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New sufficient condition for Hamiltonian graphs

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Abstract

Let G be a graph, and $\delta(G)$ and $\alpha(G)$ be the minimum degree and the independence number of G, respectively. For a vertex $v \in V(G)$, $d(v)$ and $N(v)$ represent the degree of v and the neighborhood of v in G, respectively. A number of sufficient conditions for a connected simple graph G of order n to be Hamiltonian have been proved. Among them are the well known Dirac condition (1952) ($\delta(G) \geq \frac{n}{2}$) and Ore condition (1960) (for any pair of independent vertices u and v, $d(u) + d(v) \geq n$). In 1984 Fan generalized these two conditions and proved that if G is a 2-connected graph of order n and $\max\{d(x), d(y)\} \geq n/2$ for each pair of nonadjacent vertices x, y with distance 2 in G, then G is Hamiltonian. In 1993, Chen proved that if G is a 2-connected graph of order n, and if $\max\{d(x), d(y)\} \geq n/2$ for each pair of nonadjacent vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha(G) - 1$, then G is Hamiltonian. In 1996, Chen, Egawa, Liu and Saito further showed that if G is a k-connected graph of order n, and if $\max\{d(v) : v \in S\} \geq n/2$ for every independent set S of G with $|S| = k$ which has two distinct vertices $x, y \in S$ such that the distance between x and y is 2, then G is Hamiltonian. In this paper, we generalize all the above conditions and prove that if G is a k-connected graph of order n, and if $\max\{d(v) : v \in S\} \geq n/2$ for every independent set S of G with $|S| = k$ which has two distinct vertices $x, y \in S$ satisfying $1 \leq |N(x) \cap N(y)| \leq \alpha(G) - 1$, then G is Hamiltonian.

Keywords: Hamiltonian graphs; Dirac condition; Ore condition; Fan condition; Chen condition

1. Introduction

We consider finite and simple graphs in this paper; undefined notations and terminology can be found in [1]. In particular, we use $V(G)$, $E(G)$, $\kappa(G)$, $\delta(G)$ and $\alpha(G)$ to denote the vertex set, edge set, connectivity, minimum degree and independence number of G, respectively. If G is a graph and $u, v \in V(G)$, then a path in G from u to v is called a (u, v)-path of G. If $v \in V(G)$ and H is a subgraph of G, then $N_H(v)$ denotes the set of vertices in H that are adjacent to v in G. Thus, $d_H(v)$, the degree of v relative to H, is $|N_H(v)|$. We also write $d(v) = d_G(v)$ and $N(v) = N_G(v)$. If C and H are subgraphs of G, then $N_C(H) = \cup_{u \in V(H)} N_C(u)$, and $G - C$ denotes the subgraph of G induced by $V(G) - V(C)$. For vertices $u, v \in V(G)$, the distance between u and v, denoted $d(u, v)$, is the length of a shortest (u, v)-path in G, or ∞ if no such path exists.

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Let $C_m = x_1x_2\cdots x_mx_1$ denote a cycle of order m. Define

$$N^+_{C_m}(u) = \{x_{i+1} : x_i \in N_{C_m}(u)\}, N^-_{C_m}(u) = \{x_{i-1} : x_i \in N_{C_m}(u)\},$$

and define $N^{\pm}_{C_m}(u) = N^+_{C_m}(u) \cup N^-_{C_m}(u)$, where subscripts are taken modulo m. A subset $S \subseteq V(G)$ is said to be an *essential* independent set if S is an independent set in G and there exist two distinct vertices $x, y \in S$ with $d(x, y) = 2$.

The following sufficient conditions to assure the existence of a Hamiltonian cycle in a simple graph G of order $n \geq 3$ are well known.

Theorem 1.1 (*Dirac [4]*). *If $\delta(G) \geq n/2$, then G is Hamiltonian.*

Theorem 1.2 (*Ore [6]*). *If $d(u)+d(v) \geq n$ for each pair of nonadjacent vertices $u, v \in V(G)$, then G is Hamiltonian.*

Theorem 1.3 (*Fan [5]*). *If G is a 2-connected graph and if $\max\{d(u), d(v)\} \geq n/2$ for each pair of vertices $u, v \in V(G)$ with $d(u, v) = 2$, then G is Hamiltonian.*

Theorem 1.4 (*Chen [2]*). *If G is a 2-connected graph and if $\max\{d(u), d(v)\} \geq n/2$ for each pair of vertices $u, v \in V(G)$ with $1 \leq |N(u) \cap N(v)| \leq \alpha(G) - 1$, then G is Hamiltonian.*

Theorem 1.5 (*Chen et al. [3]*). *If G is a k-connected ($k \geq 2$) graph and if $\max\{d(v) : v \in S\} \geq n/2$ for every independent set S of order k such that S has two distinct vertices x, y with $d(x, y) = 2$, then G is Hamiltonian.*

The purpose of this paper is to unify and extend the theorems above. We shall prove the following result.

Theorem 1.6. *If G is a k-connected ($k \geq 2$) graph of order n, and if $\max\{d(v) : v \in S\} \geq n/2$ for every independent set S of order k, such that S has two distinct vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha(G)-1$, then G is Hamiltonian.*

The proof of Theorem 1.6 will be given in Section 2. It is straightforward to verify that if a graph G satisfies the hypothesis of any one of Theorems 1.1–1.5, then it will also satisfy the hypothesis of Theorem 1.6. In Section 3, we shall show that there exist Hamiltonian graphs satisfying the hypothesis of Theorem 1.6, but whose Hamiltonicity cannot be assured by any one of Theorems 1.1–1.5. In this sense Theorem 1.6 extends Theorems 1.1–1.5.

2. Proof of Theorem 1.6

For a cycle $C_m = x_1x_2\cdots x_mx_1$, we write $[x_i, x_j]$ to denote the section $x_ix_{i+1}\cdots x_j$ of the cycle C_m, where subscripts are taken modulo m. For notational convenience, $[x_i, x_j]$ will denote the (x_i, x_j)-path $x_ix_{i+1}\cdots x_j$ of C_m, as well as the vertex set of this path. If C_1 and C_2 are cycles of a graph G such that $V(C_1) \subset V(C_2)$ and $|V(C_2)| > |V(C_1)|$, then we say that C_2 *extends* C_1. Suppose that P_1 is an (x, y)-path of G and P_2 is a (y, z)-path of G such that $V(P_1) \cap V(P_2) = \{y\}$, then we use P_1P_2 to denote the (x, z)-path of G induced by the edges $E(P_1) \cup E(P_2)$. If $V(P_1) \cap V(P_2) = \{x, y\}$ and $x = z$, then P_1P_2 denotes the cycle of G induced by the edges $E(P_1) \cup E(P_2)$. We need to establish some lemmas.

Lemma 2.1. *Suppose that $C_m = x_1x_2\cdots x_mx_1$ is a cycle of a graph G, H is a component of $G - V(C_m)$, and x_i, x_j are distinct vertices in $N_{C_m}(H)$. If G does not have a cycle extending C_m, then each of the following holds.*

(i) $\{x_{i-1}, x_{i+1}, x_{j-1}, x_{j+1}\} \cap N_{C_m}(H) = \emptyset$.
(ii) $x_{i+1}x_{j+1} \notin E(G)$ *and* $x_{i-1}x_{j-1} \notin E(G)$.
(iii) *If* $x_tx_{j+1} \in E(G)$ *for some vertex* $x_t \in [x_{j+2}, x_i]$, *then* $x_{t-1}x_{i+1} \notin E(G)$ *and* $x_{t-1} \notin N_{C_m}(H)$.
(iv) *If* $x_tx_{j+1} \in E(G)$ *for some vertex* $x_t \in [x_{i+1}, x_j]$, *then* $x_{t+1}x_{i+1} \notin E(G)$.
(v) *No vertex of* $G - (V(C_m) - V(H))$ *is adjacent to both* x_{i+1} *and* x_{j+1}.
(vi) *If* $x \in V(H)$ *such that* $xx_i \in E(G)$, *then* $\{x\} \cup N^+_{C_m}(H)$ *must be an independent set.*

Proof. (i), (ii) and (v) follow immediately from the assumption that G does not have a cycle extending C_m. It remains to show that (iii), (iv) and (vi) must also hold. Since $x_i, x_j \in N_{C_m}(H)$, $\exists x'_i, x'_j \in V(H)$ such that $x_ix'_i, x_jx'_j \in E(G)$. Let P' denote an (x'_i, x'_j)-path in H.

Suppose that (iii) fails. Then there exists a vertex $x_t \in \{x_{j+2}, x_{j+3}, \ldots, x_i\}$ satisfying $x_t x_{j+1} \in E(G)$. If $x_{t-1}x_{i+1} \in E(G)$, then $x_i P' x_j [x_{j-1}, x_{t-1}][x_{t-2}, x_t][x_{t+1}, x_i]$ is a longer cycle than C_m, contrary to the assumption that C_m is longest. Hence $x_t x_{j+1} \notin E(G)$. Next we assume that x_{t-1} is adjacent to some vertex $x'_{t-1} \in V(H)$. Let P'' denote an (x'_{t-1}, x'_j)-path in H. Then $x_j P'' x_{t-1}[x_{t-2}, x_t][x_{t+1}, x_j]$ is a cycle extending C_m, a contradiction. Hence (iii) must hold. The proof for (iv) is similar, and so it is omitted.

To prove (vi), assume to the contrary, that $G[\{x\} \cup N^+_{C_m}(H)]$ has an edge e. By Lemma 2.1(i), $N^+_{C_m}(H)$ is an independent set. Hence $e = xx_{j+1}$ for some $x_j \in N_{C_m}(H)$, and so $\exists x'_j \in V(H)$ such that $x_j x'_j \in E(G)$. Let P denote an (x'_j, x)-path in H. Then $x[x_{j+1}, x_j]P$ is a cycle extending C_m. This completes the proof of the lemma. □

Lemma 2.2. *Let G be a 2-connected graph of order n, $C_m = x_1x_2\cdots x_m x_1$ be a cycle of G, and H be a component of $G - V(C_m)$. If one of the following holds,*

(i) *there exist two distinct vertices $x_i, x_j \in V(C_m)$ with x_{i+1}, x_{j+1} in $N^+_{C_m}(H)$ such that $d(x_{i+1}) \geq n/2$ and $d(x_{j+1}) \geq n/2$, or*

(ii) *there exists a vertex $x_{i+1} \in N^+_{C_m}(H)$ and a vertex $y \in V(H)$ such that $d(x_{i+1}) \geq n/2$ and $d(y) \geq n/2$,*

then there exists a cycle C^* extending C_m.

Proof. Suppose, to the contrary, that

$$G \text{ does not have a cycle } C^* \text{ extending } C_m. \tag{1}$$

First we assume that (i) holds. Then $\exists x'_i, x'_j \in V(H)$ such that $x_i x'_i, x_j x'_j \in E(G)$. Since H is connected, H has an (x'_i, x'_j)-path P.

We define a map $f : N_G(x_{j+1}) - \{x_j\} \mapsto V(G)$ as follows: $\forall v \in N_G(x_{j+1}) - \{x_j\}$,

$$f(v) = \begin{cases} v & \text{if } v \notin V(C_m) \\ v^+ = x_{t+1} & \text{if } v = x_t \in [x_{i+1}, x_{j-1}] \\ v^- = x_{t-1} & \text{if } v = x_t \in [x_{j+2}, x_i]. \end{cases}$$

Claim 1. f is an injection, and $\forall v \in N_G(x_{j+1}) - \{x_j\}$, $f(v) \notin N_G(x_{i+1}) \cup \{x_{x+1}\}$.

It is straightforward to verify that f is an injection. Firstly, by (1), $\forall v \in V(H)$, $x_{i+1}f(v) = x_{i+1}v \notin E(G)$. If $v = x_t \in [x_{i+1}, x_{j-1}]$ with $x_t x_{j+1} \in E(G)$, then by Lemma 2.1(iv), $x_{t+1}x_{i+1} \notin E(G)$. If $\exists x_t \in [x_{j+2}, x_{i-1}]$ with $x_t x_{j+1} \in E(G)$, then by Lemma 2.1(iii), $x_{t-1}x_{i+1} \notin E(G)$. Finally, the definition of f shows that $\forall v \in N_G(x_{j+1}) - \{x_j\}$, $f(v) \neq x_{i+1}$. This proves Claim 1.

By Claim 1, $f(N_G(x_{j+1}) - \{x_j\}) \cap N_G(x_{i+1}) = \emptyset$. By Lemma 2.1(i), $f(N_G(x_{j+1}) - \{x_j\}) \cap V(H) = \emptyset$. Therefore, $N_G(x_{i+1}) \cap (f(N_G(x_{j+1}) - \{x_j\}) \cup V(H) \cup \{x_{i+1}\}) = \emptyset$. Since f is an injection, $|f(N_G(x_{j+1}) - \{x_j\})| = |N_G(x_{j+1})| - 1 = d(x_{j+1}) - 1$. It follows that

$$d(x_{i+1}) = |N_G(x_{i+1})| \leq n - |f(N_G(x_{j+1}))| - |V(H)| - |\{x_{i+1}\}| \leq n - d(x_{j+1}) - |V(H)|,$$

and so $d(x_{i+1}) + d(x_{j+1}) \leq n - 1$, contrary to the assumption that both $d(x_{i+1}) \geq n/2$ and $d(x_{j+1}) \geq n/2$. Thus if (i) holds, there exists a cycle C^* extending C_m.

The proof for the case when (ii) holds is similar, and so it is omitted. □

Lemma 2.3. *Let G be a k-connected ($k \geq 2$), $C_m = x_1x_2\cdots x_m x_1$ be a cycle of G with $|V(C_m)| < |V(G)|$ such that G does not have a cycle extending C_m, and let H be a component of $G - V(C_m)$. Let $x \in V(H)$ be a vertex satisfying $|N_{C_m}(x)| \geq 1$ and $d(x) < n/2$. Then*

(i) $|N_{C_m}(H)| \geq k$.

Moreover, if $x_i \in N_{C_m}(x)$, then each of the following holds.

(ii) $1 \leq |N(x) \cap N(x_{i+1})| \leq \alpha(G) - 1$.

Proof. Since C_m is a longest cycle with $|V(C_m)| < |V(G)|$, $V(C_m) - N_{C_m}(H) \neq \emptyset$, and so $N_{C_m}(H)$ separates $V(H)$ and $V(C_m) - N_{C_m}(H)$. Since G is k-connected, $|N_{C_m}(H)| \geq \kappa(G) \geq k$. This proves Lemma 2.3(i).

By Lemma 2.1(vi), $\{x\} \cup N^+_{C_m}(H)$ must be an independent set. Thus $|N_{C_m}(H)| = |N^+_{C_m}(H)| \leq \alpha(G) - 1$, and so $1 \leq |N(x) \cap N(x_{i+1})| \leq \alpha(G) - 1$. This proves Lemma 2.3(ii). □

Lemma 2.4. *Let G be a k-connected ($k \geq 2$), $C_m = x_1x_2\cdots x_mx_1$ be a cycle of G with $|V(C_m)| < |V(G)|$ such that G does not have a cycle extending C_m, and H be a component of $G - V(C_m)$. If G satisfies the hypothesis of Theorem 1.6, then $d(x) \geq n/2$ for every $x \in V(H)$ with $|N_{C_m}(x)| \geq 1$.*

Proof. By contradiction, we assume that there exists an $x \in V(H)$ satisfying $|N_{C_m}(x)| \geq 1$ and $d(x) < n/2$.

We assume that $x_i \in N_{C_m}(x)$. By Lemma 2.3(i), $\exists V^* \subseteq N^+_{C_m}(H)$ with $x_{i+1} \in V^*$ and with $|V^*| = k - 1$. By Lemma 2.1(vi), $\{x\} \cup N^+_{C_m}(H)$ is an independent set. It follows by Lemma 2.3(ii) that $V^* \cup \{x\}$ is a k-element set satisfying the hypothesis of Theorem 1.6. Since $d(x) < n/2$, by the hypothesis of Theorem 1.6, there must be a vertex (say x_{h+1}) in V^* with $d(x_{h+1}) \geq n/2$. By Lemma 2.2(i), every vertex of $N^+_{C_m}(H) - \{x_{h+1}\}$ must have degree less than $n/2$. Since G is k-connected ($k \geq 2$), there must be some vertex $y \in V(H)$ and $x_j \in V(C_m)$ with $x_{j+1} \in N^+_{C_m}(y) - \{x_{h+1}\}$ (possibly $y = x$).

By Lemma 2.3 (with y replacing x in Lemma 2.3), we have $1 \leq |N(y) \cap N(x_{j+1})| \leq \alpha(G) - 1$. Hence pick a subset $X \subseteq N^+_{C_m}(H) - \{x_{h+1}, x_{j+1}\}$ with $|X| = k - 2$ and let $V^{**} = X \cup \{y, x_{j+1}\}$. Then V^{**} also satisfies the hypothesis of Theorem 1.6. Hence there must be a vertex $u \in V^{**}$ such that $d(u) \geq n/2$. Since $d(x_{h+1}) \geq n/2$ and $d(u) \geq n/2$, it follows by Lemma 2.2(i) that G has a cycle C^* extending C_m, contrary to the assumption that C_m has no extension in G. □

Lemma 2.5. *Let G be a k-connected ($k \geq 2$) graph, $C_m = x_1x_2\cdots x_mx_1$ be a cycle of G with $|V(C_m)| < |V(G)|$ such that G has no cycle extending C_m, and let H be a component of $G - V(C_m)$. If $d(x) \geq n/2$ for every $x \in V(H)$ with $|N_{C_m}(x)| \geq 1$, then $H = G - V(C_m)$.*

Proof. By contradiction, we assume that $G - V(C_m)$ has at least two distinct components H and H^*.

Since G is connected, there must exist a vertex $y \in V(H^*)$ adjacent to some vertex of C_m. By Lemma 2.4, $d(y) \geq n/2$. It follows that $d(x)+d(y) \geq n$. On the other hand, if a vertex $x \in V(H)$ is adjacent to some $x_i \in V(C_m)$, then x is not adjacent to x_{i+1} and x_{i-1}. Hence we have $d(x) \leq |V(C_m)|/2 + |V(H) - \{x\}|$. Similarly, we have $d(y) \leq |V(C_m)|/2 + |V(H^*) - \{y\}|$. It follows that $d(x) + d(y) \leq [|V(C_m)|/2 + |V(H) - \{x\}|] + [|V(C_m)|/2 + |V(H^*) - \{y\}|] \leq n - 2$, a contradiction. Hence we must have $H = G - V(C_m)$. □

Proof of Theorem 1.6. Suppose, to the contrary, that G is not Hamiltonian. Let $C_m = x_1x_2\cdots x_mx_1$ be a longest cycle of G, and let H be a component of subgraph $G - V(C_m)$. By Lemma 2.4, we may assume that

$$d(x) \geq n/2, \forall x \in V(H) \quad \text{with } |N_{C_m}(x)| \geq 1. \tag{2}$$

By Lemma 2.5, $H = G - V(C_m)$. Without loss of generality, assume that for some $x \in V(H)$, there exists an $x_i \in N_{C_m}(x)$.

By Lemma 2.3(i) and since $k \geq 2$,

$$|N_{C_m}(H)| \geq 2. \tag{3}$$

Choose $x_i, x_j \in N_{C_m}(H)$ such that

$$\{x_{i+1}, x_{i+2}, \ldots x_{j-1}\} \cap N_{C_m}(H) = \emptyset. \tag{4}$$

If $|V(H)| = 1$, then $C_m = C_{n-1}$ and $|N_{C_m}(H)| = d(x) \geq n/2$. Then there must exist $x_i, x_{i+1} \in N_G(x) \cap V(C_{n-1})$, and so G is Hamiltonian, a contradiction. Hence we must have

$$|V(H)| \geq 2. \tag{5}$$

$\forall x \in V(H)$, $N_G(x) \cup \{x\} \subseteq V(H) \cup N_{C_m}(H)$, and so by (2),

$$|V(H)| + |N_{C_m}(H)| \geq |N_G(x) \cup \{x\}| \geq d(x) + 1 \geq n/2 + 1. \tag{6}$$

It follows by (4)–(6) that

$$\begin{aligned}
|\{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\}| &\leq (n - |V(H)| - |N_{C_m}(H)|)/|N_{C_m}(H)| \\
&\leq [n - (|V(H)| + |N_{C_m}(H)|)]/|N_{C_m}(H)| \\
&\leq [n - (1 + n/2)]/|N_{C_m}(H)| \leq (n/2 - 1)/|N_{C_m}(H)| \\
&\leq (|V(H)| + |N_{C_m}(H)| - 2)/|N_{C_m}(H)| \\
&\leq |V(H)|/|N_{C_m}(H)| + 1 - 2/|N_{C_m}(H)| < |V(H)|.
\end{aligned}$$

In other words,

$$|\{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\}| < |V(H)|. \tag{7}$$

Recall that we have chosen $x_i, x_j \in N_{C_m}(H)$ satisfying (3). Let $x_i', x_j' \in V(H)$ with $x_i x_i', x_j x_j' \in E(G)$, and let P be an (x_i', x_j')-path of H. Then $L = x_j[x_{j+1}, x_i]Px_j$ is a cycle of G. Choose a cycle C' of G such that

$$V(L) \subseteq V(C') \text{ and } |V(C')| \text{ is maximized.} \tag{8}$$

By (8) and by applying Lemmas 2.3–2.5 to the cycle C', we obtain the following Claim 2.

Claim 2. Let H' be a component of $G - V(C')$.

(i) Then $\exists x' \in V(H')$ with $N_{C'}(x) \neq \emptyset$ and with $d(x') \geq n/2$.
(ii) $G - V(C') = H'$ has only one component.

Claim 3. $V(H') \cup \{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\} = \emptyset$, or $V(H') \cup V(H) = \emptyset$.

By Lemma 2.5, $H = G - V(C_m)$, and so $V(H') \subseteq \{x_{i+1}, x_{i+2}, \ldots x_{j-1}\} \cap V(H)$. If for some $x', x'' \in V(H')$ such that $x' \in \{x_{i+1}, x_{i+2}, \ldots x_{j-1}\}$ and $x'' \in V(H)$, then since H' is connected, H' has an (x', x'')-path Q. Since $V(H') \subseteq \{x_{i+1}, x_{i+2}, \ldots x_{j-1}\} \cap V(H)$, there is an edge in Q joining a vertex in $\{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\}$ and a vertex in $V(H)$, contrary to (3). This proves Claim 3.

If $V(H') \cap \{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\} = \emptyset$, then by Claim 2(ii), $V(C_m) \subseteq V(C')$. By (8), $V(H) \cap V(C') \neq \emptyset$, and so C' extends C_m, contrary to the assumption that C_m is a longest cycle. Thus by Claim 3, $V(H') \cap V(H) = \emptyset$, and so $V(H') \subseteq \{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\}$. By (7), $|V(H)| > |\{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\}| \geq |V(H')|$. By Lemma 2.5 and Claim 2(ii), we have

$$|V(C')| = n - |V(H')| > n - |V(H)| = |V(C_m)|,$$

contrary to the assumption that C_m is a longest cycle. Therefore, the proof of Theorem 1.6 is complete. □

3. Examples

The purpose of this section is to show that there exist Hamiltonian graphs satisfying the hypothesis of Theorem 1.6, but whose Hamiltonicity cannot be assured by any one of Theorems 1.1–1.5.

Let H and K be two vertex disjoint graphs. As in [1], $H \cup K$ denotes the disjoint union of H and K, and $H \vee K$ denotes the graph obtained from the disjoint union of H and K by adding all the edges in $\{uv : u \in V(H) \text{ and } v \in V(K)\}$. Similarly, for two disjoint vertex subsets Z_1 and Z_2, we define $Z_1 \vee Z_2$ to be the graph whose vertex set is $Z_1 \cup Z_2$ and whose edge set is $\{v_1v_2 : v_1 \in Z_1 \text{ and } v_2 \in Z_2\} \cup (E(Z_1) \cup E(Z_2))$. If Z denotes a vertex subset, let $K(Z)$ denote the complete graph whose vertex set is Z. If $Z' \subseteq E(H)$ and Z'' is an edge set not in H but the two ends of each edge in Z'' are in H, then $H - Z' + Z''$ denotes the graph obtained from H by deleting the edges in Z' and adding the edges in Z''.

Let $H_1, H_2, \ldots, H_6$ be vertex disjoint graphs each of which is isomorphic to a K_3. For each i with $1 \leq i \leq 6$, pick a vertex $x_i \in V(H_i)$. Let $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ denote a vertex set disjoint from $\cup_{i=1}^{6} V(H_i)$, and let K_6^c denote the edgeless graph with $V(K_6^c) = Y$. Define edge subsets as follows:

$$W_1' = E(\{x_6\} \vee (Y - \{y_6\}))$$
$$W_2' = E(\{y_6\} \vee (\cup_{j=1}^{3}(V(H_j) - \{x_j\})))$$
$$W_3' = E(K(\{x_1, x_2, x_3, x_4, x_5\}))$$
$$W_4' = E(K(\cup_{j=4}^{6} V(H_j) - \{x_6\}))$$
$$W_5' = E((V(H_5) - \{x_5\}) \vee (\cup_{j=1}^{3}(V(H_j) - \{x_j\}))).$$

Define

$$L = (K_6^c \vee (\cup_{i=1}^{6} H_i)) - (W_1' \cup W_2') + (W_3' + W_4' + W_5').$$

Table 1
Degrees of all the vertices in L

The vertex v	$N_L(v)$	$d(v)$	Remark
x_i	$(V(H_i) - \{x_i\}) \cup Y \cup (X - \{x_i, x_6\})$	12	$1 \le i \le 3$
x_i', x_i''	$(V(H_i) - \{v\}) \cup (Y - \{y_6\}) \cup \{x_5', x_5''\}$	9	$1 \le i \le 3$
x_4	$(\cup_{j=4}^{6} V(H_j) \cup Y \cup X) - \{x_4, x_6\}$	16	
x_4', x_4''	$(\cup_{j=4}^{6} V(H_j) \cup Y) - \{v, x_6\}$	13	
x_5	$(\cup_{j=4}^{6} V(H_j) \cup Y \cup X) - \{x_5, x_6\}$	16	
x_5', x_5''	$(\cup_{j=1}^{6} V(H_j) \cup Y) - ((X - \{x_5\}) \cup \{v\})$	19	
x_6	$\{x_6', x_6'', y_6\}$	3	
x_6', x_6''	$(\cup_{j=4}^{6} V(H_j)) \cup Y - \{v\}$	14	
y_i	$\cup_{j=1}^{6} V(H_j) - \{x_6\}$	17	$1 \le i \le 5$
y_6	$(\cup_{i=4}^{6} V(H_i)) \cup \{x_1, x_2, x_3\}$	12	

For each i with $1 \le i \le 6$, let $V(H_i) = \{x_i, x_i', x_i''\}$. Then, for each vertex $v \in V(L)$, the neighborhood of v can be expressed as in Table 1.

We have the following observations.

Proposition 3.1. *Each of the following holds.*

(i) $|V(L)| = 24$ *and* $\kappa(L) = 3$, *and* $\alpha(L) = 6$.
(ii) L *satisfies the hypothesis of Theorem* 1.6 *with* $k = 3$.
(iii) L *does not satisfy the hypothesis of any one of Theorems* 1.1–1.5.

Proof. (i) We only need to show that $\alpha(L) = 6$. Since Y is an independent set in L, we have $\alpha(L) \ge 6$. Let $S \subset V(L)$ be an independent set of L. Note that for each i with $1 \le i \le 5$, $N_L(y_i) = \cup_{i=1}^{6} V(H_i) - \{x_6\}$, and $N_L(y_6) = \cup_{i=4}^{6} V(H_i) \cup \{x_1, x_2, x_3\}$. Therefore, if for some $y_i \in S$ with $1lei \le 5$, then $S \subseteq Y \cup \{x_6\}$, and so as $y_6 x_6 \in E(L)$, $|S| \le 6$. Hence we assume that $Y \cap S \subseteq \{y_6\}$. If $y_6 \notin S$, then since each H_i is a complete graph, $|S \cap V(H_i)| \le 1$, and so $|S| \le 6$. Hence $y_6 \in S$. Hence in this case, $((Y - \{y_6\}) \cup N_L(y_6)) \cap S = \emptyset$. But for $i = 1, 2, 3$, since H_i is complete, $|S \cap V(H_i)| \le 1$, and so $|S| \le 4$. This implies that $\alpha(L) = 6$, and so proves (i).

(ii) As shown in Table 1, $(\cup_{i=1}^{3} V(H_i) - X) \cup \{x_6\}$ is the set of all vertices of degree less than $|V(L)|/2 = 12$ in L. For any two distinct, nonadjacent vertices $u, v \in (\cup_{i=1}^{3} V(H_i) - X)$, we have, by Table 1,

$$|N_L(u) \cap N_L(v)| \ge |Y - \{y_6\}| + |\{x_5', x_5''\}| \ge 7.$$

Moreover, $\forall v \in (\cup_{i=1}^{3} V(H_i) - X) \cup \{x_6\}$, by Table 1,

$$N_L(x_6) \cap N_L(v) = \emptyset.$$

It follows that if $S \subset V(L)$ is an independent with $|S| = 3$ such that for some $u, v \in S$, it holds that $1 \le |N_L(u) \cap N_L(v)| \le \alpha(L) - 1 = 5$, then one vertex in S must have degree at least $12 = |V(L)|/2$. Hence L satisfies the condition of Theorem 1.6.

(iii) By Table 1, it is clear that the hypotheses of Theorems 1.1–1.3 cannot be satisfied. Let $S_1 = \{x_1', x_6, y_6\}$. As

$$N_L(x_1') \cap N_L(y_6) = \{x_1, x_5', x_5''\}, \qquad \text{and} \qquad |N_L(x_6)| \le 3,$$

S_1 is a vertex set of L such that $\forall u, v \in S_1$, $|N_L(u) \cap N_L(v)| \le \alpha(L) - 1$. As $d_L(x_6) = 3$ and $d_L(x_1') = 8$, L does not satisfy the condition of Theorem 1.4.

Let $S_2 = \{z_1, z_2, z_3\} \subset (\cup_{i=1}^{3} V(H_i) - X)$ be an independent set of L. Then as $z_1, z_2, z_3 \in N_L(y_1)$, the distance between z_1, z_2 in L is 2. By Table 1, $\max\{d_L(v) : v \in S_2\} < 11$, and so L does not satisfy the condition of Theorem 1.5. □

The same construction also works for an arbitrary integer $k \ge 3$. Let $H_1, H_2, \ldots, H_{2k}$ be vertex disjoint graphs, each of which is isomorphic to a K_k. For each i with $1 \le i \le 2k - 1$, pick two distinct vertices $x_i', x_i'' \in V(H_i)$, and

Table 2
Degrees of all the vertices in L_k

The vertex v	$N_{L_k}(v)$	$d(v)$	Remark
$v \in V(H_i) - \{x_i', x_i''\}$	$(\cup_{j=1}^{2k-1}(V(H_j) - \{x_j', x_j''\})) \cup Y \cup (V(H_i) - \{v\})$	$2k(k-1)$	$1 \le i \le k$
x_i', x_i''	$(V(H_i) - \{v\}) \cup (Y - \{y_6\}) \cup \{x_{2k-1}', x_{2k-1}''\}$	$3k$	$1 \le i \le k$
$v \in V(H_i) - \{x_i', x_i''\}$	$(\cup_{j=1}^{k}(V(H_j) - \{x_j', x_j''\})) \cup (\cup_{j=k+1}^{2k} V(H_j)) \cup Y - \{v, x_{2k}\}$	$2(k^2-1)$	$k+1 \le i \le 2k-2$
x_i', x_i''	$\cup_{j=k+1}^{2k} V(H_j) \cup Y - \{v, x_{2k}\}$	k^2+2k-2	$k+1 \le i \le 2k-2$
$v \in V(H_{2k-1}) - \{x_{2k-1}', x_{2k-1}''\}$	$(\cup_{j=1}^{k}(V(H_j) - \{x_j', x_j''\})) \cup (\cup_{j=k+1}^{2k} V(H_j)) \cup Y - \{v, x_{2k}\}$	$2(k^2-1)$	
x_{2k-1}', x_{2k-1}''	$(\cup_{j=1}^{k}\{x_j', x_j''\} \cup (\cup_{j=k+1}^{2k} V(H_j) \cup Y) - \{v, x_{2k}\})$	k^2+4k-2	
x_{2k}	$\{y_6\} \cup V(H_{2k}) - \{x_{2k}\}$	k	
$v \in V(H_{2k}) - \{x_{2k}\}$	$(\cup_{j=k+1}^{2k} V(H_j)) \cup Y - \{v\}$	k^2+2k-1	
y_i	$\cup_{j=1}^{2k} V(H_j) - \{x_{2k}\}$	$2k^2-1$	$1 \le i \le 2k-1$
y_{2k}	$(\cup_{j=k+1}^{2k}(V(H_j))) \cup (\cup_{j=1}^{k}(V(H_j) - \{x_j', x_j''\}))$	$2k^2-2k$	

let $x_{2k} \in V(H_{2k})$. Let $Y = \{y_1, y_2, \ldots, y_{2k}\}$ denote a vertex set disjoint from $\cup_{i=1}^{2k} V(H_i)$, and let K_{2k}^c denote the edgeless graph with $V(K_{2k}^c) = Y$. Define edge subsets as follows:

$$W_1 = E(\{x_{2k}\} \vee (Y - \{y_{2k}\})),$$
$$W_2 = E(\{y_{2k}\} \vee (\cup_{j=1}^{k}(\{x_j', x_j''\}))),$$
$$W_3 = E(K(\cup_{j=1}^{2k-1}(V(H_j) - \{x_j', x_j''\}))),$$
$$W_4 = E(K(\cup_{j=k+1}^{2k} V(H_j) - \{x_{2k}\})),$$
$$W_5 = E((\{x_{2k-1}', x_{2k-1}''\}) \vee (\cup_{j=1}^{k}\{x_j', x_j''\})).$$

Define

$$L_k = (K_{2k}^c \vee (\cup_{i=1}^{2k} H_i)) - (W_1 \cup W_2) + (W_3 + W_4 + W_5).$$

Note that $L_3 = L$ above. For each vertex $v \in V(L_k)$, the neighborhood of v can be expressed as in Table 2.

Imitating the proof for Proposition 3.1, we have the following Proposition 3.2.

Proposition 3.2. *Each of the following holds.*

(i) $|V(L_k)| = 2k(k+1)$ *and* $\kappa(L_k) = k$, *and* $\alpha(L_k) = 2k$.
(ii) L_k *satisfies the hypothesis of Theorem* 1.6 *with* $\kappa(L_k) = k$.
(iii) L_k *does not satisfy the hypothesis of any one of Theorems* 1.1–1.5.

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