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Quadrangularly connected claw-free graphs☆

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Abstract

A graph G is *quadrangularly connected* if for every pair of edges e_1 and e_2 in $E(G)$, G has a sequence of l-cycles $(3 \leq l \leq 4)$ $C_1, C_2, \ldots, C_r$ such that $e_1 \in E(C_1)$ and $e_2 \in E(C_r)$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$ for $i = 1, 2, \ldots, r-1$. In this paper, we show that every quadrangularly connected claw-free graph without vertices of degree 1, which does not contain an induced subgraph H isomorphic to either G_1 or G_2 such that $N_1(x, G)$ of every vertex x of degree 4 in H is disconnected is hamiltonian, which implies a result by Z. Ryjáček [Hamiltonian circuits in N_2-locally connected $K_{1,3}$-free graphs, J. Graph Theory 14 (1990) 321–331] and other known results.

Keywords: Cycle; Claw-free graph; Quadrangularly connected

1. Notation and terminology

We use [1] for notation and terminology not defined here, and consider finite simple graphs only. Let G be a graph. Denote by $G[S]$ the induced subgraph of G on the subset S of $V(G)$. For a vertex v of G, the neighborhood of v, i.e., the induced subgraph on the set of all vertices that are adjacent to v, will be called the neighborhood of the *first type* of v in G and denoted by $N_1(v, G)$, or briefly, $N(v)$. For notational convenience, we shall use $N_G(v)$ to denote both the induced subgraph and the set of vertices adjacent to v in G. We define the neighborhood of the *second type* of v in G (denoted by $N_2(v, G)$, or briefly, $N_2(v)$) as the subgraph of G induced by the edge subset $\{e = xy \in E(G) : v \notin \{x, y\}$ and $\{x, y\} \cap N(v) \neq \emptyset\}$. We say that a vertex v is *locally connected* if $N(v)$ is connected; and G is *locally connected* if every vertex of G is locally connected. Analogously, a vertex v is *N_2-locally connected* if its second type neighborhood $N_2(v)$ is connected; and G is called *N_2-locally connected* if every vertex of G is N_2-locally connected. It follows from the definitions that every locally connected graph is N_2-locally connected. A cycle of length k is called a k-cycle. Given a cycle $C = (x_1x_2 \ldots x_kx_1)$ of a graph G, we fix an orientation of C, let $x_i^+ = x_{i+1}$ and $x_i^- = x_{i-1}$, and let $C[x_i, x_j] = (x_ix_{i+1}, \ldots, x_{j-1}, x_j) = \{x_i, x_{i+1}, \ldots, x_{j-1}, x_j\}$, $C(x_i, x_j) = C[x_i, x_j] - \{x_i, x_j\}$ and

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$C^-[x_j, x_i] = (x_j x_{j-1} \dots x_i)$. An edge $e = uv$ is called a chord of C if $e \notin E(C)$ and u and v are on C. If a cycle C has no chords, then we say that C is chord-free. Obviously, 3-cycles are chord-free.

The *line graph* of a graph G, denote it by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. A graph G is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. Obviously, the line graph of a graph is claw-free.

2. Introduction

The following theorem gives the hamiltonicity of locally connected graphs.

Theorem 2.1 (*Oberly and Sumner [6]*). *Every connected locally connected claw-free graph on at least three vertices is hamiltonian.*

A pair of edges e_1 and e_2 in $E(G)$ is called triangularly adjacent if G has a sequence of 3-cycles $C_1, C_2, \dots, C_r$ such that $e_1 \in E(C_1)$ and $e_2 \in E(C_r)$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$ for $i = 1, 2, \dots, r-1$. A graph G is *triangularly connected* if every pair of edges e_1 and e_2 in $E(G)$ is triangularly adjacent. Obviously, every connected, locally connected graph is triangularly connected (see [2]). But the converse is not true. Recently, Shao [2] generalized the above theorem as follows.

Theorem 2.2 (*Shao [2]*). *Every triangularly connected claw-free graph on at least three vertices is hamiltonian.*

A graph G is *vertex pancyclic* if it contains cycles of all possible length through every vertex. They [2] actually proved that a graph with the same conditions as Theorem 2.2 is vertex pancyclic. Ryjáček [7] strengthened Theorem 2.1 as follows in 1990, and Li [5] improved Theorem 2.3 using N^2-locally connectedness.

Theorem 2.3 (*Ryjáček [7]*). *Let G be a connected, N_2-locally connected claw-free graph without vertices of degree* 1, *which does not contain an induced subgraph H isomorphic to either G_1 or G_2* (Fig. 1) *such that $N_1(x, G)$ of every vertex x of degree* 4 *in H is disconnected. Then G is hamiltonian.*

Li [4] obtained the following theorem.

Theorem 2.4 (*Li [4]*). *Let G be a connected, N_2-locally connected claw-free graph with $\delta(G) \geqslant 3$, which does not contain an induced subgraph H isomorphic to either G_1 or G_2* (Fig. 1). *Then G is vertex pancyclic.*

A pair of edges e_1 and e_2 in $E(G)$ is called quadrangularly connected if G has a sequence of chord-free l-cycles $(3 \leqslant l \leqslant 4)$ $C_1, C_2, \dots, C_r$ such that $e_1 \in E(C_1)$ and $e_2 \in E(C_r)$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$ for $i = 1, 2, \dots, r-1$.

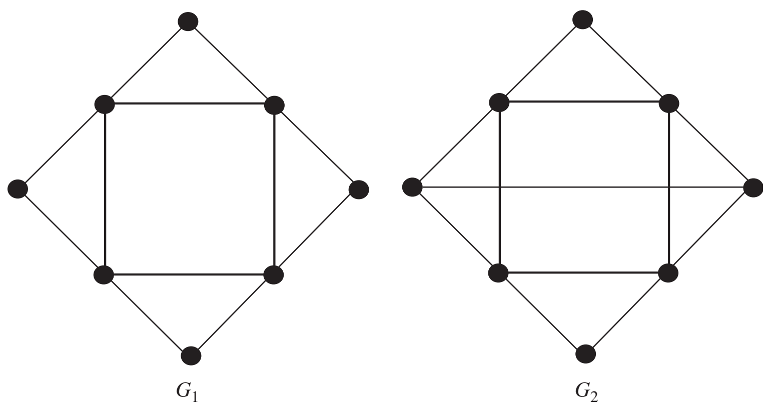

Fig. 1.

A graph G is *quadrangularly connected* if every pair of edges in $E(G)$ is quadrangularly adjacent. Obviously, if a graph is triangularly connected then it is quadrangularly connected. From the definition, we easily prove the following proposition that every connected N_2-locally connected graph is quadrangularly connected. But the converse is not true. For example, let $k>4$ be an integer, and let $H_1, H_2, \ldots, H_k$ be complete graphs of order at least 4. We construct a graph G by the following process: for each $i=1,2,\ldots,k-1$, choose an edge $e'_i=v'_{i1}v'_{i2}$ in H_i and an edge $e''_i=v''_{i1}v''_{i2}$ in H_{i+1}, and add two new edges $e^1_i=v'_{i1}v''_{i1}$ and $e^2_i=v'_{i2}v''_{i2}$ to form a quadrangle of G, and identify a vertex v_1 in H_1 with a vertex v_k in H_k to form a vertex v in G such that $V(H_1)\cap V(H_k)=\{v\}$ and $v\notin\{v'_{11}, v'_{12}, v''_{(k-1)1}, v''_{(k-1)2}\}$. Then G is quadrangularly connected, but v is not N_2-locally connected because $N_2(v)$ does not induce a connected subgraph of G. Note that G is not triangularly connected because e'_1 and e''_1 are not triangularly adjacent.

Define a relation on $E(G)$ such that $e\sim e'$ if and only if e and e' are quadrangularly connected in G. Then $\sim$ is transitive and an equivalence relation. Thus G is quadrangularly connected if and only if $E(G)$ has only one equivalence class.

Proposition 2.5. *Every connected N_2-locally connected graph G is quadrangularly connected.*

Proof. Let e_1 and e_2 be any two edges in $E(G)$. If e_1 and e_2 are both in the neighborhood of the second type of v in G or v is a common vertex of e_1 and e_2 (i.e., $e_1, e_2\in E(G[N_2(v)\cup\{v\}])$), then it is easy to check that $e_1\sim e_2$ since every edge of $E(G[N_2(v)\cup\{v\}])$ is contained in a 3-cycle or a 4-cycle in $G[N_2(v)\cup\{v\}]$ from the connectedness of $N_2(v)$. Hence e_1 and e_2 belong to two distinct second type neighborhoods of two distinct vertices v_1 and v_2 such that $e_1\in E(G[N_2(v_1)\cup\{v_1\}])$ and $e_2\in E(G[N_2(v_2)\cup\{v_2\}])$. Since G is connected, there is a path $P=x_0x_1x_2\ldots x_k$ connecting $N_2(v_1)$ and $N_2(v_2)$ such that $x_0, x_1\in N_2(v_1)\cup\{v_1\}$ and $x_{k-1}, x_k\in N_2(v_2)\cup\{v_2\}$ but x_i $(i=2,\ldots,k-2)$ are not in $N_2(v_1)\cup N_2(v_2)\cup\{v_1, v_2\}$. For $i=1,2,\ldots,k-1$, the edges $x_{i-1}x_i$ and x_ix_{i+1} are in the same neighborhood of the second type of x_i in G, so the edges $x_{i-1}x_i$ and x_ix_{i+1} are quadrangularly connected. Note that $e_1\sim x_0x_1$ and $e_2\sim x_{k-1}x_k$. Thus, we know from transitivity of $\sim$ that $e_1\sim e_2$. Therefore, we have completed the proof of Proposition 2.5. □

In this paper, we show the following result which generalizes Theorems 2.1–2.3. We postpone its proof to the next section.

Theorem 2.6. *Every quadrangularly connected claw-free graph without vertices of degree* 1, *which does not contain an induced subgraph H isomorphic to either G_1 or G_2 such that $N_1(x, G)$ of every vertex x of degree* 4 *in H is disconnected is hamiltonian.*

The following was conjectured by Ryjáček [7] and recently proved affirmatively in [3].

Theorem 2.7 (*Lai et al. [3]*). *Every* 3-*connected N_2-locally connected claw-free graph is hamiltonian.*

We naturally ask whether we may replace N_2-locally connectedness by quadrangularly connectedness in Theorem 2.7? and so make the following conjecture which would generalize Theorem 2.7 if it is true. Let $C_1=(x_1x_2\ldots x_8x_1)$ and $C_2=(y_1y_2\ldots y_8y_1)$ be two 8-cycles, and let H be a graph whose vertex set is $V(C_1)\cup V(C_2)$ and whose edge set is $E(C_1)\cup E(C_2)\cup\{x_iy_i : i=1,2,\ldots,8\}\cup\{x_1x_5\}$. Then H is triangle-free, and the edge x_1x_5 is not contained in a 4-cycle in H. The line graph $L(H)$ of H is 3-connected, quadrangularly connected and claw-free. This example may be useful for considering the following Conjecture 2.8.

Conjecture 2.8. Every 3-connected quadrangularly connected claw-free graph is hamiltonian.

Conjecture 2.9. Every graph satisfying the conditions of Theorem 2.6 is vertex pancyclic.

3. Proof of Theorem 2.6

In this section, we will provide the proof of Theorem 2.6. Let $S_{34}(G)$ denote the graph whose vertex set is the set of all 3-cycles and chord-free 4-cycles, and $C_1, C_2\in V(C_{34}(G))$ are adjacent in $S_{34}(G)$ if $E(C_1)\cap E(C_2)\neq\emptyset$. Then, from the definition of quadrangularly connectedness, we have established the following fact.

Proposition 3.1. *A graph G is quadrangularly connected if and only if both of the following hold*:

(1) *For any edge* $e \in E(G)$, *there is a cycle* $C_e \in V(S_{34}(G))$ *such that* $e \in E(C_e)$, *and*
(2) $S_{34}(G)$ *is connected.*

In our proof, we need to use the Ryjáček's closure concept in claw-free graphs [8]. Let G be a graph such that for every $x \in V(G)$, $G[N(x)]$ is either a clique or an union of two disjoint cliques. We call such a graph a *closed claw-free graph*. Then by Lemma 1 in [8], we know that G is the line graph of a triangle-free graph H.

Theorem 3.2 (*Ryjáček [8]*). *If G is a claw-free graph, then there is a closed claw-free graph* $\mathrm{cl}(G)$ (*called the closure of G*) *such that*

(1) *G is a spanning subgraph of* $\mathrm{cl}(G)$, *and*
(2) *the length of a longest cycle in both G and* $\mathrm{cl}(G)$ *is the same.*

The following fact is easy to prove, and will be used in our proof.

Proposition 3.3. *If a claw-free graph G is quadrangularly connected, then so is* $\mathrm{cl}(G)$.

Proof. Let G be a quadrangularly connected claw-free graph and $v \in V(G)$. Let $z, z' \in N_G(v)$ be two nonadjacent vertices in G, and let e denote an edge not in G which joins z and z'. Then, in $G+e$, e lies in a triangle containing vz, and so $e \sim vz$ in $G+e$. For any edge $e' \in E(G)$, since G is quadrangularly connected, $vz \sim e'$ in G, and so $vz \sim e'$ in $G+e$. As $\sim$ is transitive, $e \sim e'$ in $G+e$. Therefore, we have proved Proposition 3.3. □

Proof of Theorem 2.6. Let G be a graph satisfying the conditions of Theorem 2.6. If G is not hamiltonian, then consider the closure $\mathrm{cl}(G)$ of G. By Proposition 3.3, $\mathrm{cl}(G)$ is also quadrangularly connected. From Theorem 3.2(2), without loss of generality assume that $\mathrm{cl}(G)=G$. Then G is a closed claw-free graph. Let C be a longest cycle of G. Then there is a vertex v such that $v \notin V(C)$ and v is adjacent to some vertex x on C. Let

$$\Gamma = \{e \in E(G) : e \text{ is incident with exactly one vertex in } V(C)\}.$$

Then $xv \in \Gamma$. Note that, for any edge $e \in \Gamma$ (for example, $e = xv$, $x \in V(C)$ and $v \notin V(C)$), we have that $x^+v, x^-v \notin E(G)$ since otherwise G has a longer cycle than C. Thus $x^+x^- \in E(G)$ since $G[x, x^+, x^-, v] \neq K_{1,3}$ (the first vertex x in the set $\{x, x^+, x^-, v\}$ is always the center of a claw in the following proof), and we have the following fact.

Claim 1. *For any edge* $e \in \Gamma$, *e is contained in a* 3*-cycle* $C_0 \in S_{34}(G)$.

Proof. Let e_1 be any edge of Γ, and without loss of generality assume that $e_1 = xv$, $x \in V(C)$ and $v \notin V(C)$. Consider the pair of edges $vx = e_1$ and $xx^+ = e_2$. From the conditions of Theorem 2.6, there is a sequence of 3-cycles or chord-free 4-cycles $C_0, C_1, \ldots, C_k$ such that $e_1 \in E(C_0)$ and $e_2 \in E(C_k)$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$ for $i = 0, 1, \ldots, k$. If $|V(C_0)| = 3$, we are done. Thus $|V(C_0)| = 4$ and let $C_0 = (xvx'x''x)$. □

Since $x^+v, x^-v \notin E(G)$, e_1 and e_2 are not contained in some same 3-cycle. We have that e_1 and e_2 are not contained in the same 4-cycle C' since otherwise, let $C' = (vxx^+yv)$. Then $y \in V(C)$ since otherwise G has a longer cycle than C by replacing xx^+ with $xvyx^+$, and so $y^+v, y^-v \notin E(G)$. It follows that $y^+y^- \in E(G)$ since G is claw-free and $G[y, y^+, y^-, v] \neq K_{1,3}$, and then G has a longer cycle than C by replacing y^-yy^+ with the edge y^-y^+ and xx^+ with $xvyx^+$, a contradiction. Thus, e_1 and e_2 are not contained in some same 4-cycle. In order to prove Claim 1, we first establish the following fact.

Claim 1.1. $E(C_0) \cap E(C) = \emptyset$.

Proof. Otherwise, $|E(C_0) \cap E(C)| = 1$. We have that $x'x'' \in E(C)$ because xx^+ and xv are not at the same cycle in $S_{34}(G)$. Thus we can obtain a longer cycle than C by replacing x^-xx^+ with x^-x^+ and $x'x''$ with $x'vxx''$, a contradiction. So Claim 1.1 is true. □

Claim 1.2. *$x'' \in V(C)$, and $x^-x'', x^+x'' \in E(G)$.*

Proof. If $x'' \notin V(C)$, then $xx'' \in \Gamma$, and so $x''x^+, x''x^- \notin E(G)$. Note that $x''v \notin E(G)$. Thus $G[x, v, x'', x^+] = K_{1,3}$, a contradiction. Since $G[x, x^-(x^+), v, x''] \neq K_{1,3}$ and $vx'', vx^+, vx^- \notin E(G)$, $x^-x'', x^+x'' \in E(G)$. □

Claim 1.3. *$x' \in V(C)$ and $x'^+x'^- \in E(G)$.*

Proof. If $x' \notin V(C)$, then $x'x'' \in \Gamma$, and so $x''^-x''^+ \in E(G)$ since $G[x'', x''^+, x''^-, x'] \neq K_{1,3}$. It follows that we can obtain a longer cycle than C by replacing $x''^-x''x''^+$ with $x''^-x''^+$ and xx^+ with $xvx'x''x^+$, a contradiction. Thus $x' \in V(C)$, and so $vx' \in \Gamma$. From $G[x', x'^+, x'^-, v] \neq K_{1,3}$ and $x'^+v, x'^-v \notin E(G)$, we have $x'^+x'^- \in E(G)$. Thus Claim 1.3 is true. □

Now we return the proof of Claim 1. By Claims 1.2 and 1.3, we have $x', x'' \in V(C)$. Without loss of generality assume that $x'' \in C(x, x')$. By Claim 1.1, $|C(x'', x')| \geqslant 1$. From $G[x', v, x'^+(x'^-), x''] \neq K_{1,3}$ and $x'^+v, x'^-v, x''v \notin E(G)$, $x'^+x'', x'^-x'' \in E(G)$. Similarly, $x^+x'', x^-x'' \in E(G)$. It follows that we easily get $|C(x^+, x'')| \geqslant 1$ and $|C(x'', x'^-)| \geqslant 1$. Note that $xx' \notin E(G)$. From $G[x'', x''^-, x, x'] \neq K_{1,3}$, $x''^-x \in E(G)$ or $x''^-x' \in E(G)$. If $x''^-x \in E(G)$, then G has a new cycle $C[x^+, x''^-]xvx'C[x'', x'^-]C[x'^+, x^-]x^+$ than C, a contradiction. Similarly, if $x''^-x' \in E(G)$, we easily get a longer cycle than C, a contradiction. Thus we have completed the proof of Claim 1. □

By Proposition 3.1(1), there is a sequence of $C_1, C_2, \ldots, C_m$ in $S_{34}(G)$ such that $E(C)$ is contained in $E(C_1) \cup E(C_2) \cup \cdots \cup E(C_m)$ and m is minimal with the property. Then, by the minimality of m, we have the following fact.

Claim 2. *$E(C_i) \cap E(C) \neq \emptyset$ for $i = 1, 2, \ldots, m$.*

For any edge $e \in \Gamma$, by Claim 1, there is a 3-cycle C_0 such that $e \in E(C_0)$. By Proposition 3.1, there is a shortest path $P = Q_0Q_1Q_2 \ldots Q_r$ in $S_{34}(G)$ from C_0 to the vertex set $\{C_1, C_2, \ldots, C_m\}$, where $Q_0 = C_0$. From the above, we know that the length r of P is at least one. Choose an edge (say e) in Γ such that r is as small as possible. without loss of generality assume that $e = xv$ and $Q_r = C_1$. Then we have the following fact.

Claim 3. *$|V(C_0) \cap V(C)| \leqslant 2$, and $E(Q_i) \cap E(C) = \emptyset$ for $i = 0, 1, \ldots, r-1$.*

Claim 4. *$r = 1$ and $|V(C_0) \cap V(C)| = 2$.*

Proof. If $r > 1$, then, since $|V(C_0) \cap V(C)| \leqslant 2$, there is a largest integer t $(0 \leqslant t \leqslant r-1)$ such that $|V(Q_t) \cap V(C)| = 2$, and by Claim 1, $|V(Q_t)| = 3$. If $t = 0$, then we are done. Thus $t \geqslant 1$. Let $Q_t = (v'u'w'v')$ and $v' \notin V(C)$ but $u' \in V(C)$. We replace the edge e by $v'u'$, we get a shorter path $P' = Q_t \ldots Q_r$ than P, which contradicts the choice of e. Thus $r = 1$. From the above, we also have $|V(C_0) \cap V(C)| = 2$. □

Let $C_0 = (xvwx)$. Then $w, x \in V(C)$, and so $vw \in \Gamma$ and $w^+w^- \in E(G)$ but $w^-v, w^+v \notin E(G)$. Note that $Q_1 = C_1$ and C is divided into two segments $C' = (xx^+ \ldots w^-w)$ and $C'' = (ww^+ \ldots x^-x)$. We further have the following fact.

Claim 5. *$|V(C_1)| = 4$.*

Proof. Otherwise, $|V(C_1)| = 3$. Since $E(C) \cap E(C_1) \neq \emptyset$ and $E(C_0) \cap E(C_1) \neq \emptyset$, we must have that one of $\{ww^-, ww^+, xx^+, xx^-\}$ belongs to $E(C) \cap E(C_1)$. For example, $ww^- \in E(C) \cap E(C_1)$. Since $vw^- \notin E(G)$, we have $C_1 = (xww^-x)$. Note that $w^+w^-, x^+x^- \in E(G)$. Thus replacing x^-xx^+ by x^-x^+ and w^-w by w^-xvw, we obtain a longer than C, a contradiction. Thus Claim 5 is true. □

Let $e_1 = yz \in E(C) \cap E(C_1)$. Then $e_1 \in \{xx^+, x^-x, w^-w, ww^+\}$ since otherwise, for example, yz is on the segment $C(x^+, w^-)$. Since $E(C_0) \cap E(C_1) \neq \emptyset$, there is an edge $e''(\neq e_1)$ such that $e'' \in E(C_0) \cap E(C_1)$ and

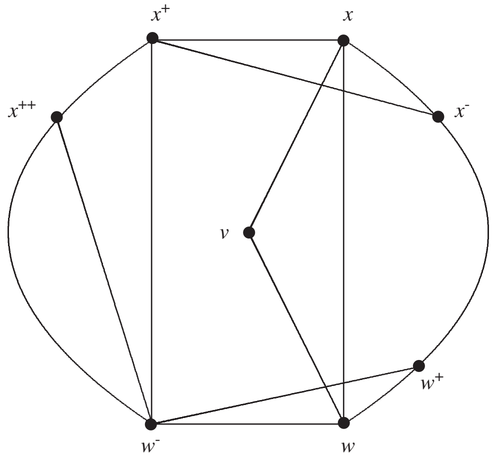

Fig. 2.

$e'' \in \{xv, vw, xw\}$. If $e'' = xv$, then $vz \in \Gamma$ and without loss of generality assume that $C_1 = (xvzyx)$. Replacing yz by $yxvz$ and x^-xx^+ by x^-x^+, we obtain a longer cycle than C, a contradiction. Similarly, if $e'' = vw$, then we can get a contradiction. If $e'' = xw$, then replacing yz with $yxvwz$, x^-xx^+ with x^-x^+ and w^-ww^+ with w^-w^+, we get a longer cycle than C. This contradiction shows $e_1 \in \{xx^+, x^-x, w^-w, ww^+\}$ (say $e_1 = w^-w$).

Note that $vw^- \notin E(G)$. If $vw \in E(C_0) \cap E(C_1)$, let $C_1 = (wvzw^-w)$. Then $z \in V(C)$ and so $vz \in \Gamma$ and $z^+z^- \in E(G)$. Without loss of generality assume that $z \in C(x^+, w^-)$, we obtain a longer cycle than C by replacing z^-zz^+ with z^-z^+ and w^-w with w^-zvw. This contradiction shows $vw \notin E(C_0) \cap E(C_1)$. Similarly, $vx \notin E(C_0) \cap E(C_1)$. Thus $xw \in E(C_0) \cap E(C_1)$. □

Let $C_1 = (xww^-zx)$. Then we have the following fact.

Claim 6. *Without loss of generality, we may assume that $z = x^+$, and $C_1 = (xww^-x^+x)$.*

Proof. Otherwise, without loss of generality assume that $z \in C(x^+, w^-)$. Then $|C(x^+, z)| \geqslant 1$. We have $z^-z^+ \notin E(G)$ since otherwise we can obtain a longer cycle than C by replacing z^-zz^+ with z^-z^+, x^-xx^+ with x^-x^+, and w^-w with w^-zxvw. Obviously, $z^-w^- \notin E(G)$ since otherwise G has a longer cycle $C[x^+, z^-]C^-[w^-, z]xvC[w, x^-]x^+$ than C. From $G[z, x, z^-, w^-] \neq K_{1,3}$, we have $xz^- \in E(G)$. Replacing x^-xx^+ with x^-x^+ and z^-z with z^-xz, we obtain a new cycle C' of the same length as C but xz is an edge on C'. Thus without loss of generality, we may assume that $z = x^+$, and so $C_1 = (xww^-x^+x)$. Therefore, Claim 6 is true. □

We next complete the proof of Theorem 2.6. By Claims 5 and 6, we have $x^+x^-, w^-w^+ \in E(G)$. It is easy to see that $|C(x^+, w^-)| \geqslant 1$ and $|C(w^+, x^-)| \geqslant 1$. Let $x^{++} = (x^+)^+$. Obviously, $x^-x^{++} \notin E(G)$ since otherwise we can obtain a longer cycle by replacing $x^-xx^+x^{++}$ with x^-x^{++} and w^-w with w^-x^+xvw. We have $x^-w^- \notin E(G)$ since otherwise G has a longer cycle $C[x, w^-]C^-[x^-, w]vx$ than C. Similarly, $x^+w^+ \notin E(G)$. From $G[x^+, x^-, x^{++}, w^-] \neq K_{1,3}$, we have $x^{++}w^- \in E(G)$ (see Fig. 2). Let $H = G[V(C_1) \cup \{x^{++}, w^+, v, x^-\}]$ (see Fig. 2). Since G is a closed claw-free graph, it is easy to check that $G[N(x)]$, $G[N(x^+)]$, $G[N(w^-)]$ and $G[N(w)]$ are not connected. We have that $vx^{++} \notin E(G)$ since otherwise G has a longer cycle $C[x^{++}, x^-]x^+xvx^{++}$than C.

If $x^-w^- \in E(G)$, then G has a longer cycle $C[w, x^-]w^-C^-[w^-, x]vw$ than C. This contradiction shows that $x^-w^- \notin E(G)$. Similarly, we can prove that

$$x^-w^-, w^+x^{++}, x^-x^{++}, vx^-, w^+v, w^+x^- \notin E(G).$$

Note that it is possible that $x^-w^+ \in E(G)$. Thus H is isomorphic to G_1 or G_2 in Fig. 1, a contradiction and the proof of Theorem 2.6 has been completed. □

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