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Some problems related to hamiltonian line graphs

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ABSTRACT. Part of this paper summarizes some of the recent developments in the study of hamiltonian line graphs and the related hamiltonian claw-free graphs. The last section of this paper solves some problems on the hamiltonian like indices from a paper by Clark and Wormald in 1983.

1. Definitions and Terminology

Graphs considered here are finite and loopless. Unless otherwise noted, we follow [**2**] for notations and terms. As in [**2**], $\kappa(G)$, $\kappa'(G)$ and $\delta(G)$ represent the *connectivity*, *edge-connectivity*, and the *minimum degree* of a graph G, respectively.

DEFINITION 1.1. A graph G is *nontrivial* if $E(G) \neq \emptyset$.

DEFINITION 1.2. A vertex cut X of G is *essential* if $G - X$ has at least two nontrivial components.

DEFINITION 1.3. For an integer $k > 0$, a graph G is *essentially k-connected* if G does not have an essential cut X with $|X| < k$.

DEFINITION 1.4. An edge cut Y of G is *essential* if $G - Y$ has at least two nontrivial components.

DEFINITION 1.5. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge cut Y with $|Y| < k$.

DEFINITION 1.6. Let G be a graph and let $X \subseteq E(G)$ be an edge subset. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops.

For convenience, we use G/e for $G/\{e\}$ and $G/\emptyset = G$; and if H is a subgraph of G, we write G/H for $G/E(H)$.

DEFINITION 1.7. For a graph G, $O(G)$ denotes the set of all vertices of odd degree in G.

DEFINITION 1.8. A graph G is *even* if $O(G) = \emptyset$, is *eulerian* if G is both even and connected, and is *supereulerian* if G contains a spanning eulerian subgraph.

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DEFINITION 1.9. A subgraph H of G is *dominating* if $E(G - V(H)) = \emptyset$.

DEFINITION 1.10. A graph G is *hamiltonian* if G has a spanning cycle. A spanning cycle of G is a *Hamilton cycle* of G.

DEFINITION 1.11. A graph G is *hamiltonian connected* if $\forall u, v \in V(G)$, G has a spanning (u, v)-path.

DEFINITION 1.12. The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

The following theorem relates hamiltonian line graph $L(G)$ and dominating eulerian subgraph in G.

THEOREM 1.13. *(Harary and Nash-Williams* [10]*) Let G be a connected graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

DEFINITION 1.14. If $P = v_0 v_1 \cdots v_k$ denotes a trail (or path, respectively), of G, and if $E(P) = \{e_1, e_2, \cdots, e_k\}$ is an edge set such that for $i = 1, 2, \cdots, k$, $e_i = v_{i-1}v_i$, then P is called a (v_0, v_k)-*trail* (or *path*, respectively) of G, and an (e_1, e_k)-trail (or path, respectively) of G. The vertices $v_1, v_2, \cdots, v_{k-1}$ are called the *internal vertices* of P. The edges e_1, e_k are called the *end edges* of P.

DEFINITION 1.15. A trail P of G is *dominating* if every edge of G is incident with an internal vertex of P.

With similar arguments (see page 74 in [10]), the following can be proved.

THEOREM 1.16. *Let G be a connected graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian connected if and only if for any pair of edges $e_1, e_2 \in E(G)$, G has a dominating (e_1, e_2)-trail.*

DEFINITION 1.17. For a graph G, an induced subgraph H isomorphic to $K_{1,3}$ is called a *claw* of G, and the only vertex of degree 3 of H is the *center* of the claw.

DEFINITION 1.18. A graph G is *claw free* if it does not contain a claw.

Beineke [1] and Robertson [11] independently proved that a graph G is a line graph if and only if G does not contain a list of 9 graphs as induced subgraphs. Šoltés [31], Lai and Šoltés [15] indicated for highly connected graphs, this forbidden list can be reduced to fewer graphs. But in any case, a line graph cannot have $K_{1,3}$ as an induced subgraph.

The purpose of this article is to summarize some of the recent development on the study of hamiltonian line graphs, hamiltonian claw-free graphs, and to solve some of the problems from an earlier study by Clark and Wormald [7].

2. Sufficient Condition with Local Connectivity

DEFINITION 2.1. A vertex v is *locally connected* if $N(v)$ is connected; and G is *locally connected* if every vertex of G is locally connected.

THEOREM 2.2. *(D. J. Oberly and D. P. Sumner* [26]*) Every connected, locally connected claw-free graph is hamiltonian.*

DEFINITION 2.3. A graph is *vertex pancyclic* if given any vertex $v \in V(G)$, G has cycles C_i of length i containing v, for each $3 \leq i \leq |V(G)|$.

THEOREM 2.4. *(L. Clark* [**6**]*, R. H. Shi* [**30**]*, and C.-Q. Zhang* [**36**]*) Every connected, locally connected claw-free graph is vertex pancyclic.*

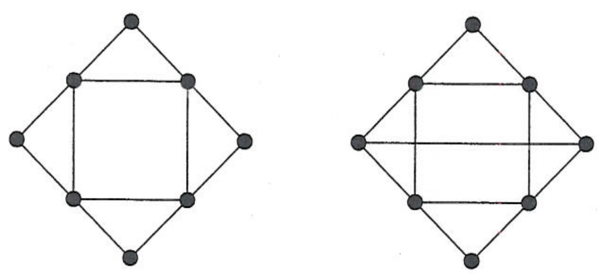
DEFINITION 2.5. For a vertex $v \in V(G)$, define

$N_2(v, G)=\{e \in E(G) : e$ has at least one end in $N(v, G)$, but not incident with $v\}$.

DEFINITION 2.6. A vertex v is *locally N_2-connected* in G if $N_2(v, G)$ induces a connected subgraph in G.

DEFINITION 2.7. A graph G is *locally N_2-connected* if every vertex of G is locally N_2-connected.

THEOREM 2.8. *(Ryjáček* [**27**]*) Let G be a connected, N_2-locally connected claw-free graph without vertices of degree 1, which does not contain an induced subgraph H isomorphic to either G_1 or G_2 (Figure below) such that $N_1(x, G)$ of every vertex x of degree 4 in H is disconnected. Then G is hamiltonian.*



G_1 G_2

Figure 1

The following was conjectured by Ryjáček [**28**].

THEOREM 2.9. *(H.-J. Lai, Y. Shao and M. Zhan* [**16**]*) Every 3-connected, locally N_2-connected claw-free graph is hamiltonian.*

DEFINITION 2.10. For an integer $k \geq 1$, and a vertex $v \in V(G)$,

$$N^k(v, G) = \{x \in V(G) : dist_G(v, x) \in \{1, 2, \cdots, k\}\}.$$

DEFINITION 2.11. A vertex v is *locally N^k-connected* in G if $N^k(v, G)$ induces a connected subgraph in G.

DEFINITION 2.12. A graph G is *locally N^k-connected* if every vertex of G is locally N^k-connected.

CONJECTURE 2.13. (Li [**22**]) Every 3-connected, locally N^2-connected claw-free graph is hamiltonian.

We now give a counterexample to Conjecture 2.13 as follows:

EXAMPLE 2.14. Let P_{10} denote the Petersen graph. For an integer $k > 0$ and let $P_{10}(k)$ denote the graph obtained from P_{10} by adding k pendant edges at each vertex of P_{10}. Let $G(k) = L(P_{10}(k))$ be the line graph of $P_{10}(k)$. For an example when $k = 3$ see Figure 2.

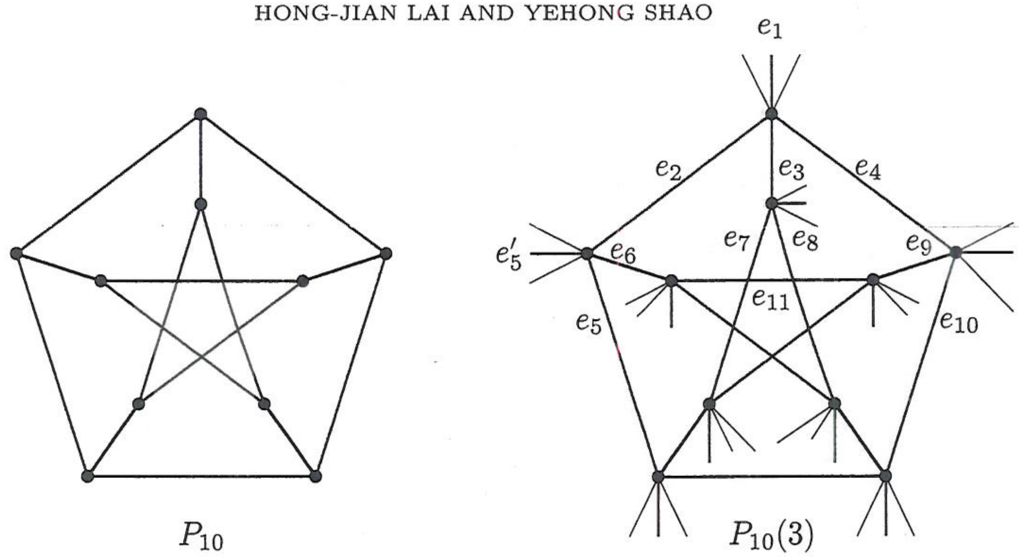


Figure 2 The Petersen graph P_{10} and $P_{10}(3)$

By a well known result of Harary and Nash-Williams [**10**], $G(k)$ does not have a hamilton cycle. On the other hand, we shall check that each $G(k)$ is claw-free 3-connected (hence 3-edge-connected) and N^2-locally connected. Since line graphs are claw-free, $G(k)$ must also be claw-free. Since $P_{10}(k)$ does not have an essential edge cut with size less than 3, $\kappa(G(k)) \geq 3$. It remains to check that each $G(k)$ is N^2-locally connected.

We shall use the notation in Figure 2 to show that $G(k)$ is locally N^2-connected, by the symmetry of the Petersen graph, it suffices to show that both vertices e_1 and e_2 are locally N^2-connected.

Let v_1, v_2 and v_3 denote the vertices in $P_{10}(k)$ that are incident with both e_1 and e_2, both e_2 and e_6, and both e_3 and e_7, respectively. For each vertex $v \in V(P_{10}(k))$, let $K(v)$ denote the complete graph in $G(k)$ induced by the edge incident with v in $P_{10}(k)$.

Since e_1 is a pendant edge in $P_{10}(k)$, it lies in a complete subgraph $K(v_1)$ of $G(k)$ containing e_2, e_3, e_4. Any vertices that are of distance 2 from e_1 in $G(k)$ must be a vertex adjacent to one of e_2, e_3 and e_4. Therefore, e_1 is a locally N^2-connected vertex in $G(k)$, see Figure 3.

It is not difficult to check that e_2 is also a locally N^2-connected vertex in $G(k)$.

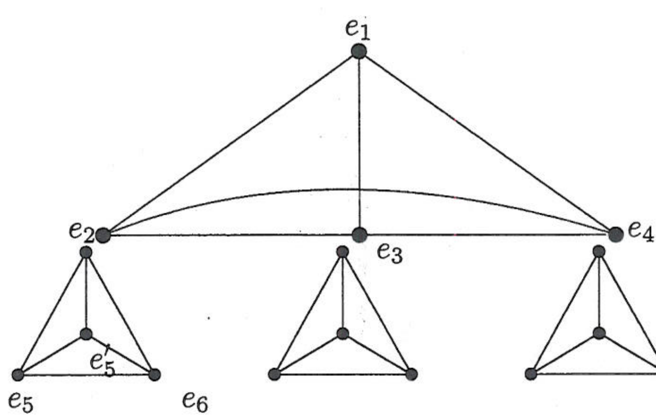


Figure 3 An illustration of the proof that e_1 is locally N^2-connected in $G(k)$

THEOREM 2.15. *(H.-J. Lai, M. Li, Y. Shao and L. Xiong* [**20**]*)*

(1) *Every 2-edge-connected, locally N^2-connected claw-free graph has a spanning Eulerian subgraph with maximum degree at most 4.*
(2) *Every 3-edge-connected, locally N^3-connected claw-free graph has a spanning Eulerian subgraph with maximum degree at most 4.*

DEFINITION 2.16. A graph G is *triangularly connected* if for every pair of edges $e_1, e_2 \in E(G)$, G has a sequence of 3-cycles $C_1, C_2, \cdots, C_l$ such that $e_1 \in C_1, e_2 \in C_l$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$, $(1 \leq i \leq l-1)$.

Every connected, locally connected graph is triangularly connected, but triangularly connected graphs may not be locally connected.

THEOREM 2.17. *(H.-J. Lai, L. Miao, Y. Shao and L. Wan [**29**]) Every triangularly connected claw-free graph is vertex pancyclic.*

CONJECTURE 2.18. (M. Li, L. Xiong and H.-J. Lai, [**23**]) Every 3-connected 4-cycle connected claw-free graph G with $|E(G)| \geq 3$ is hamiltonian.

We believe that such graphs may even be vertex pancyclic.

3. Degree Conditions

When $\kappa(H) = 2$, Kuipers and Veldman [**13**], and independently Favaron, Flandrin, Li and Ryjáček [**8**], proved that if H is a 2-connected claw-free graph with sufficiently large order ν, and if $\delta(H) \geq \frac{\nu+c}{6}$ (where c is a constant), then H is hamiltonian except a member of ten well-defined families of graphs. When $\kappa(H) = 3$, the following have been proved and proposed.

THEOREM 3.1. *(Kuipers and Veldman[**13**]) If H is a 3-connected claw-free simple graph with sufficiently large order ν, and if $\delta(H) \geq \frac{\nu+29}{8}$, then H is hamiltonian.*

THEOREM 3.2. *(Favaron and Fraisse [**9**]) If H is a 3-connected claw-free simple graph with order ν, and if $\delta(H) \geq \frac{\nu+37}{10}$, then H is hamiltonian.*

CONJECTURE 3.3. (Kuipers and Veldman [**13**], see also [**9**]) Let H be a 3-connected claw-free simple graph of order ν with $\delta(H) \geq \frac{\nu+6}{10}$. If ν is sufficiently large, then H is hamiltonian.

THEOREM 3.4. *(H.-J. Lai, Y. Shao and M. Zhan [**18**]) If H is 3-connected claw-free simple graph with $\nu \geq 196$, and if $\delta(H) \geq \frac{\nu+5}{10}$, then either H is hamiltonian, or $\delta(H) = \frac{\nu+5}{10}$ and $cl(H)$ is the line graph of G obtained from the Petersen graph P_{10} by adding $\frac{\nu-15}{10}$ pendant edges at each vertex of P_{10}.*

4. Matthews, Sumner and Thomassen Conjectures

CONJECTURE 4.1. (Thomassen [**32**]) Every 4-connected line graph is hamiltonian.

CONJECTURE 4.2. (Matthews and Sumner [**24**]) Every 4-connected claw-free graph is hamiltonian.

In 1986, Zhan proved:

THEOREM 4.3. *(Zhan [**34**]) If G is a 4-edge-connected graph, then the line graph $L(G)$ is hamiltonian connected.*

THEOREM 4.4. *(Ryjáček [**28**])*

(1) *Conjecture 4.1 and Conjecture 4.2 are equivalent.*
(2) *Every 7-connected claw-free graph is hamiltonian.*

THEOREM 4.5. *(Chen, Lai, Lai, and Weng* [**14**]*) Every 4-connected line graph of a claw free graph is hamiltonian.*

This has been improved by Kriesell.

THEOREM 4.6. *(Kriesell* [**12**]*) Every 4-connected line graph of a claw free graph is hamiltonian connected.*

DEFINITION 4.7. Let C_4 denote a 4-cycle in K_5. The graph $K_5 - E(C_4)$ is called an *hourglass.*

DEFINITION 4.8. A graph G is *hourglass free* if G does not have an induced subgraph isomorphic to $K_5 - E(C_4)$.

THEOREM 4.9. *(Broersma, Kriesell and Ryjáček* [**3**]*) Every 4-connected hourglass free line graph is hamiltonian connected.*

These results suggest that there may be a more general theorem behind them. In fact, the following is recently proved.

THEOREM 4.10. *(Lai, Shao, Yu and Zhan* [**19**]*) Let G be a connected graph with $|V(G)| \geq 4$. The core of this graph G, denoted by G_0, is obtained by deleting all the vertices of degree 1 and contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$. If every 3-edge-cut of the core G_0 has at least one edge lying in a cycle of length at most 3 in G_0, then the following statements are equivalent.*
(1) *$L(G)$ is hamiltonian connected.*
(2) $\kappa(L(G)) \geq 3$.

COROLLARY 4.11. Let G be a graph with $|V(G)| \geq 4$. Suppose that $L(G)$ is hourglass free in which every 3-cut of $L(G)$ is not an independent set. Then $L(G)$ is hamiltonian-connected if and only if $\kappa(L(G)) \geq 3$.

DEFINITION 4.12. A set $B \subset V(G)$ is a *dominating* set if every vertex of G belongs to B or has a neighbor in B.

DEFINITION 4.13. The size of a minimum dominating set of G will be called *dominating number* of G and is denoted by $\gamma(G)$. If $\gamma(G) \leq k$, then G is *k-dominated.*

DEFINITION 4.14. A graph G is *almost claw free* if the vertices that are centers of claws in G are independent and if the neighborhoods of the center of each claw in G is 2-dominated.

Note that every claw free graph is an almost claw free graph and there exist almost claw free graphs that are not claw-free.

COROLLARY 4.15. Every 4-connected line graph of an almost claw free graph is hamiltonian-connected.

We conclude this section with another result and a conjecture in this direction.

THEOREM 4.16. *(Lai, Shao, Wu and Zhou* [**21**]*) Every 3-connected, essentially 11-connected line graph is hamiltonian.*

CONJECTURE 4.17. (H.-J. Lai and L. Šoltés, [**31**]) Every 7-connected claw-free graph is hamiltonian-connected.

5. Problems Related to the Hamiltonian Like Indices

DEFINITION 5.1. For a nontrivial connected graph G, we define $L^0(G) = G$ and for any integer $k > 0$, $L^k(G) = L(L^{k-1}(G))$.

DEFINITION 5.2. The *hamiltonian index* $h(G)$ of G is the smallest positive integer k such that $L^k(G)$ is hamiltonian.

The concept of hamiltonian index was first introduced by Chartrand and Wall [**5**], who showed that (Theorem A of [**5**]) if a connected graph G is not a path, then $L^k(G)$ is defined for any positive integer k. For this reason, we shall assume, throughout this section, that the graph G under discussion is simple, connected and not a path.

Clark and Wormald developed the idea and introduced the hamiltonian like indices.

DEFINITION 5.3. A graph is *edge-hamiltonian* (eh) if each edge lies on a Hamilton cycle.

DEFINITION 5.4. A graph is *pancyclic* (pc) if G has a cycle of length k, for each k with $3 \leq k \leq |V(G)|$.

DEFINITION 5.5. A graph is *vertex pancyclic* (vp) if for every vertex $v \in V(G)$, G has a cycle containing v and of length k, for each k with $3 \leq k \leq |V(G)|$.

DEFINITION 5.6. A graph is *edge pancyclic* (ep) if for every edge $e \in E(G)$, G has a cycle containing e and of length k, for each k with $3 \leq k \leq |V(G)|$.

DEFINITION 5.7. A graph is *hamiltonian connected* (hc) if for every pair of vertices u and v of G, G has a spanning (u, v)-path.

DEFINITION 5.8. For a property $\mathcal{P}$ and a connected nonempty graph G which is not a path, define the $\mathcal{P}$*-index* of G, denoted $\mathcal{P}(G)$, as

$$\mathcal{P}(G) = \begin{cases} \min\{k : L^k(G) \text{ has property } \mathcal{P}\} & \text{if at least one such integer } k \text{ exists} \\ \infty & \text{otherwise} \end{cases}$$

Clark and Wormald [**7**] showed that if a connected, nontrivial graph G is not a path nor a cycle, then the indices $h(G), eh(G), pc(G), vp(G), ep(G), hc(G)$ exist as finite numbers.

The most studied index is $h(G)$, the hamiltonian index. By Theorem 1.13, if a graph G is hamiltonian, then $L(G)$ is also hamiltonian. Therefore, a question arises: for any of the properties $\mathcal{P} \in \{eh, pc, vp, ep, ep, hc\}$, if G has property $\mathcal{P}$, does $L(G)$ also have property $\mathcal{P}$? We in this section will answer this question.

DEFINITION 5.9. For a trail P defined in Definition 1.14, let

$$\partial(P) = \{e \in E(G) : \ e \text{ is incident with an internal vertex of } P\}.$$

If $\partial(P) = E(G)$, then P is a dominating trail. We put the following facts in the form of a lemma, which follows directly from the definition of a line graph.

LEMMA 5.10. *Let $P = v_0v_1 \cdots v_k$ denote a (possibly closed) trail of G, and let $E(P) = \{e_1, e_2, \cdots, e_k\}$ such that for $i = 1, 2, \cdots, k$, $e_i = v_{i-1}v_i$. Suppose that $X \subseteq \partial(P) - E(P)$ can be represented by $X = \{x_1, \cdots, x_{n_1}, x_{n_1+1}, \cdots, x_{n_{k-1}}\}$, such* *that for each* $i = 0, 1, 2, \cdots, k-2$, $x_{n_i+1}, \cdots, x_{n_{i+1}}$ *are incident with the vertex* v_{i+1}, *where we define* $n_0 = 0$. *Then the sequence*

$$e_1, x_1, \cdots, x_{n_1}, e_2, x_{n_1+1}, \cdots, x_{n_{k-1}}, e_k$$

is a path L *in* $L(G)$ *with* $V(L) = E(P) \cup X$.

PROPOSITION 5.11. If G is hamiltonian connected, then $L(G)$ is also hamiltonian connected.

PROOF. By Theorem 1.16, it suffices to show that for any edges $e_1, e_2 \in E(G)$, G has dominating (e_1, e_2)-trail.

We assume that $e_1 = u_1v_1$ and $e_2 = u_2v_2$. Since G is hamiltonian connected, G has a hamiltonian (u_1, u_2)-path P. If $e_1, e_2 \in E(P)$, then since P is a (u_1, u_2)-path, e_1 and e_2 must be the two end edges of the path and so P is a dominating (e_1, e_2)-trail of G. If $e_1, e_2 \notin E(P)$, then $G[E(P) \cup \{e_1, e_2\}]$ is a dominating (e_1, e_2)-trail of G. If $|E(P) \cap \{e_1, e_2\}| = 1$, then we may assume that $e_1 \in E(P)$ and $e_2 \notin E(P)$. Since P is a (u_1, u_2)-path, e_1 must be an end edge of P, and so $G[E(P) \cup \{e_2\}]$ is a dominating (e_1, e_2)-trail of G. □

PROPOSITION 5.12. If G is pancyclic, the $L(G)$ is also pancyclic.

PROOF. For any integer k with $3 \le k \le |E(G)|$, we need to show that $L(G)$ has a cycle of length k.

First assume that $3 \le k \le |V(G)|$. Since G is pancyclic, G has a cycle C with $|E(C)| = k$. By Lemma 5.10, $L(C)$ is a cycle of length k in $L(G)$. Thus we assume that $|V(G)| + 1 \le k \le |E(G)|$. Since G is pancyclic, G has a Hamilton cycle C, and so there exists an edge subset $X \subseteq E(G) - E(C)$ such that $|E(C) \cup X| = k$. It then follows by Lemma 5.10 that $L(G)[E(C) \cup X]$ contains a cycle of length k. Hence $L(G)$ must also be pancyclic. □

PROPOSITION 5.13. If G is edge-hamiltonian, the $L(G)$ is also edge-hamiltonian.

PROOF. Let $f = e_1e_2$ denote an arbitrary edge in $L(G)$, where e_1, e_2 are both adjacent to a vertex u in G.

Suppose first that u has degree 2 in G. Since G is edge-hamiltonian, G has a Hamilton cycle C such that $e_1 \in E(C)$. It follows that C is a dominating cycle of G. Since u has degree 2 in G, $e_2 \in E(C)$. Therefore, e_1 and e_2 are adjacent in C and G has no other edges adjacent to u. It follows by Lemma 5.10 that $L(G)$ has a Hamilton cycle containing the edge $f = e_1e_2$.

Hence we assume that u has degree at least 3 in G. The there exists an edge $e_3 \in E(G) - \{e_1, e_2\}$ such that e_3 is also incident with u in G. Since G is edge-hamiltonian, G has a Hamilton cycle C such that $e_3 \in E(C)$. Since C is a cycle containing e_3 and since e_1, e_2, e_3 are all incident with the same vertex u, we may assume that $e_2 \notin E(C)$. It then follows by Lemma 5.10 that $L(G)$ has a Hamilton cycle containing the edge $f = e_1e_2$. □

PROPOSITION 5.14. If G is vertex pancyclic, the $L(G)$ is also vertex pancyclic.

PROOF. For any integer k with $3 \le k \le |E(G)|$, and for any $e \in V(L(G)) = E(G)$, we need to show that $L(G)$ has a cycle L_k of length k such that $e \in V(L_k)$. Let $e = uv$ such that $deg_G(u) \ge deg_G(v)$. We may assume that G is not a 3-cycle, and since G is vertex pancyclic, $deg_G(u) \ge 3$. Therefore, e lies in a 3-cycle of $L(G)$. We assume that $k \ge 4$.

Since G is vertex pancyclic, G has a cycle C_k with $|E(C_k)| = k$, for $k = 3, 4, \cdots, |V(G)|$, such that $u \in V(C_k)$. If $e \in E(C_k)$, then by Lemma 5.10, $L_k = L(C_k)$ is a cycle of length k containing e in $L(G)$.

Therefore, we assume that $e \notin E(C_k)$. As $k \geq 4$, G has a cycle C_{k-1} of length $k-1$ containing u. Since $deg_G(u) \geq 3$, $|\partial(C_{k-1})| \geq k$ and $e \in \partial(C_{k-1})$. It follows by Lemma 5.10 that $L(G)$ has a cycle of length k containing e.

When $|E(G)| \geq k \geq |V(G)| + 1$, since G has a Hamilton cycle C containing u, a similar argument shows that $L(G)$ has a cycle of length k containing the edge e. Hence $L(G)$ must also be vertex pancyclic. □

PROPOSITION 5.15. If G is edge pancyclic, then $L(G)$ is also edge pancyclic.

PROOF. For any integer k with $3 \leq k \leq |E(G)|$, and for any $e_1, e_2 \in V(L(G)) = E(G)$ with $f = e_1e_2 \in E(L(G))$, we need to show that $L(G)$ has a cycle L_k of length k such that $f \in E(L_k)$. Let u denote the vertex incident to both e_1 and e_2.

We first assume u has degree at least $d \geq 3$ in G, and so that G has an edge $e_3 \in E(G) - \{e_1, e_2\}$. Thus $L(G)$ has a k-cycle containing the edge $f = e_1e_2$, for each $k = 3, 4, \cdots, d$.

Let $k \geq d + 1 \geq 4$ be an integer. Assume that $k \leq |V(G)|$. Since G is edge pancyclic, G has a cycle C_{k-1} of length $k-1$ such that $e_3 \in E(C_{k-1})$. If $e_1 \in E(C_{k-1})$, then $e_2 \notin E(C_{k-1})$. Since $e_2 \in \partial(C_{k-1})$, it follows by Lemma 5.10 that $L(G)$ has a cycle of length k containing $f = e_1e_2$. The case when $e_2 \in E(C_{k-1})$ can be similarly proved. Therefore, we assume that $e_1, e_2 \notin E(C_{k-1})$. It follows that $d \geq 4$ and G has an edge $e_4 \in E(G) - \{e_1, e_2, e_3\}$ incident with u in G. Since G is edge-pancyclic, G has a cycle C_{k-2} of length $k-2$ containing e_3. Therefore, $|\partial(C_{k-2})| \geq k$ and $e_1, e_2, e_3, e_4 \in \partial(C_{k-2})$. By Lemma 5.10, $L(G)$ has a cycle of length k containing the edge $f = e_1e_2$. When $|E(G)| \geq k \geq |V(G)| + 1$, since G has a Hamilton cycle C containing e_3, a similar argument shows that $L(G)$ has a cycle of length k containing the edge $f = e_1e_2$.

It remains to prove the case when u has degree 2 in G. In this case, any cycle of G containing e_1 must also contain e_2. Using this fact and the same argument above, we can similarly prove that $L(G)$ has a cycle of length k containing $f = e_1e_2$, for any k with $3 \leq k \leq |E(G)|$. □

There were a few open problems posted at the end of the paper of Clark and Wormald [**7**].

DEFINITION 5.16. For a property $\mathcal{P}$ and integers a, b, with $1 \leq a \leq b$, define

$$\mathcal{P}(a,b) = \begin{cases} \max\{\mathcal{P}(G) : \kappa'(G) \geq a \text{ and } \delta(G) \geq b\} & \text{if the max exists} \\ \infty & \text{otherwise} \end{cases}.$$

Clark and Wormald determined most of the values of $\mathcal{P}(a, b)$ in [**7**]. At the end, they raised this question: What are the values of $h(a, b), eh(a, b)$, and $hc(a, b)$ when $4 \leq a \leq b$?

To answer this question, we shall apply the following theorem by Nash-Williams and Tutte.

THEOREM 5.17. *(Nash-Williams [**25**] and Tutte [**33**]) Let $k > 0$ be an integer. A graph G has k edge-disjoint spanning trees if and only if for any partition $\{V_1, V_2, \cdots, V_t\}$ of $V(G)$, the number of edges in G joining distinct sets of these V_i's is at least $k(t-1)$.*

Using this powerful theorem of Nash-Williams and Tutte, Zhan proved the following.

THEOREM 5.18. *(Zhan [35]) If G is 4-edge-connected, then for any edges $e_1, e_2 \in E(G)$, $G - \{e_1, e_2\}$ has two edge-disjoint spanning trees.*

Catlin [4] proved the following.

THEOREM 5.19. *(Catlin [4]) If G has two edge-disjoint spanning trees, then for any two vertices $u, v \in V(G)$, G has a spanning (u, v)-trail.*

It follows that if $\kappa'(G) \geq \kappa(G) \geq 4$, then by Theorem 5.18 and Theorem 5.19, for any pair of edges $e_1 = u_1v_1, e_2 = u_2v_2 \in E(G)$, the graph $G - \{e_1, e_2\}$ has a spanning (u_1, u_2)-trail C, which can be augmented by adding the edges e_1 and e_2 to result in a dominating (e_1, e_2)-trail of G. By Theorem 1.16, $L(G)$ is hamiltonian connected. This indicates that when $4 \leq a \leq b$, $hc(a, b) \leq 1$. By the definition of these indices, when $4 \leq a \leq b$, we have

$$h(a, b) \leq eh(a, b) \leq hc(a, b) \leq 1.$$

For any integer $m \geq 5$, since $K_{m,4}$ is 4-connected without a Hamilton cycle, $h(4, b) \geq 1$, for any $b \geq 4$. Therefore, when $4 \leq a \leq b$, we have

$$h(a, b) = eh(a, b) = hc(a, b) = 1.$$

This answers the question in [7].

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