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Note

An inequality for the group chromatic number of a graph

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Abstract

We give an inequality for the group chromatic number of a graph as an extension of Brooks' Theorem. Moreover, we obtain a structural theorem for graphs satisfying the equality and discuss applications of the theorem.

Keywords: Brooks' Theorem; Group coloring; Group chromatic number

1. Introduction

Let $\delta(G)$, $\Delta(G)$ and $\chi(G)$ denote the minimum degree, the maximum degree and the chromatic number of G, respectively. A graph G is *unicyclic* if G contains only one cycle.

Brooks' Theorem [1] is well known as follows.

Theorem 1.1. *If G is a connected graph, then*

$$\chi(G) \leqslant \Delta(G) + 1$$

with equality if and only if G is complete or an odd cycle.

Let $\rho(G)$ be the largest eigenvalue of the vertex-adjacency matrix of G, i.e., the matrix whose i, j entry is 1 if vertices i and j are connected and 0 otherwise. Wilf [6] improved Brooks' Theorem by replacing $\Delta(G)$ by $\rho(G)$ and proved that $\chi(G) \leqslant \rho(G) + 1$ with equality if and only if G is an odd cycle or a complete graph. Let $\gamma(G)$ be a real function on a graph G satisfying the following two properties:

(P1) If H is an induced subgraph of G, then $\gamma(H) \leqslant \gamma(G)$.
(P2) $\gamma(G) \geqslant \delta(G)$ with equality if and only if G is regular.

Szekeres and Wilf [5] extended Theorem 1.1 by replacing $\Delta(G)$ by $\gamma(G)$, as follows.

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Theorem 1.2. *If $\gamma(G)$ is a real function on a graph G with properties* (P1) *and* (P2), *then*

$$\chi(G) \leqslant \gamma(G) + 1.$$

Cao [2] observed Szekeres and Wilf's result and proved that the equality holds if and only if G is an odd cycle or a complete graph. Unfortunately, there is a minor error in his result in the sense that there exists a counterexample as shown below.

For a connected graph G, we define

$$\gamma(G) = \begin{cases} 0 & \text{if } G = K_1, \\ 1 & \text{if } G = K_2, \\ 2 & \text{if } G \text{ is unicyclic or a tree and } G \neq K_2, \\ \Delta(G) & \text{otherwise.} \end{cases}$$

For a disconnected graph G, we define $\gamma(G) = \max\{\gamma(G_i) : G_i \text{ is a connected component of } G\}$. It follows that $\gamma(G)$ satisfies (P1) and (P2). Therefore, when G is a connected unicyclic graph with an odd girth and G is not isomorphic to a cycle, we have $\chi(G) = \gamma(G) + 1$, but G is neither a cycle nor a complete graph.

One asks when the equality in Theorem 1.2 holds. Moreover, when the equality holds for a graph G, what kind of graph G is. In this note, we will discuss these questions in group coloring. One easily obtains similar results in coloring.

We end this section with some important terminology. Graphs in this note are finite and simple. Unless otherwise stated, notations and terminology in graph theory are standard. Denote by $H \subseteq G$ the fact that H is an induced subgraph of G and by $N(v)$ the set of neighbors of v in G.

The concept of group coloring was first defined in [3]. Let G be a graph and A an Abelian group. Denote by $F(G, A)$ the set of all functions from $E(G)$ to A and by D an orientation of $E(G)$. For $f \in F(G, A)$, an (A, f)-coloring of G under the orientation D is a function $c : V(G) \mapsto A$ such that for every directed edge uv from u to v, $c(u) - c(v) \neq f(uv)$. G is *A-colorable* under the orientation D if for any function $f \in F(G, A)$, G has an (A, f)-coloring. Jaeger et al. [3] proved that A-colorability is independent of the choice of the orientation.

The *group chromatic number* of a graph G, denoted by $\chi_g(G)$, is defined to be the minimum m for which G is A-colorable for any Abelian group A of order at least m. It follows that for any graph G, $\chi(G) \leqslant \chi_g(G)$.

2. Main results

The group coloring version of Brooks' Theorem was proved in [4], as follows.

Theorem 2.1. *For any connected graph G*,

$$\chi_g(G) \leqslant \Delta(G) + 1,$$

where equality holds if and only if $\Delta(G) = 2$ and G is a cycle; *or $\Delta(G) \geqslant 3$ and G is complete*.

Theorem 2.1 tells us that when $\Delta(G) = 2$, the equality holds if and only if G is a cycle, which is different from Theorem 1.1.

Motivated by Theorem 2.1, the authors first consider to generalize this extension of Brooks' Theorem. For a graph G, we now introduce the concept of $\chi_g(G)$(or $\chi(G)$)-semi-critical graphs as follows. A graph G is defined to be $\chi_g(G)$(or $\chi(G)$)-*semi-critical* if $\chi_g(G - v) < \chi_g(G)$(or $\chi(G - v) < \chi(G)$), for every vertex $v \in G$ with $d(v) = \delta(G)$. It is easy to see that a cycle is semi-critical and so is a complete graph.

For a graph G, there exists a subgraph H such that H is a $\chi_g(H)$-semi-critical subgraph. Moreover, $\chi_g(G) = \chi_g(H)$. $\chi_g(G)$-semi-critical graph has the following properties which play a key role in the proof of Theorem 2.4.

Lemma 2.2. *Let G be a graph*, $v \in V(G)$, *and* $H = G - v$. *If* $d_G(v) < \chi_g(H)$, *then* $\chi_g(G) = \chi_g(H)$.

Proof. We assume, without loss of generality, that G is oriented such that all the edges incident with v are oriented from v. Let A be an Abelian group with $|A| \geqslant \chi_g(H)$ and $f \in F(G, A)$. It follows that for $f|_H : E(H) \mapsto A$, there is a function $c : V(H) \mapsto A$ such that $c(x) - c(y) \neq f(xy)$ for each directed edge from x to y. Let $N(v) = \{v_1, v_2, \dots, v_l\}$, where $l = d(v)$. Since $l < \chi_g(H) \leqslant |A|$, there is an $a \in A - \bigcup_{i=1}^{l} \{c(v_i) + f(vv_i)\}$. Define $c_1 : V(G) \mapsto A$ by

$$c_1(u) = \begin{cases} c(u) & \text{if } u \in V(H), \\ a & \text{if } u = v, \end{cases}$$

which implies that G is A-colorable and hence $\chi_g(G) \leqslant \chi_g(H)$. We conclude our lemma by inequality $\chi_g(H) \leqslant \chi_g(G)$ since $H \subseteq G$. $\square$

Lemma 2.3. *If G is $\chi_g(G)$-semi-critical with $\chi_g(G) = k$, then $d_G(v) \geqslant k - 1$ for all $v \in V(G)$.*

Proof. Suppose, to the contrary, that G has a vertex v_0 with $d_G(v_0) = \delta(G) \leqslant k - 2$. Since G is a χ_g-semi-critical graph with $\chi_g(G) = k$, we have $\chi_g(G - v_0) \leqslant k - 1$. If $\chi_g(G - v_0) = k - 1$, by Lemma 2.2, $\chi_g(G) = \chi_g(G - v_0) = k - 1$, contrary to that $\chi_g(G) = k$. We thus assume that $\chi_g(G - v_0) \leqslant k - 2$. By similar argument as in the proof of Lemma 2.2, we obtain that $\chi_g(G) \leqslant k - 1$. This contradiction proves our lemma. $\square$

Theorem 2.4. *If G is a connected graph and $\gamma(G)$ is a real function satisfying* (P1) *and* (P2), *then $\chi_g(G) \leqslant \gamma(G) + 1$. Moreover, if G is $\chi_g(G)$-semi-critical, then $\chi_g(G) = \gamma(G) + 1$ if and only if G is a cycle or a complete graph.*

Proof. Let $k = \chi_g(G)$ and let $H \subseteq G$ be a k-semi-critical induced subgraph. By (P1), $\gamma(H) \leqslant \gamma(G)$. By Lemma 2.3, $\delta(H) \geqslant k - 1$. By (P2), $k - 1 \leqslant \delta(H) \leqslant \gamma(H) \leqslant \gamma(G)$, which implies that $\chi_g(G) = k \leqslant \gamma(G) + 1$.

If G is a cycle C or a complete graph K_n, by (P2), $\gamma(C) = 2$ and $\gamma(K_n) = n - 1$. Our sufficiency follows from that $\chi_g(C) = 3$ and $\chi_g(K_n) = n$ (see [4]).

Conversely, let G be a $\chi_g(G)$-semi-critical graph such that $\chi_g(G) = \gamma(G) + 1$. By Lemma 2.3 and (P2), $\chi_g(G) - 1 \leqslant \delta(G) \leqslant \gamma(G) = \chi_g(G) - 1$, which implies $\delta(G) = \gamma(G) = \chi_g(G) - 1$. By (P2), G is regular. Therefore, $\chi_g(G) = \gamma(G) + 1 = \delta(G) + 1 = \Delta(G) + 1$. By Theorem 2.1, G is a cycle or a complete graph. $\square$

Theorem 2.4 answers the first question posed in Section 1. For vertex coloring, we easily obtain the similar results as Lemmas 2.2 and 2.3. Corollary 2.5 extends Szekeres and Wilf's result, that is Theorem 1.2. When we prove Corollary 2.5, we only need a slight change by using Theorem 1.1 instead of using Theorem 2.1.

Corollary 2.5. *If G is a connected graph and $\gamma(G)$ is a real function satisfying* (P1) *and* (P2), *then $\chi(G) \leqslant \gamma(G) + 1$. Moreover, if G is semi-critical, then $\chi(G) = \gamma(G) + 1$ if and only if G is an odd cycle or a complete graph.*

We are ready to discuss the structure of connected graphs with $\chi_g(G) = \gamma(G) + 1$, where $\gamma(G)$ is a real function satisfying (P1) and (P2). At first we define a family of graphs as follows.

For a positive integer m, define $\mathscr{F}(m)$ to be the set of connected simple graphs such that:

(1) For $m = 1$, $G \in \mathscr{F}(m)$ if and only if $G = K_1$.
(2) For $m = 2$, $G \in \mathscr{F}(m)$ if and only if $G = K_2$.
(3) For $m = 3$, $G \in \mathscr{F}(m)$ if and only if either G is a cycle, or $H = G - v$ for some $H \in \mathscr{F}(m)$ and a vertex v with $d(v) = 1$.
(4) For $m \geqslant 4$, $G \in \mathscr{F}(m)$ if and only if either $G = K_m$ or $H = G - v$ for some $H \in \mathscr{F}(m)$ and a vertex v with $d(v) \leqslant m - 2$.

The goal of the definition is to prove Theorem 2.6 with a real function $\gamma(G)$. Thus, we define $\mathscr{F}(1) = \{K_1\}$ and $\mathscr{F}(2) = \{K_2\}$. Note that if $m = 3$, $\mathscr{F}(3)$ is the set of connected unicyclic graphs and if $m \geqslant 4$, then $\mathscr{F}(m)$ is the set of connected graphs such that by the removal of all vertices of degree no more than $m - 2$, the resulting graph is K_m.

The following theorem answers the second question posed in Section 1.

Theorem 2.6. *Let G be a connected graph. If $\chi_g(G) = \gamma(G) + 1$, then $G \in \mathscr{F}(m)$ where $m = \chi_g(G)$.*

Proof. Let $m = \chi_g(G)$ and let H_0 be a $\chi_g(G)$-semi-critical induced subgraph of G. It is trivial when $\chi_g(G) = 1, 2$. We thus assume that $\chi_g(G) \geqslant 3$. It follows that $\chi_g(H_0) = \chi_g(G) = \gamma(G) + 1 \geqslant \gamma(H_0) + 1 \geqslant \chi_g(H_0)$. It implies that

$\chi_g(H_0)=\gamma(H_0)+1$ and $\gamma(H_0)=\gamma(G)$. By Theorem 2.4, H_0 is a cycle or a complete graph and hence $\delta(H_0)=m-1$.

If $\delta(G)\geqslant m-1$, then $m=\chi_g(G)=\gamma(G)+1\geqslant\delta(G)+1\geqslant m$ and hence $\gamma(G)=\delta(G)=m-1$. It follows that G is regular. Since G is connected, $G=H_0$. When $m=3$, H_0 is a cycle and hence $G\in\mathscr{F}(3)$. When $m\geqslant 4$, H_0 is a complete graph and hence $G\in\mathscr{F}(m)$.

Assume now that $\delta(G)\leqslant m-2$. It follows that when $m=3$, G cannot be a cycle and that when $m\geqslant 4$, G is not a complete graph. By Theorem 2.4, G cannot be $\chi_g(G)$ semi-critical. Therefore, there exists $v\in V(G)$ with $d(v)=\delta(G)$ such that $\chi_g(G)=\chi_g(G-v)$.

We prove that $G\in\mathscr{F}(m)$ by induction on $|V(G)|$. If $|V(G)|=|V(H_0)|+1$, then $G-v=H_0\in\mathscr{F}(m)$. By the definition of $\mathscr{F}(m)$, $G\in\mathscr{F}(m)$. Assume now that $|V(G)|>|V(H_0)|+1$. By Lemma 2.2 and by Property (P1),

$$\chi_g(G-v)=\chi_g(G)=\gamma(G)+1\geqslant\gamma(G-v)+1\geqslant\chi_g(G-v).$$

It implies that $\chi_g(G-v)=\gamma(G-v)+1$. We apply the induction hypothesis to $G-v$, $G-v\in\mathscr{F}(m)$. By the definition of $\mathscr{F}(m)$, $G\in\mathscr{F}(m)$. $\square$

Note that $\chi(G)\leqslant\chi_g(G)\leqslant\gamma(G)+1$. $\chi(G)=\gamma(G)+1$ implies $\chi_g(G)=\gamma(G)+1$. When $\chi(G)=\gamma(G)+1$, $G\in\mathscr{F}(m)$. If G is a unicyclic graph, the cycle cannot be even. Thus, the following corollary is straightforward.

Corollary 2.7. *Let G be a connected graph and $\gamma(G)$ a real function satisfying* (P1) *and* (P2). *If $\chi(G)=\gamma(G)+1$, then $G\in\mathscr{F}(m)$ for some $m\geqslant 3$ or G is a unicyclic graph with odd girth or $G=K_2$ or $G=K_1$.*

As applications of Theorem 2.6, we prove Corollaries 2.8 and 2.9, respectively. Corollary 2.8 is the group coloring version of a Cao's result [2]. The *k-degree* of vertex v in a graph G is defined to be the number of walks of length k from v in G. Denote by $T(G)$ the maximum 2-degree of G and by $\Delta_k(G)$ the maximum k-degree in a graph G which is the maximum row sum of A^k, where A is the vertex-adjacency matrix of G.

Corollary 2.8. *If G is a connected graph, then each of the following holds*:

(i) $\chi_g(G)\leqslant\sqrt{T(G)}+1$, *where equality holds if and only if G is complete or a cycle.*
(ii) $\chi_g(G)\leqslant[\Delta_k(G)]^{1/k}+1$, *where equality holds if and only if G is complete or a cycle.*

Proof. (i) It is easy to check that $\sqrt{T(G)}$ satisfies (P1) and (P2). By Theorem 2.4, $\chi_g(G)\leqslant\sqrt{T(G)}+1$. When $\chi_g(G)=\sqrt{T(G)}+1$, by Theorem 2.6, $G\in\mathscr{F}(m)$. The result follows immediately if $m=1,2$. If $m=3$, G is a unicyclic graph and hence $\chi_g(G)=3$. Since $\sqrt{T(G)}\geqslant\sqrt{2+2+x}$ with $\sqrt{T(G)}=2$ if and only if $x=0$, that is, G must be a cycle. Similarly, when $m\geqslant 4$, $G\in\mathscr{F}(m)$ and $\sqrt{T(G)}=m-1$ and hence G is a complete graph.

Conversely, if G is a complete graph or a cycle, then G is semi-critical. By Theorem 2.4, $\chi_g(G)=\sqrt{T(G)}+1$.

(ii) The proof is similar. $\square$

For a graph G, the *average degree* and the *maximum average degree* of G are defined to be $l(G)=2|E(G)|/|V(G)|$ and $L(G)=\max\{l(H): H$ is an induced subgraph of $G\}$, respectively. Clearly, $L(G)$ satisfies (P1) and (P2). Therefore, the following corollary is straightforward.

Corollary 2.9. *If G is a connected graph with maximum average degree $L(G)$, then $\chi_g(G)\leqslant L(G)+1$, where equality holds if and only if $G\in\mathscr{F}(m)$, where $m=\chi_g(G)$.*

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