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Note

Conditional colorings of graphs

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Abstract

For an integer $r > 0$, a *conditional* (k, r)*-coloring* of a graph G is a proper k-coloring of the vertices of G such that every vertex of degree at least r in G will be adjacent to vertices with at least r different colors. The smallest integer k for which a graph G has a conditional (k, r)-coloring is the rth order conditional chromatic number $\chi_r(G)$. In this paper, the behavior and bounds of conditional chromatic number of a graph G are investigated.

Keywords: Conditional coloring; Conditional chromatic number

1. Introduction

We follow the terminology and notations of [3] and consider finite and loopless graphs. For a graph G, let $\omega(G) = \max\{k : G \text{ contains a } K_k \text{ as a subgraph}\}$. As in [3], $\delta(G)$ and $\Delta(G)$ denote the minimum degree and the maximum degree of a graph G, respectively. For a vertex $v \in V(G)$, the *neighborhood* of v in G is $N_G(v) = \{u \in V(G) : u \text{ is adjacent to } v \text{ in } G\}$. Vertices in $N_G(v)$ are called *neighbors of* v.

For an integer $k > 0$, let $\overline{k} = \{1, 2, \ldots, k\}$. A *proper k-coloring* of a graph G is a map $c : V(G) \mapsto \overline{k}$ such that if $u, v \in V(G)$ are adjacent vertices in G, then $c(u) \neq c(v)$. The smallest k such that G has a proper k-coloring is the *chromatic number* of G, denoted $\chi(G)$.

This paper considers a generalization of the classical coloring as follows. For integers $k > 0$ and $r > 0$, a *proper* (k, r)*-coloring* of a graph G is a map $c : V(G) \mapsto \overline{k}$ such that both of the following hold:

(C1) If $u, v \in V(G)$ are adjacent vertices in G, then $c(u) \neq c(v)$; and
(C2) for any $v \in V(G)$, $|c(N_G(v))| \geqslant \min\{|N_G(v)|, r\}$.

For a fixed number r, the smallest k such that G has a proper (k, r)-coloring is the (rth order) *conditional chromatic number* of G, denoted $\chi_r(G)$.

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By the definition of $\chi_r(G)$, it follows immediately that $\chi(G)=\chi_1(G)$, and so $\chi_r(G)$ is a generalization of the classical graph coloring. The purpose of this paper is to investigate the behavior of $\chi_r(G)$ and to generalize certain properties on $\chi(G)$ to $\chi_r(G)$.

2. The conditional chromatic number of certain graph families

In this section, we determine the conditional chromatic number of a certain families of graphs, including complete bipartite graphs, and cycles. Throughout this section, $r > 0$ denotes an integer.

Proposition 2.1. *Let G be a connected graph. Each of the following holds*:

(i) $\chi_r(G) \geqslant \chi_{r-1}(G) \geqslant \cdots \geqslant \chi_2(G) \geqslant \chi(G)$.
(ii) $|V(G)| \geqslant \chi_r(G) \geqslant \min\{r, \Delta(G)\} + 1$.
(iii) *Let* $n \geqslant 1$ *be an integer. Then* $\chi_r(K_n) = n$.
(iv) *If* $|V(G)| \geqslant 2$ *and* $r \geqslant 2$, *then* $\chi_r(G) = 2$ *if and only if* $G \cong K_2$.
(v) *If* $|V(G)| \geqslant 2$, *then* $\chi_1(G) = 2$ *if and only if G is a bipartite graph.*

Proof. (i) If $i > j > 0$, then any (k, i)-coloring of G is also a (k, j)-coloring of G.

(ii) Let $v \in V(G)$ be a vertex with maximum degree. If $r \geqslant \Delta(G)$, then all vertices in $N_G(v) \cup \{v\}$ must be colored with different colors; if $r < \Delta(G)$, then $N_G(v) \cup \{v\}$ must be colored with at least $r + 1$ colors. On the other hand, for any r, a $|V(G)|$-coloring of G is always a $(|V(G)|, r)$-coloring of G.

(iii) follows from (ii); and (iv) follows from (ii) and (iii). (v) is well known. □

Theorem 2.2. *If G is a tree with* $|V(G)| \geqslant 3$, *then* $\chi_r(G) = \min\{r, \Delta(G)\} + 1$.

Proof. We argue by induction on $n = |V(G)|$. For $n = 3$, G is a path of three vertices with $\Delta(G) = 2$. By Proposition 2.1(v), the theorem holds with $r = 1$, and so we assume that $r \geqslant 2$. Then by Proposition 2.1(ii) and (iv), $\chi_r(G) = 3$.

Assume that $n \geqslant 4$ and that the theorem holds for smaller values of n. Let G be a tree on n vertices and let v be a vertex of degree 1 in G such that the degree of its neighbor is minimized. By induction, $\chi_r(G - v) = k = \min\{r, \Delta(G - v)\} + 1$.

If $G \not\cong K_{1,n-1}$, then $\Delta(G) = \Delta(G - v)$, and so any (k, r)-coloring of $G - v$ can be extended to a (k, r)-coloring of G by defining $c(v)$ different from the color of its only neighbor in G. Therefore, we assume that $G = K_{1,n-1}$. Then the theorem follows by Proposition 1.2(ii). □

Theorem 2.3. *Suppose that* $m \geqslant n \geqslant 2$, *then* $\chi_r(K_{m,n}) = \min\{2r, n + m, r + n\}$.

Proof. Let (X, Y) denote the bipartition of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. Let $k = \chi_r(K_{m,n})$ and let $c : V(K_{m,n}) \mapsto \overline{k}$ be a proper (k, r)-coloring.

Suppose first that $r \geqslant m$. For any $x \in X$, by (C2), $|c(N_G(x))| \geqslant r$ and so we must color Y with at least r-colors. Similarly, we must color X with at least r colors. By (C1), for any $y \in Y$, $c(x) \neq c(y)$. Thus $k \geqslant 2r$. On the other hand, if we color vertices in X with colors $\{1, 2, \ldots, r\}$ and vertices in Y with $\{r + 1, r + 2, \ldots, 2r\}$. Then this is a proper $(2r, r)$-coloring of $K_{m,n}$. Thus $\chi_r(K_{m,n}) = 2r$.

The other two cases when $r \leqslant n$ and when $n \leqslant r \leqslant m$ can be proved similarly. □

Theorem 2.4. *If* $k \geqslant r + 1$, *then* $\chi_r(K_{i_1,\ldots,i_k}) = k$ *if each* $i_j \geqslant 1$.

Proof. The unique proper coloring of $K_{i_1,\ldots,i_k}$ with k colors is also a proper (k, r)-coloring and so $\chi_r(K_{i_1,\ldots,i_k}) = \chi(K_{i_1,\ldots,i_k}) = k$. □

Theorem 2.5. *Let* $n \geqslant 3$ *be an integer and* C_n *denote a cycle of n vertices. If* $r \geqslant 2$, *then*

$$\chi_r(C_n) = \begin{cases} 5 & \text{if } n = 5, \\ 3 & \text{if } n \equiv 0 \pmod 3, \\ 4 & \text{otherwise.} \end{cases}$$

Proof. Let $C_n = v_1 v_2 \cdots v_n v_1$, and let $k = \chi_r(C_n)$. By Proposition 2.1(ii), $\chi_r(C_n) \geqslant 3$.

Suppose first that $n \equiv 0 \pmod 3$. Define $c : V(C_n) \mapsto \overline{3}$ by

$$c^{-1}(1) = \{v_i : i \equiv 1 \pmod 3\},$$
$$c^{-1}(2) = \{v_i : i \equiv 2 \pmod 3\},$$
$$c^{-1}(3) = \{v_i : i \equiv 0 \pmod 3\}. \tag{1}$$

Then c is a proper $(3, r)$-coloring and so $\chi_r(C_n) = 3$.

Next, we assume that $n = 5$. Let $c : V(C_5) \mapsto \overline{k}$ be a proper (k, r)-coloring. Without loss of generality, we may assume that $c(v_i) = i$ for $i = 1, 2, 3$. By (C1) and (C2) at v_3, $c(v_4) \notin \{2, 3\}$. If $c(v_4) = 1$, then both neighbors of v_2 would have the same color, violating (C2) at v_5. Thus $c(v_4) \notin \{1, 2, 3\}$, and so we may assume that $c(v_4) = 4$. By (C1), $c(v_5) \notin \{1, 4\}$. By (C2) at both v_1 and v_4, $c(v_5) \notin \{2, 3\}$. Therefore, we must have $c(v_5) \notin \{1, 2, 3, 4\}$, and so $k \geqslant 5$. On the other hand, Proposition 2.1(ii) implies that $k \leqslant 5$. Hence $\chi_r(C_5) = 5$. The same argument also shows that $\chi_r(C_4) = 4$.

Finally, we assume that $n > 5$ and $n \not\equiv 0 \pmod 3$. By contradiction, we assume that $k = 3$. Let $c : V(C_n) \mapsto \overline{3}$ be a proper $(3, r)$-coloring. Without loss of generality, we may assume that $c(v_i) = i$ for $i = 1, 2, 3$. Then it forces that (1) must hold. If $n \equiv 1 \pmod 3$, then we would have $c(v_1) = 1 = c(v_n)$, contrary to (C1); if $n \equiv 2 \pmod 3$, then we would have $c(v_2) = 2 = c(v_n)$, a violation of (C2) at v_1. Therefore, we must have $k \geqslant 4$.

To show that $k = 4$, it suffices to construct a proper $(4, r)$-coloring of C_n. Suppose that $n \equiv 1 \pmod 3$. Define $c : V(C_n) \mapsto \overline{4}$ by $c^{-1}(1) = \{v_i : i \equiv 1 \pmod 3 \text{ and } i < n\}$, $c^{-1}(2) = \{v_i : i \equiv 2 \pmod 3\}$, $c^{-1}(3) = \{v_i : i \equiv 0 \pmod 3\}$, and $c(v_n) = 4$. Then c is a proper $(4, r)$-coloring of C_n. Thus $\chi_r(C_n) = 4$ in this case.

Suppose then that $n \equiv 2 \pmod 3$. Define $c : V(C_n) \mapsto \overline{4}$ by $c^{-1}(1) = \{v_i : i = 1 \text{ or both } n > i > 4 \text{ and } i \equiv 2 \pmod 3\}$, $c^{-1}(2) = \{v_i : i = 2 \text{ or both } i > 4 \text{ and } i \equiv 0 \pmod 3\}$, $c^{-1}(3) = \{v_i : i = 3 \text{ or both } i > 4 \text{ and } i \equiv 1 \pmod 3\}$, and $c(v_4) = c(v_n) = 4$. Then, as $n > 5$, c is a proper $(4, r)$-coloring of C_n, and so $\chi_r(C_n) = 4$ also. □

3. Comparison of $\chi_r(G)$ and $\chi(G)$

Proposition 2.1(i) indicates that $\chi_2(G) \geqslant \chi(G)$. In this section, we consider the problem when $\chi_2(G) = \chi(G)$, and the problem whether there exists a constant upper bound for $\chi_2(G) - \chi(G)$ that holds for all graphs G.

Defined a graph G as *normal* if $\chi_2(G) = \chi(G)$. As examples, if $n > 2$ is odd and a multiple of three, then C_n is normal; any other cycle is not normal. Any complete graph is normal. The only normal trees are K_1 and K_2.

Lemma 3.1. *If any vertex of degree greater than one is in a triangle, then G is normal.*

Proof. If a vertex is in a triangle, then its two neighbors in the triangle are adjacent and by the adjacency condition must be colored differently in any proper coloring of G. Thus, any proper coloring of G is also a dynamic coloring of G, and so $\chi_2(G) = \chi(G)$. □

The condition presented in Lemma 3.1, while sufficient for a graph to be normal, is not necessary. This is demonstrated by the following theorem, in which a method used to construct triangle-free graphs [3, Theorem 8.7, p. 129] is shown to also produce normal graphs when the initial graph is a normal graph.

Theorem 3.2. *For every $k \geqslant 1$, there exists a normal, triangle-free, k-chromatic graph.*

Proof. Let $G_1 = K_1$, $G_2 = K_2$, and $G_3 = C_9$. Suppose that $k \geqslant 3$, and assume that a normal, triangle-free, k-chromatic graph G_k has been obtained. Let $n = |V(G_k)|$.

Construct G_{k+1} from G_k by adding $n + 1$ vertices $\{u_1, \ldots, u_n, v\}$ to the vertices $\{v_1, \ldots, v_n\}$ of G_k and by joining u_i to each vertex v_j to which v_i is adjacent; v is joined to each u_i.

Assume a proper k-coloring of G_k is given. Then color u_i the same as v_i and color v a $(k + 1)$st color. Then the proof that G_{k+1} is triangle free and $\chi(G_{k+1}) = k + 1$ is the same as the proof given in [3].

Suppose that for some $k \geqslant 3$, every k-coloring of G_k is also a $(k, 2)$-coloring of G_k. We shall show that every $(k+1)$-coloring of G_{k+1} is also a $(k + 1, 2)$-coloring. Assume that a k-coloring of G_k is given. Then each vertex v_i of G_k has some neighbors of different colors, where $k \geqslant 3$. Since the neighbors of v_i are also neighbors of u_i, then u_i has some neighbors of different colors in G_{k+1}. Since each u_i is colored the same as v_i, which are not all colored the same, then v, being adjacent to each u_i, has some neighbors of different colors in G_{k+1}. Therefore, a $(k+1)$-coloring of G_{k+1} is also a $(k+1, 2)$-coloring of G_{k+1}. By induction, $\chi_2(G_k) = \chi(G_k) = k$ for all $k > 0$. □

Theorem 3.3. *Let $n = |V(G)| \geqslant 3$. If $\delta(G) > \lfloor n/2 \rfloor$, then G is normal. This bound on $\delta(G)$ is best possible.*

Proof. Suppose $n \geqslant 3$. For any vertex v, a neighbor of v not adjacent to another neighbor of v would be adjacent to at most $n - \delta(G) \leqslant \lceil n/2 \rceil - 1 < \delta(G)$ vertices. Thus, any two adjacent vertices are in a triangle. Hence, by Lemma 3.1, G is normal.

To see that this bound is best possible, we examine the graph $G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ for $n \geqslant 4$. Then $\delta(G) = \lfloor n/2 \rfloor$. By Theorem 2.3, both $\chi(G) = 2$ and $\chi_2(G) \geqslant 4$, and so G is not normal. □

We now turn to the problem whether the gap $\chi_2(G) - \chi(G)$ can be bounded. Let G be a graph and let $e = uv$ be an edge of G with ends $u, v \in V(G)$. An *elementary subdivision* of e is to replace the edge e by a path uv_ev of length 2, where v_e is a newly added vertex. For each integer $k \geqslant 3$, let SK_k denote the graph obtained from the complete graph K_k by applying an elementary subdivision to each of the edges in K_k. Thus for a fixed $k \geqslant 3$, SK_k is a bipartite graph with a bipartition (X, Y) where $|X| = k$ and $|Y| = \binom{k}{2}$, such that each vertex in Y is adjacent to exactly two vertices in X, and distinct vertices in Y are adjacent to distinct pairs of vertices in X. Thus, $d(v) = k - 1$ for each $v \in X$.

In a conditional coloring of SK_k, any two vertices of X must be colored with different colors, as (C2) must be satisfied at every vertex in Y. Hence, $\chi_2(\mathrm{SK}_k) \geqslant k$. If $X = \{x_1, \dots, x_k\}$, then the coloring $c(x_i) = i$, $c(y) \in \{1, \dots, k\}$ and $c(y) \neq i, j$ if $y \in Y$ is adjacent to x_i and x_j, is a proper $(k, 2)$-coloring of SK_k, and so $\chi_2(\mathrm{SK}_k) = k$.

For an integer $r \geqslant 2$, by Proposition 2.1(i), $\chi_r(G) - \chi(G) \geqslant \chi_2(G) - \chi(G)$, and so the example above also shows that the gap $\chi_r(G) - \chi(G)$ can be arbitrarily big.

For $r \geqslant 2$, we can similarly define that a graph G is *r-normal* if $\chi_r(G) = \chi(G)$. Note that Lemma 3.1 can be extended as follows:

Lemma 3.4. *If any vertex v of a graph G is contained in K_k for some $k \geqslant \min\{r, d(v)\} + 1$, then G is r-normal.*

Proof. Let $k = \chi_r(G)$. For any proper k-coloring c of G, $|c(N(v))| \geqslant \min\{r, d(v)\}$ by (C1). Thus, c satisfies (C2) and hence is also a proper (k, r)-coloring. Thus, $\chi_r(G) = \chi(G)$ and so G is r-normal. □

Proposition 3.5. *The only r-normal graphs for all $r \geqslant 2$ are any complete graph and any odd cycle of length a multiple of three.*

Proof. By Proposition 2.1(ii), for any graph G, $\chi_r(G) \geqslant \min\{r, \Delta(G)\} + 1$ and by Brooks' Theorem [4], $\Delta(G) + 1 \geqslant \chi(G)$. Thus G can be r-normal for all $r \geqslant 2$ only if $\chi_r(G) = \chi(G) = \Delta(G) + 1$. By Brooks' Theorem, the only graphs satisfying $\chi(G) = \Delta + 1$ are odd cycles and complete graphs. Thus, the only graphs that are r-normal for all $r \geqslant 2$ are C_n, for n odd and a multiple of three, and complete graphs. □

Proposition 3.6. *Let G be a graph with $n = |V(G)|$ and let $r \geqslant 2$ be an integer. If $\delta \geqslant \lfloor (r-1)n/r \rfloor + 1$, then G is r-normal. The lower bound on $\delta(G)$ is best possible.*

Proof. It suffices by Lemma 3.4 to show that any vertex is contained in a complete subgraph of $r + 1$ vertices. Suppose $\delta \geqslant \lfloor (r-1)n/r \rfloor + 1$.

For any vertex v_1, there exists a vertex v_2 not in the set S_{v_1} of vertices nonadjacent to v_1, and in general there exists a vertex v_t not in the set $\bigcup_{i=1}^{t-1} S_{v_i}$ (so that each v_i in $\{v_1, \dots, v_t\}$ is adjacent to any other vertex in $\{v_1, \dots, v_t\}$) if $\sum_{i=1}^{t-1} |S_{v_i}| < n$. Since $|S_{v_i}| \leqslant n - (\lfloor (r-1)n/r \rfloor + 1) = \lfloor n/r \rfloor - 1$, then $\sum_{i=1}^{r} |S_{v_i}| \leqslant r(\lfloor n/r \rfloor - 1) < n$.

If $n \geqslant r + 2$, then $G = K_{i_1, \dots, i_r}$, where $i_1, \dots, i_j = \lfloor n/r \rfloor, i_{j+1}, \dots, i_r = \lceil n/r \rceil$ and $j = \lceil n/r \rceil r - n$, has $\delta(G) = \lfloor (r-1)n/r \rfloor$. Also, $\chi_r(G) \geqslant r + 1$ since, otherwise, $\chi_r(G) = \chi(G) = r$ and since G is colored uniquely with r color classes, then $|c(N(v))| = r - 1 < \min\{r, d(v)\}$ for any v, a contradiction. □

4. Claw-free graphs

A graph G is $K_{1,3}$*-free* (also known as *claw-free*) if it does not have an induced subgraph isomorphic to $K_{1,3}$. For $k \geqslant 4$, SK_k contains an induced $K_{1,3}$, one of the smallest and simplest graphs G for which $\chi_2(G)$ and $\chi(G)$ differ. This suggests considering as a possible class of graphs for which $\chi_2(G) - \chi(G)$ is bounded.

Lemma 4.1. *Suppose G is connected and $K_{1,3}$-free. If $\chi(G) = 2$, then $\chi_2(G) \leqslant 4$ with $\chi_2(G) = 4$ only if G is a cycle of even length and not a multiple of* 3.

Proof. Suppose $\chi(G)=2$ and G is $K_{1,3}$-free. Then $\Delta(G) \leqslant 2$, since otherwise any vertex of degree at least 3 is contained in K_3, and so $\chi(G) \geqslant 3$.

If each vertex has degree 2, then G is an even cycle, since $\chi(G) = 2$. By Theorem 2.5, $\chi_2(G) \leqslant 4$, and $\chi_2(G) = 4$ only if the cycle also has length not a multiple of 3.

Otherwise, each vertex has degree 1 or degree 2, so that G is a path. By Theorem 2.2, $\chi_2(G) = 2$ or $\chi_2(G) = 3$. □

Theorem 4.2. *If G is a connected and $K_{1,3}$-free, then $\chi_2(G) \leqslant \chi(G) + 2$, and equality holds if and only if G is a cycle of length* 5 *or of even length not a multiple of* 3.

Proof. By Theorem 2.5, the upper bound holds as stated for any cycle C_n.

Assume henceforth that G is not a cycle. Define an *arc* of G to be a path for which all the internal vertices have degree two in G. Let l denote the maximum length of an arc in G, and let u and v typically denote the end vertices of such an arc $P_{u,v}$. Let G' denote the subgraph of G induced by $(V(G) - V(P_{u,v})) \cup \{u, v\}$. Let the neighborhood $N_u = G[N_{G'}(u) \cup \{u\}]$ of u be the subgraph of G induced by u and its adjacent vertices in G'. Define $N_v = G[N_{G'}(v) \cup \{v\}]$ similarly. Since G is $K_{1,3}$-free, if $l \geqslant 3$, then N_u and N_v are complete, which must also hold for $l = 2$ if u and v are nonadjacent.

The proof is by induction on $n = |V(G)|$. The result is easily verified for $n \leqslant 3$.

Suppose $l = 1$. Then G has no arcs of length at least two and hence no vertices of degree two. Thus, any vertex of degree greater than one is in some K_3, since G is $K_{1,3}$-free. Hence, by Lemma 4.1, $\chi_2(G) = \chi(G)$.

Suppose $l=2$. If $\chi(G')=1$, then G' consists of the disjoint vertices u and v, whence $G = P_{uv}$ and $\chi_d(G) = 3 = \chi(G) + 1$.

Suppose that $l = 2$ and $\chi(G') = 2$. By Lemma 4.1, each component of G' must be a path or an even cycle. Since G is $K_{1,3}$-free, G' must be a K_2, and so $G = K_3$. Thus $\chi_2(G) = \chi(G) = 3$.

Suppose that $l = 2$ and $\chi(G') \geqslant 3$. Hence, $\chi(G') = \chi(G)$.

Let $k' = \chi_2(G')$. Suppose some $(k', 2)$-coloring c of G' has $c(u) \neq c(v)$. Then $k' \geqslant 3$, and since $l = 2$ implies that only for u, v adjacent in G can $d_G(u) = 2$ or $d_G(v) = 2$, then coloring the internal vertex w of $P_{u,v}$ any color different from $c(u)$ and $c(v)$ extends c to a $(k', 2)$-coloring of G, showing $\chi_2(G) = \chi_2(G')$. Since $l = 2$, G' is not a cycle of length greater than three, hence also not an even cycle. Thus, $\chi_2(G) = \chi_2(G') \leqslant \chi(G') + 1 = \chi(G) + 1$, so that $\chi_d(G) \leqslant \chi(G) + 1$.

Suppose any $(k', 2)$-coloring c of G' has $c(u) = c(v)$. Since N_u, N_v are complete subgraphs and $|V(N_u)| \geqslant 3$ or $|V(N_v)| \geqslant 3$, then $|V(N_u)| = \chi_2(G')$ or $|V(N_v)| = \chi_2(G')$, since otherwise G' could be recolored by recoloring u to be in $c(G') - c(N_u)$ or, respectively, by recoloring v to be in $c(G') - c(N_v)$, yielding a $(k', 2)$-coloring c' of G' having $c'(u) \neq c'(v)$. Thus, $\chi_2(G') = \omega(G')$ and, since $\omega(G) \leqslant \chi(G) \leqslant \chi_2(G)$ for any graph, then $\chi_2(G') = \chi(G')$. Since $\chi_2(G) \leqslant \chi_2(G') + 1$, then $\chi_2(G) \leqslant \chi(G') + 1 = \chi(G) + 1$.

Suppose $l \geqslant 3$. Then both N_u and N_v must be complete graphs.

Suppose $k' = \chi_2(G') \leqslant 3$. Since the remaining vertices of P_{uv} may be colored with four colors (including the colors used in $c(G')$) to extend any $(k', 2)$-coloring c of G' to a $(4, 2)$-coloring of G, then $\chi_2(G) \leqslant 4$. By Lemma 4.1, $\chi(G) = 2$ when $\chi_2(G) = 4$ only if G is a cycle of even length not a multiple of three.

Suppose $k' = \chi_2(G') \geqslant 4$. Then $\chi(G') = 1$ is not possible, since then $G = P_{uv}$ and $\chi_2(G')) = 1$. Consider $\chi(G') = 2$. If not connected, G' has two components. If a component of G' is nontrivial, then it would be a path or a cycle, whence N_u or N_v is incomplete, a contradiction. Thus both components of G' must be trivial, and so $G = P_{uv}$. If G' is connected and hence a path of length at least three or a cycle of length at least four, then N_u and N_v are incomplete.

Consider $\chi(G')\geqslant 3$. Then $\chi(G)=\chi(G')$. Also, $\chi_2(G)\leqslant\chi_2(G')$, since any $(k',2)$-coloring c of G' can be extended to a $(k',2)$-coloring of G by coloring the remaining vertices of P_{uv} with colors of $c(G')$ so that at most four colors of $c(G')$ would color P_{uv}. If G' is a cycle, then $G'=K_3$ to ensure N_u and N_v are complete; in this case, $\chi_2(G)=4$ and $\chi(G)=3$. Otherwise, $\chi_2(G')\leqslant\chi(G')+1$ by the induction hypothesis. Thus, $\chi_2(G)\leqslant\chi_2(G')\leqslant\chi(G')+1=\chi(G)+1$, and so $\chi_2(G)\leqslant\chi(G)+1$. □

5. Upper bounds

Proposition 2.1(ii) provides a trivial upper bound for $\chi_r(G)$. We first consider some cases when $\chi_r(G)=|V(G)|$.

Proposition 5.1. *For any $r\geqslant 2$, a graph G with $\chi_r(G)=n$ if and only if any two nonadjacent vertices of G are adjacent to a vertex of degree at most r.*

Proof. If the stated condition is not satisfied for vertices u and w, then a coloring c of G of $n-1$ colors in which only u and w are colored the same clearly satisfies (C1) and (C2) since, for any v not adjacent to both u and w, $|c(N(v))|=|N(v)|=d(v)\geqslant\min\{r,d(v)\}$ and, for any v adjacent to both u and w, $d(v)>r$ and so $|c(N(v))|=|N(v)|-1=d(v)-1\geqslant\min\{r,d(v)\}$. Thus, $\chi_r(G)\leqslant n-1$.

Suppose $k=\chi_r(G)\leqslant n-1$. Then some (k,r)-coloring c of G has $c(u)=c(w)$ for two nonadjacent vertices u and w. Thus, u and w are not adjacent to any vertex v such that $d(v)\leqslant r$, since otherwise $|c(N(v))|\leqslant d(v)-1<\min\{r,d(v)\}$. □

Proposition 5.1 can be useful for specifying particular graphs G satisfying $\chi_r(G)=n$ for $r\geqslant 2$. For example, P_3, C_4, C_5, and K_n are immediately seen to satisfy the condition of Proposition 5,1. Hence, each of them satisfies $\chi_r(G)=n$ for $r\geqslant 2$.

Since all graphs of $n=4$ vertices other than P_3 have, for any pair of vertices, a common neighbor (of degree at most $\Delta(G)\leqslant n-1=3$), then precisely all five graphs of four vertices other than P_3 satisfy $\chi_r(G)=n$ for $r\geqslant 3$.

Similarly, for $n=5$ and $r\geqslant 4$, the only graphs not satisfying $\chi_r(G)=n$ are precisely those in which some nonadjacent vertices have no common vertices, i.e., K_3 with an end of P_2 adjoined, C_4 with an end of P_1 adjoined, C_4+e with an end of P_1 adjoined to a low-degree vertex, and any tree other than $K_{1,4}$.

Proposition 5.1 allows us to deduce that the only trees satisfying $\chi_r(G)=n$ are $K_{1,n-1}$ for $n\leqslant r+1$, and that $\chi_r(K_n-e)=n$ if and only if $n\leqslant r+1$.

Proposition 5.2. *Let G be a connected graph with $n=|V(G)|\geqslant 2$, and let $r>0$ be an integer. If $\chi_r(G)=n$, then $G=K_n$ or $n\leqslant r^2+1$. If $n=r^2+1$, then any incomplete graph G with $\chi_r(G)=n$ must be r-regular.*

Proof. Suppose $\chi_r(G)=n$. If $G\neq K_n$, then G has two nonadjacent vertices u and w, which by Proposition 5.1 are adjacent to some vertex v, $d(v)\leqslant r$.

Let $N'(v)=V-N(v)-\{v\}$. Since any x in $N'(v)$ is not adjacent to v, then by Proposition 5.1, x and v are adjacent to some y_x in $N(v)$ with $d(y_x)\leqslant r$. Let $Y=\{y_x : x\in N'(v)\}$.

Thus, $|N'(v)|+|Y|\leqslant\sum_Y d(z)\leqslant|Y|r$, so that $|N'(v)|\leqslant\sum_Y d(z)-1\leqslant(r-1)|Y|\leqslant(r-1)d(v)$, since $|Y|\leqslant|N(v)|=d(v)$. Since $d(v)\leqslant r$ and $|N'(v)|=n-1-d(v)\geqslant n-r-1$, then $n-r-1\leqslant|N'(v)|\leqslant(r-1)r$, which gives $n\leqslant r^2+1$.

If $n=r^2+1$, then $|Y|=d(v)$ (hence $Y=N(v)$) and $\sum_Y d(z)-1=(r-1)r$, so that $d(z)=r$ for each z in $Y=N(v)$. Since $u\in N(v)$ was an arbitrary vertex of degree less than $n-1$, then all vertices have been shown to have degree $r\leqslant n-2$ or degree $n-1$. Since v and all vertices in $N(v)$ have degree r and since all vertices in $N'(v)=V-N(v)-\{v\}$ have degree at most $n-2$, then all vertices in G have degree r. □

Using Proposition 5.2, it is now simple to specify all the graphs that satisfy $\chi_2(G)=n$. Suppose $G\neq K_n$. Then $n\leqslant 5$. If $n=5$, then G must be 2-regular, and hence G must be C_5; $\chi_2(C_5)=5$. If $n=4$, then if G contains K_3 and another vertex v, then $\chi_2(G)=3$, since v may be colored the same as a vertex it is not adjacent to. The remaining graphs for $n=4$ yield $\chi_2(K_{1,3})=3$, $\chi_2(P_4)=3$, and $\chi_2(C_4)=4$. If $n=3$, $\chi_2(P_3)=3$. Thus, only P_3, C_4, C_5, and K_n satisfy $\chi_2(G)=n$.

If $\chi_3(G)=n$ and $G \neq K_n$, then $n \leqslant 10$ by Proposition 5.2. For $n=10$, $\chi_3(G)=10$ for the Petersen graph, which is 3-regular. Many other graphs, such as the W_4 and W_5 (wheels on 5 and 6 vertices, respectively) can be seen to satisfy Proposition 5.2 and thus have $\chi_3(G)=n$.

The task of specifying all the graphs satisfying $\chi_r(G)$ for a particular r only becomes more and more difficult with increasing r and would not be treated here for any $r \geqslant 3$, although Propositions 5.1 and 5.2 remain helpful tools for discovering many such graphs.

Before proving a theorem giving an upper bound for $\chi_r(G)$ in terms of $\Delta(G)$, we first prove a theorem crucial in proving the upper bound, and also interesting in itself. First, we define the distance of a vertex v from a color class to be the minimum of the distances from v to vertices in that color class. Any graph G has a proper $\chi(G)$-coloring such that some vertex is adjacent to any other color class. This is not true for $(\chi_r(G), r)$-colorings for any $r \geqslant 2$, as shown by C_4 or C_5. However, there is a similar property for conditional colorings, which we now show.

Theorem 5.3. *Any graph G has a $(\chi_r(G), r)$-coloring such that some vertex is within distance two of any other color class.*

Proof. Let $k=\chi_r(G)$. If not, of all such (k, r)-colorings of G, let c be one having a color class V_1 of minimum size. Recolor some v in V_1 the color j of a color class V_j at a distance of at least three from v, so that c' has color classes $V_1'=V_1-v$, $V_j'=V_j \cup \{v\}$, and $V_i'=V_i$ for $i \neq 1, j$.

Then c' satisfies the adjacency condition, since v is not adjacent to any vertex of V_j. Also, c' satisfies the multiple-adjacency condition, since V_j at a distance of at least three from v implies that any u adjacent to v is not adjacent to any vertex in V_j' other than v. Thus, c' is also a (k, r)-coloring of G.

Hence, either $|V_1|=1$ and c' has $\chi_r(G)-1$ colors, or some vertex is within distance two of any other color class of c', or no such vertex exists but c' has a smaller color class $V_1'=V_1-v$ than c, a contradiction in each case. □

Proposition 5.4. *For $r \geqslant 2$, $\chi_r(G) \leqslant \Delta(G)+r^2-r+1$ if $\Delta(G) \leqslant r$.*

Proof. Let $k=\chi_r(G)$. By Theorem 5.3, G has a (k, r)-coloring with some vertex v within distance two of any other color class. Thus, $\chi_r(G)=1+n_1+n_2$, where n_i is the number of color classes at distance i from v.

Since $\Delta(G) \leqslant r$, then $n_1=d(v) \leqslant \Delta(G)$ and $n_2 \leqslant \Delta(G)(\Delta(G)-1) \leqslant r(r-1)$. So, $\chi_r(G)=1+n_1+n_2 \leqslant \Delta(G)+r(r-1)+1=\Delta(G)+r^2-r+1$. □

When $r=1$, the well known Brooks coloring theorem gives the bound $\chi(G) \leqslant \Delta(G)+1$. An analogue of Brooks Theorem for the conditional chromatic number $\chi_2(G)$ was proved in [9].

6. Remarks

Conditional colorings are natural generalizations of the notion of graph vertex coloring. Therefore, it is natural to investigate what vertex coloring results can be generalized to conditional colorings. In [9], the analogous of Brooks Theorem for the case when $r=2$ is proved. It will be interested to find the Brooks Theorem for conditional coloring with a generic value of r.

The upper bound of the conditional chromatic number $\chi_r(G)$ for graphs G embedded on surfaces is also of particular interests. The famous 4-Color-Theorem [1,2,12] and the Heawood formula [7] provide complete answers to the case when $r=1$. For $r=2$, Lai and Poon [10] showed that for a planar graph G, $\chi_2(G) \leqslant 5$. As $\chi_2(C_5)=5$, this bound is best possible. They also conjectured that C_5 is the only planar graph with the second order of conditional chromatic number equal to 5. For larger values of r, this remains to be investigated.

Since $\chi_2(G) \leqslant 5$ for a planar graph G, it would be interested to know that what kind of planar graphs will have the second order of conditional chromatic number equal to 4. A recent result by Meng et al. [11] shows that the second order of conditional chromatic number of Pseudo-Harlin graphs is at most 4.

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