lemma 3.3. *Let X be an edge cut of G_0' such that $G_0' - X$ has two components G_1' and G_2', let $i \in \{1, 2\}$, and let $e_i \in E(G_i')$. Each of the following holds.*

(i) *If C_1, C_2 are two 3-cycles of G, then $C_1 = C_2$ if and only if $|E(C_1) \cap E(C_2)| \geq 2$.*
(ii) *If $e \in E(G)$, then $|E(e)| \geq 2k$.*
(iii) *If $e \in E(G_0')$, then $|E(e) \cap U| \geq k$.*
(iv) *If $U \cup X$ is an edge cut of G, then $|(U \cup X) \cap C| = 2$, for any $C \in \mathcal{C}(e_i)$.*
(v) *If $U \cup X$ is an edge cut of G, then $|(U \cup X) \cap E(e_i)| \geq 2k$.*
(vi) *Fix an $i \in \{1, 2\}$. If $U \cup X$ is an edge cut of G and if $\{r_1, r_2, \ldots, r_l\} \subseteq E(G_i')$ induces a $K_{1,l}$ in G_i', then $|(U \cup X) \cap (\cup_{j=1}^{l} E(r_j))| \geq (l+1)k$.*

Proof. Statement (i) follows from the assumption that G is a simple graph. By Lemma 3.3 (i) and by the assumption that G is k-triangular, $|E(e)| \geq 2k$. Since $e \in E(G_0')$ and since G_0' is reduced, e lies in no 3-cycle of G_0', and so by (1), each member in $\mathcal{C}(e)$ must intersect U. Thus $|E(e) \cap U| \geq |\mathcal{C}(e)| \geq k$. This proves (ii) and (iii).

Since $e_1, e_2 \in E(G_0')$, for any $C \in \mathcal{C}(e_i)$ $(i = 1, 2)$, $C \cap (U \cup X) \neq \emptyset$. Since $U \cup X$ is an edge cut, Lemma 3.3(iv) follows from the fact that $|C \cap (U \cup X)|$ must be an even number. Lemma 3.3 (v) and (vi) follow from Lemma 3.3 (iv) and (iii). □

Lemma 3.4. *G_0' does not contain a 2 edge cut X of G_0' such that each side of $G_0' - X$ contains an edge or a nontrivial vertex or a vertex that is adjacent to a vertex in $D_1(G - U)$. In particular, $D_2(G_0')$ is an independent set.*

Proof. By contradiction, suppose that G_0' has an edge cut X with $|X| = 2$ and let G_1' and G_2' denote the two components of $G_0' - X$, and let G_1 and G_2 be the two components of $G_0 - X$ such that the reduction of G_i is G_i', $1 \leq i \leq 2$. Note that each G_i' contains an edge or a nontrivial vertex or a vertex that is adjacent to a vertex in $D_1(G - U)$. Since $L(G)$ is $(s+2)$-connected, and since $|U| = s$, it must be the case that $U \cup X$ is an edge cut of G such that $G - U = G_0$ and such that $G - (U \cup X) = G_1 \cup G_2$. We have the following observations. □

Claim 1. For each $i \in \{1, 2\}$, G_i' does not have an edge joining two vertices in $D_2(G_i') \cup D_3(G_i')$.

Suppose not. Then there exist $v_1, v_2 \in D_2(G_i') \cup D_3(G_i')$ such that $e_0 = v_1v_2 \in E(G)$. Let X_1 be the set of edges in $G_i' - e_0$ incident with v_1 or v_2. Then X_1 is an essential edge cut of G_i'. Since $L(G)$ is $(s+2)$-connected, and since $|X_1| \le 4$, there are at least $s - 2$ edges in $U \cup X$ joining a vertex in the preimages of v_1 or v_2 to a vertex in G_{3-i}, and so there are at most 2 edges in $U \cup X$ not incident with a vertex in the preimages of v_1 and v_2. Since G_i' contains no cycles of length less than 4 [Theorem 2.2 (iii)], we can choose v_1 and v_2 so that G_i' has an edge e_1 not incident with v_1 nor v_2. By Lemma 3.3(v), $|(U \cup X) \cap E(e_1)| \ge 2k$. Since e_1 is not adjacent with v_1 nor v_2, and since there are at most 2 edges in $U \cup X$ not incident with a vertex in the preimages of v_1 and v_2, we have $2 \ge 2k$, and so it must be $k = 1$. Hence $s \le 2$ and $|U \cup X| \le |U| + |X| = s + 2 \le 4$. By Lemma 3.3 (iv), each edge in $E(G_i')$ must be adjacent to two edges in $U \cup X$, and by Theorem 2.2 (iii), G_i' cannot have a 2 or 3-cycle. It follows by Lemma 3.2 that G_i' must be a path with at most 4 vertices such that the two edges in X are incident with the two ends of the path in G_i', respectively. Therefore, G_0' is a cycle, contrary to Lemma 3.1 (ii). This proves Claim 1.

Claim 2. For each $i \in \{1, 2\}$, $\Delta(G_i') \le 3$.

Suppose that for some $i \in \{1, 2\}$, $l = \Delta(G_i') \ge 4$. Then G_i' has a vertex v which is adjacent to some vertices in $\{v_1, v_2, v_3, \ldots, v_l\}$ in G_i'. By $l \ge 4 > |X| = 2$, by the fact that G_i' does not have a cycle of length less than 4 [Theorem 2.2 (iii)] and since G_0' is 2-edge-connected (Lemma 3.2), G_i' must have a vertex $u \in V(G_i') - \{v, v_1, \ldots, v_l\}$ such that u is adjacent to two vertices $u_1, u_2 \in V(G_i') - \{v, v_1, v_2\}$, renaming the vertices if needed. Therefore, each of $\{v, v_1, v_2\}$ and $\{u, u_1, u_2\}$ induces a $K_{1,2}$ in G_i' and these two $K_{1,2}$'s are disjoint. Since $X \subseteq E(G_0')$, for any $e \in \{vv_1, vv_2, uu_1, uu_2\}$, by Theorem 2.2 (iii), $C(e) \cap X = \emptyset$. By Lemma 3.3(vi), we have $6k \le |(U \cup X) \cap \cup_{j=1}^2 E(uu_j)| + |(U \cup X) \cap \cup_{j=1}^2 E(vv_j)| = |U| = s \le \max\{2k, 6k - 16\}$, a contradiction. This proves Claim 2.

Note that by Claim 2, G_i' must have an edge joining two vertices in $D_2(G_i') \cup D_3(G_i')$, contrary to Claim 1. This proves the lemma.

Let $\tilde{G}$ be a graph obtained from G_0' by contracting exactly one edge incident with each vertex in $D_2(G_0')$. By Lemma 3.4, $\tilde{G}$ is 3-edge-connected. The following is straightforward.

Lemma 3.5. *If $\tilde{G}$ has an eulerian subgraph L' such that $V(L')$ contains all nontrivial vertices as well as all vertices that are adjacent to a vertex in $D_1(G - U)$, then each of the following holds.*

(i) *$G - U$ has a DES, and*

(ii) *$L(G) - U$ is hamiltonian.*

Proof. Note that L' can be lifted to an eulerian subgraph L in G'_0, by adding edges (whenever necessary) that had been contracted in the process of getting $\tilde{G}$ from G'_0. By the definition of $\tilde{G}$ and by Lemma 3.4, L is a DES of G'_0 that contains all nontrivial vertices as well as all vertices that are adjacent to a vertex in $D_1(G - U)$. It follows by Theorem 2.2(ii) that $G - U$ has a DES. By Theorem 2.1, $L(G) - U$ is hamiltonian. □

Lemma 3.6. *$\tilde{G}$ cannot be contracted to the Petersen graph.*

Proof. By contradiction, assume that $\tilde{G}$ can be contracted to P_{10}, the Petersen graph. Note that for any $z \in V(P_{10})$, P_{10} has a cycle containing all vertices in $V(P_{10}) - z$. If there is one vertex $z \in V(P_{10})$ which is a trivial vertex and is not adjacent to a vertex in $D_1(G - U)$, then any cycle of this P_{10} containing $V(P_{10} - z)$ corresponds to a DES of $G - U$, contrary to Lemma 3.1. Therefore, every vertex of P_{10} is either a nontrivial vertex or adjacent to a vertex in $D_1(G - U)$. Let $v_0 \in V(P_0)$ and let X denote the set of the 3 edges incident with v_0 in P_{10}. Then X is an essential edge cut of $G - U$. It follows by the assumption that $L(G)$ is $(s+2)$-connected that all but at most one edge in U are linking a vertex in one side of $G - (X \cup U)$ to a vertex in the other side. Let $U' \subseteq U$ be a subset such that $|U - U'| \leq 1$ and such that $U' \cup X$ is an edge cut of G.

Note that $P_{10} - v_0$ has 6 edges $e_1, e_2, \ldots, e_6 \in E(P_{10} - v_0)$ such that $\{e_i, e_{i+3}\}$ induces a subgraph H_i isomorphic to a $K_{1,2}$ in $P_{10} - v_0$, where $1 \leq i \leq 3$, and such that the H_i's are mutually vertex disjoint. By Lemma 3.3 (iv), by $|U - U'| \leq 1$ and by the fact that P_{10} has no 4-cycle, the only edge in $U - U'$ may be adjacent to at most two members in $\{e_1, \cdots, e_6\}$. Therefore by Lemma 3.3 (vi) (with $l = 2$), we have $|(\cup_{i=1}^{6} E(e_i)) \cap (U' \cup X)| \geq 9k - 2$. By $|U - U'| \leq 1$ again and by $|X| = 3$, $|(\cup_{i=1}^{6} E(e)) - U'| \leq 4$. It follows that

$$\begin{aligned} 9k - 2 \leq |\cup_{i=1}^{6} E(e_i)| \leq |U'| + 4 = s + 4 &\leq \max\{2k, 6k - 16\} + 4 \\ &= \max\{2k + 4, 6k - 12\}, \end{aligned}$$

contrary to the assumption that $k \geq 1$. □

We shall derive at a contradiction by showing that $\tilde{G}$ is supereulerian. Let $\tilde{G}'$ denote the reduction of $\tilde{G}$. By Theorem 2.2(1), we may assume, by contradiction, that $\tilde{G}'$ is not supereulerian. Note that $\tilde{G}'$ is a 3-edge-connected reduced graph.

Claim 1. $k \geq 5$ and so $s \leq 6k - 16$.

If not, then $k \leq 4$, and so $s \leq 2k$. By Theorem 2.4 and by Lemma 3.6, we may assume that $|V(\tilde{G}')| \geq 14$. It follows by Theorem 2.3 that $\tilde{G}$ has a matching M with $|M| = 6$. Note that M is also a matching of G'_0. By Lemma 3.3 (iii), for each $e \in M$, $|E(e) \cap U| \geq k$. Since $M \subseteq E(\tilde{G}') \subseteq E(G'_0)$, every edges in M lies in no 3-cycles in $\tilde{G}'$ [by Theorem 2.2 (iii)], and so every edge in M must be adjacent to k

edges in U. Denote $M = \{e_1, e_2, \dots, e_6\}$. Construct a new graph H with 7 vertices $u_0, u_1, \dots, u_6$, such that each u_i, $1 \le i \le 6$ represents the edge $e_i \in M$. There are t_{ij} edges joining u_i and u_j if and only if there are t_{ij} edges in U which are adjacent to both e_i and e_j; and each u_i is linked to u_0 with t_i edges if and only if U has t_i edges that are adjacent only to e_i and not to any other edges in M. Since M is a matching, any edge in U cannot be adjacent to more than 2 edges in M, and so we may assume that $U = E(H)$. By Lemma 3.3 (iii), every vertex in $V(H) - \{u_0\}$ has degree at least k, and so it follows that $4k \ge 2s = 2|U| = 2|E(H)| \ge 6k$, a contradiction. This contradiction proves Claim 1.

Claim 2. G does not have an independent set $\{v_1, v_2, \dots, v_6\}$ such that $v_1, v_2, \dots, v_6$ are in the preimages of 6 distinct vertices in $D_3(\tilde{G}')$.

By contradiction, we assume that such vertices $v_1, \dots, v_6$ exist. For notational convenience, we also use v_i $(1 \le i \le 6)$ to denote the vertex in $D_3(\tilde{G})$ whose preimage contains v_i. Assume that each v_i is incident with an edge $e_i \in E(\tilde{G}') \subseteq E(G_0')$. By Lemma 3.3 (iii), $|E(e_i) \cap U| \ge k$. Let $E_i \subset E(e_i) \cap U$ denote the edges in $G - e_i$ incident with the vertex v_i together with possibly two more edges in $E(e_i) \cap U$ that will form a member in $\mathcal{C}(e_i)$ with the two edges in $\tilde{G}'$ that are incident with v_i (recall that $v_i \in D_3(\tilde{G}')$). Thus $|E_i| \ge k$. Since all the v_i are mutually nonadjacent in $\tilde{G}'$, and since G is simple, $|E_i \cap E_j| \le 1$ whenever $i \ne j$, and so there are at most 15 edges which are lying in at most two members of the E_i's. It follows that $6k - 15 \le \left|\bigcup_{i=1}^{6} E_i\right| \le |U| = s$, contrary to Claim 1. This proves Claim 2.

By Theorem 2.2 (iv) and by Claim 2, we may assume that there exist $v_1, v_2, v_3, v_4 \in V(G)$ which are in the preimages of 4 vertices in $D_3(\tilde{G}')$, respectively, such that $e_1 = v_1v_2, e_2 = v_3v_4 \in E(\tilde{G}')$. Let $e_i', e_i'' \in E(\tilde{G}')$ be the edges incident with the vertex in $D_3(\tilde{G}')$ whose preimage contains v_i, $(1 \le i \le 4)$.

Then $X_1 = \{e_1', e_1'', e_2', e_2''\}$ is an essential edge cut of $\tilde{G}'$, and so $X_1 \cup U$ must contain an essential edge cut of G. Therefore, there exists a set $U_1 \subset U$ such that $X_1 \cup U_1$ is an essential edge cut of G. Similarly, $X_2 = \{e_3', e_3'', e_4', e_4''\}$ is an essential edge cut of $\tilde{G}'$, and so there exists a set $U_2 \subset U$ such that $X_2 \cup U_2$ is an essential edge cut of G. Since $L(G)$ is $(s+2)$-connected, $|U_i| + 4 = |U_i \cup X_i| \ge s + 2$, and so $|U_i| \ge s - 2$. It follows by the Principle of Inclusion and Exclusion that $|U_1 \cap U_2| \ge |U_1| + |U_2| - |U| \ge s - 4$. Note that every edge in $U_1 \cap U_2$ must have its ends in the preimages of the vertices containing v_1, v_2, v_3, v_4.

Since $\tilde{G}'$ is 3-edge-connected with at least 10 vertices [Theorem 2.2 (iv)], there exists an edge $e \in E(\tilde{G}')$ that is not adjacent to e_1 nor e_2. By Lemma 3.3(iii), $k \le |E(e) \cap U| = |E(e) \cap (U - U_1 \cap U_2)| \le |U - U_1 \cap U_2| \le 4$, contrary to Claim 1. Therefore $\tilde{G}'$ must be supereulerian, and so by Theorem 2.2 and by Theorem 2.4, $\tilde{G}$ must be supereulerian. Thus by Lemma 3.5, $L(G) - U$ must be hamiltonian. This proves Theorem 1.2.

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