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Every 3-connected, essentially 11-connected line graph is Hamiltonian

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Abstract

Thomassen conjectured that every 4-connected line graph is Hamiltonian. A vertex cut X of G is essential if $G - X$ has at least two non-trivial components. We prove that every 3-connected, essentially 11-connected line graph is Hamiltonian. Using Ryjáček's line graph closure, it follows that every 3-connected, essentially 11-connected claw-free graph is Hamiltonian.

Keywords: Line graph; Claw-free graph; Super-Eulerian graphs; Collapsible graph; Hamiltonian graph; Dominating Eulerian subgraph; Essential connectivity

1. Introduction

We use [1] for terminology and notations not defined here, and consider finite graphs without loops. In particular, $\kappa(G)$ and $\kappa'(G)$ represent the *connectivity* and *edge-connectivity* of a graph G. A graph is *trivial* if it contains no edges. A vertex cut X of G is *essential* if $G - X$ has at least two non-trivial components. For an integer $k > 0$, a graph G is *essentially k-connected* if G does not have an essential cut X with $|X| < k$. An edge cut Y of G is *essential* if $G - Y$ has at least two non-trivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge cut Y with $|Y| < k$.

For a graph G, let $O(G)$ denote the set of odd degree vertices of G. A graph G is *Eulerian* if G is connected with $O(G) = \emptyset$, and G is *super-Eulerian* if G has a spanning Eulerian subgraph. Let $X \subseteq E(G)$ be an edge subset. The *contraction* G/X is the graph obtained from G by identifying

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the two ends of each edge in X and then deleting the resulting loops. When $X = \{e\}$, we also use G/e for $G/\{e\}$. For an integer $i > 0$, define

$$D_i(G) = \{v \in V(G): \deg_G(v) = i\}.$$

For any $v \in V(G)$, define

$$E_G(v) = \{e \in E(G): e \text{ is incident with } v \text{ in } G\}.$$

Let H_1, H_2 be subgraphs of a graph G. Then $H_1 \cup H_2$ is a subgraph of G with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$; and $H_1 \cap H_2$ is a subgraph of G with vertex set $V(H_1) \cap V(H_2)$ and edge set $E(H_1) \cap E(H_2)$. If V_1, V_2 are two disjoint subsets of $V(G)$, then $[V_1, V_2]_G$ denotes the set of edges in G with one end in V_1 and the other end in V_2. When the graph G is understood from the context, we also omit the subscript G and write $[V_1, V_2]$ for $[V_1, V_2]_G$. If H_1, H_2 are two vertex disjoint subgraphs of G, then we also write $[H_1, H_2]$ for $[V(H_1), V(H_2)]$.

The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. From the definition of a line graph, if $L(G)$ is not a complete graph, then a subset $X \subseteq V(L(G))$ is a vertex cut of $L(G)$ if and only if X is an essential edge cut of G. In 1986, Thomassen proposed the following conjecture.

Conjecture 1.1. (Thomassen [5]) *Every* 4*-connected line graph is Hamiltonian.*

A graph that does not have an induced subgraph isomorphic to $K_{1,3}$ is called a *claw-free* graph. It is well known that every line graph is a claw-free graph. Matthews and Sumner proposed a seemingly stronger conjecture.

Conjecture 1.2. (Matthews and Sumner [3]) *Every* 4*-connected claw-free graph is Hamiltonian.*

The best results towards these conjectures so far were obtained by Zhan and Ryjáček. A graph G is *Hamiltonian connected* if for every pair of vertices u and v in G, G has a spanning (u, v)-path.

Theorem 1.3. (Zhan [7]) *Every* 7*-connected line graph is Hamiltonian connected.*

Theorem 1.4. (Ryjáček [4])

(i) *Conjecture* 1.1 *and Conjecture* 1.2 *are equivalent.*
(ii) *Every* 7*-connected claw-free graph is Hamiltonian.*

In this paper, we consider the following problem: For 3-connected claw-free graphs, can high essential connectivity guarantee the existence of a Hamiltonian cycle? This leads us to prove the following Theorem 1.5. However, what is the smallest positive integer k such that every 3-connected, essentially k-connected claw-free graph is Hamiltonian? This question remains to be answered. It is well known that the line graph of the graph obtained by subdividing each edge of the Petersen graph exactly once is a 3-connected claw-free graph without a Hamiltonian cycle. Thus Corollary 1.6 below suggests that $4 \leqslant k \leqslant 11$. We fail to construct examples to show that there exists a 3-connected essentially 4-connected non-Hamiltonian claw-free graph, and we conjecture that $k = 4$.

Theorem 1.5. *Every* 3*-connected, essentially* 11*-connected line graph is Hamiltonian.*

Ryjáček [4] introduced the line graph closure of a claw-free graph and used it to show that a claw-free graph G is Hamiltonian if and only if its closure $\mathrm{cl}(G)$ is Hamiltonian, where $\mathrm{cl}(G)$ is a line graph. With this argument and using the fact that adding edges will not decrease the connectivity of a graph, The following corollary is obtained.

Corollary 1.6. *Every* 3*-connected, essentially* 11*-connected claw-free graph is Hamiltonian.*

2. Reductions

Catlin in [2] introduced collapsible graphs. A graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph H_R such that $O(H_R) = R$. Note that when $R = \emptyset$, a spanning connected subgraph H with $O(H) = \emptyset$ is a spanning Eulerian subgraph of G. Thus every collapsible graph is *super-Eulerian*. Catlin [2] showed that any graph G has a unique subgraph H such that every component of H is a maximally collapsible subgraph of G and every non-trivial collapsible subgraph of G is contained in a component of H. The contraction G/H is called the *reduction* of G. A graph G is *reduced* if it is the reduction of itself. The following summarizes some of the former results concerning collapsible graphs.

Theorem 2.1. *Let G be a connected graph. Let $F(G)$ denote the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. Each of the following holds.*

(i) (*Catlin* [2]) *If H is a collapsible subgraph of G, then G is collapsible if and only if G/H is collapsible*; *G is super-Eulerian if and only if G/H is super-Eulerian.*
(ii) (*Catlin, Theorem* 8 *of* [2]) *If G is reduced and if $|E(G)| \geqslant 3$, then $\delta(G) \leqslant 3$, and $2|V(G)| - |E(G)| \geqslant 4$.*
(iii) (*Catlin, Theorem* 5 *of* [2]) *A graph G is reduced if and only if G contains no non-trivial collapsible subgraphs. As cycles of length less than* 4 *are collapsible, a reduced graph does not have a cycle of length less than* 4.

Let G be a connected, essentially 3-edge-connected graph such $L(G)$ is not a complete graph. The *core* of this graph G, denoted by G_0, is obtained by deleting all the vertices of degree 1 and contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$.

Lemma 2.2. (Shao [6]) *Let G be a connected, essentially* 3*-edge-connected graph G.*

(i) *G_0 is uniquely defined, and $\kappa'(G_0) \geqslant 3$.*
(ii) *If G_0 is super-Eulerian, then $L(G)$ is Hamiltonian.*

A subgraph of G isomorphic to a $K_{1,2}$ or a 2-cycle is called a 2*-path* or a P_2 *subgraph* of G. An edge cut X of G is a P_2*-edge-cut* of G if at least two components of $G - X$ contain 2-paths. By the definition of a line graph, for a graph G, if $L(G)$ is not a complete graph, then $L(G)$

is essentially k-connected if and only if G does not have a P_2 edge cut with size less than k. Since the core G_0 is obtained from G by contractions (deleting a pendant edge is equivalent to contracting the same edge), every P_2-edge-cut of G_0 is also a P_2-edge-cut of G. Hence we have the following.

Lemma 2.3. *Let $k > 2$ be an integer, and let G be a connected, essentially 3-edge-connected graph. If $L(G)$ is essentially k-connected, then every P_2-edge-cut of G_0 has size at least k.*

3. Proof of Theorem 1.5

Throughout this section, we assume that G is a graph such that $L(G)$ is 3-connected, essentially 11-connected, and that $L(G)$ is not a complete graph. Let G_0 denote the core of G and G_0' denote the reduction of G_0. We shall show that $G_0' = K_1$, and so G_0 is collapsible, which implies that G_0 is super-Eulerian. Hence by Lemma 2.2, $L(G)$ is Hamiltonian.

By contradiction, we assume that G_0' is a non-trivial graph. By Theorem 2.1(iii),

$$G_0' \text{ does not have a cycle of length less than 4.} \tag{1}$$

Since $L(G)$ is 3-connected, G is essentially 3-edge-connected. By Lemma 2.2, G_0' is 3-edge-connected. By Theorem 2.1(ii), $D_3(G_0') \neq \emptyset$.

Lemma 3.1. *For each $u, v, w \in V(G_0')$ such that $P = uvw$ is a 2-path in $V(G_0')$, the edge cut $X = [\{u, v, w\}, V(G_0') - \{u, v, w\}]_{G_0'}$ is a P_2-edge-cut of G_0' and $|X| \geqslant 11$.*

Proof. Suppose that $G_0' - X$ has components $H_1, H_2, \ldots, H_c$ with $c \geqslant 2$ and with $H_1 = G_0'[\{u, v, w\}]$ denoting a 2-path of G_0'. To show that X is a P_2-edge-cut of G_0', it suffices to show that for some $i \geqslant 2$,

$$\left|V(H_i)\right| \geqslant 2 \quad \text{and} \quad \left|E(H_i)\right| \geqslant 2. \tag{2}$$

Suppose first that for some $i \geqslant 2$, $|V(H_i)| = 1$. Since H_i is a component of $G_0' - X$, $|[\{u, v, w\}, V(H_i)]_{G_0'}| \geqslant \kappa'(G_0') \geqslant 3$, and so G_0' would have a cycle of length at most 3, contrary to (1). Similarly, suppose that for some $i \geqslant 2$, we have $E(H_i) = \{xy\}$. Then by $\kappa'(G_0') \geqslant 3$, each of x and y has degree at least 3 in G_0' and so $|[\{u, v, w\}, V(H_i)]_{G_0'}| \geqslant 4$. It follows again that G_0' would have a cycle of length at most 3, contrary to (1). This proves (2).

Thus X is a P_2-edge-cut of G_0'. Since $L(G)$ is essentially 11-connected, $|X| \geqslant 11$. □

Lemma 3.2. *Every component of $G_0'[D_3(G_0')]$ contains at most 2 vertices.*

Proof. By contradiction, we assume that one component of $G_0'[D_3(G_0')]$ contains at least 3 vertices, and so this component has three vertices u, v, w such that $G_0'[\{u, v, w\}]$ is connected. Thus $X = [\{u, v, w\}, V - \{u, v, w\}]$ is a P_2-edge-cut of G_0'. Since $u, v, w \in D_3(G_0')$, $|X| \leqslant 5$, contrary to Lemma 3.1. □

Define a real valued function

$$f(x) = \frac{x - 4}{x}, \quad \text{over the interval } [3, \infty).$$

For each $v \in G_0'$, define $l(v) = f(\deg_{G_0'}(v))$. Note that (i) of Lemma 3.3 below is a fact from Calculus and (ii) of Lemma 3.3 follows from (i) of Lemma 3.3.

Lemma 3.3. *Each of the following holds.*

(i) $f(x)$ *is an increasing function.*
(ii) *If* $\deg_{G'_0}(v) \geqslant k$*, then* $l(v) \geqslant f(k)$.

Lemma 3.4. *Suppose that* $v \in D_3(G'_0)$ *is an isolated vertex of* $G'_0[D_3(G'_0)]$ *such that* v_1, v_2, v_3 *are the vertices adjacent to* v *in* G'_0*. Then* $l(v_1) + l(v_2) + l(v_3) \geqslant 1$.

Proof. Since v is an isolated vertex in $D_3(G'_0)$, $v_i \notin D_3(G'_0)$. Relabelling the vertices if needed, we may assume that

$$4 \leqslant \deg_{G'_0}(v_1) \leqslant \deg_{G'_0}(v_2) \leqslant \deg_{G'_0}(v_3). \tag{3}$$

For $i, j \in \{1, 2, 3\}$, by Lemma 3.1, $\deg_{G'_0}(v_i) + \deg_{G'_0}(v_j) - 2 + 1 = |[\{v, v_i, v_j\}, V(G'_0) - \{v, v_i, v_j\}]| \geqslant 11$, and so

$$\deg_{G'_0}(v_i) + \deg_{G'_0}(v_j) \geqslant 12. \tag{4}$$

If $\deg_{G'_0}(v_1) \geqslant 6$, then by (3) and by Lemma 3.3(ii), $l(v_1) + l(v_2) + l(v_3) \geqslant 3f(6) = 1$. Suppose then that $\deg_{G'_0}(v_1) = 5$. Then by (4), both $\deg_{G'_0}(v_2) \geqslant 7$ and $\deg_{G'_0}(v_3) \geqslant 7$. It follows by Lemma 3.3(ii) that $l(v_1) + l(v_2) + l(v_3) \geqslant f(5) + 2f(7) \geqslant 1$. Finally, we assume that $\deg_{G'_0}(v_1) = 4$. Then by (4), both $\deg_{G'_0}(v_2) \geqslant 8$ and $\deg_{G'_0}(v_3) \geqslant 8$. It follows by Lemma 3.3(ii) that $l(v_1) + l(v_2) + l(v_3) \geqslant f(4) + 2f(8) = 1$. □

Lemma 3.5. *Suppose that* $v, w \in D_3(G'_0)$ *and* $vw \in E(G'_0)$*. If* v_1, v_2, w *are the vertices adjacent to* v *in* G'_0 *and if* v_3, v_4, v *are the vertices adjacent to* w *in* G'_0*, then*

(i) v_1, v_2, v_3, v_4 *are mutually distinct vertices, and*
(ii) *both* $l(v_1) + l(v_2) \geqslant 1$ *and* $l(v_3) + l(v_4) \geqslant 1$.

Proof. If $|\{v_1, v_2, v_3, v_4\}| \leqslant 4$, then G'_0 could contain a cycle of length at most 3, contrary to Theorem 2.1(iii). Thus Lemma 3.5(i) follows.

For $i \in \{1, 2, 3, 4\}$, by Lemma 3.1, $\deg_{G'_0}(v_i) - 1 + 3 = |[\{v, w, v_i\}, V - \{v, w, v_i\}]| \geqslant 11$, and so

$$\deg_{G'_0}(v_i) \geqslant 9. \tag{5}$$

It follows by (5) and Lemma 3.3(ii) that both $l(v_1) + l(v_2) \geqslant 2f(9) \geqslant 1$ and $l(v_3) + l(v_4) \geqslant 2f(9) \geqslant 1$. □

Let $d_i = |D_i(G'_0)|$, for each $i \geqslant 3$. By Lemmas 3.2, 3.4 and 3.5, and writing $E = E(G'_0)$ and $D_i = D_i(G'_0)$, we have

$$\begin{aligned} d_3 = \sum_{v \in D_3} 1 &\leqslant \sum_{v \in D_3} \sum_{uv \in E, u \notin D_3} l(u) = \sum_{u \notin D_3} \sum_{uv \in E, v \in D_3} l(u) = \sum_{i \geqslant 4} \sum_{u \in D_i} \sum_{uv \in E, v \in D_3} l(u) \\ &\leqslant \sum_{i \geqslant 4} \sum_{u \in D_i} i \cdot f(i) = \sum_{i \geqslant 4} \sum_{u \in D_i} (i - 4) = \sum_{i \geqslant 4} (i - 4) \cdot d_i. \end{aligned} \tag{6}$$

It follows by (6) that

$$2\big(2|V(G)| - |E(G)|\big) = 4|V(G)| - 2|E(G)| = \sum_{i \geqslant 3} (4-i) \cdot d_i$$
$$= d_3 - \sum_{i \geqslant 4} (i-4) \cdot d_i \leqslant 0,$$

contrary to Theorem 2.1(ii). Thus $G_0' = K_1$ and G_0 is super-Eulerian. By Lemma 2.2, $L(G)$ is Hamiltonian. This completes the proof of Theorem 1.5.

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