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Journal of Combinatorial Theory, Series B 96 (2006) 493–504

Journal of Combinatorial Theory Series B

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Hamiltonicity in 3-connected claw-free graphs

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Received 26 March 2003

Available online 18 November 2005

Abstract

Kuipers and Veldman conjectured that any 3-connected claw-free graph with order ν and minimum degree $\delta \geqslant (\nu + 6)/10$ is Hamiltonian for ν sufficiently large. In this paper, we prove that if H is a 3-connected claw-free graph with sufficiently large order ν, and if $\delta(H) \geqslant (\nu + 5)/10$, then either H is Hamiltonian, or $\delta(H) = (\nu + 5)/10$ and the Ryjáček's closure $cl(H)$ of H is the line graph of a graph obtained from the Petersen graph P_{10} by adding $(\nu - 15)/10$ pendant edges at each vertex of P_{10}.

Keywords: Claw-free graphs; Hamiltonian; Collapsible graphs

1. Introduction

We use [1] for terminology and notations not defined here, and consider loopless finite simple graphs only. Let G be a graph. If $S \subseteq V(G)$, $G[S]$ is the subgraph induced in G by S. The *degree* and *neighborhood* of a vertex x of G are respectively denoted by $d_G(x)$ and $N_G(x)$, and the *minimum degree*, the *independence number*, the *edge independence number*, the *connectivity* and the *edge connectivity* of G are denoted by $\delta(G)$, $\alpha(G)$, $\alpha'(G)$, $\kappa(G)$ and $\kappa'(G)$, respectively. An edge $e = uv$ is called a *pendant edge* if either $d_G(u) = 1$ or $d_G(v) = 1$. We use $H \subseteq G$ to denote the fact that H is a subgraph of G. For $H \subseteq G$, $x \in V(G)$ and $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, denote $N_H(x) = N_G(x) \cap V(H)$, $d_H(x) = |N_H(x)|$, $N_H(A) = \bigcup_{v \in A} N_H(v)$, $[A, B]_G = \{uv \in E(G) \mid u \in A,\ v \in B\}$, and $G - A = G[V(G) - A]$. When $A = \{v\}$, we use $G - v$ for

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0095-8956/$ – see front matter
doi:10.1016/j.jctb.2005.10.003

$G - \{v\}$. If $H \subseteq G$, then for an edge subset $X \subseteq E(G) - E(H)$, we write $H + X$ for $G[E(H) \cup X]$. For each $i = 0, 1, 2, \ldots$, denote $D_i(G) = \{v \in V(G) \mid d_G(v) = i\}$.

A subgraph H of G is *dominating* if $G - V(H)$ is edgeless. A vertex $v \in G$ is called a *locally connected vertex* if $G[N_G(v)]$ is connected. We denote C_n an n-cycle and denote $O(G)$ the set of all vertices in G with odd degrees. A graph G is *Eulerian* if $O(G) = \emptyset$ and G is connected.

Let $X \subseteq E(G)$. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If K is a subgraph of G, then we write G/K for $G/E(K)$. If K is a connected subgraph of G, and if v_K is the vertex in G/K onto which K is contracted, then K is called the *preimage* of v_K, and is denoted by $PI(v_K)$. A vertex v in a contraction of G is *nontrivial* if $PI(v)$ has at least one edge.

The *line graph* of a graph G, denote by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. Let H be the line graph $L(G)$ of a graph G. The order $\nu(H)$ of H is equal to the number $m(G)$ of edges of G, and $\delta(H) = \min\{d_G(x) + d_G(y) - 2 \mid xy \in E(G)\}$. If $L(G)$ is k-connected, then G is *essentially k-edge-connected*, which means that the only edge-cut sets of G having less than k edges are the sets of edges incident with some vertex of G. Harary and Nash-Williams showed that there is a closed relationship between a graph and its line graph concerning Hamilton cycles.

Theorem 1.1. (Harary and Nash-Williams [8]) *The line graph $H = L(G)$ of a graph G is Hamiltonian if and only if G has a dominating Eulerian subgraph.*

A graph H is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. In [14], Ryjáček defined the *closure $cl(H)$* of a claw-free graph H to be one obtained by recursively adding edges to join two nonadjacent vertices in the neighborhood of any locally connected vertex of H, as long as this is possible.

Theorem 1.2. (Ryjáček [14]) *Let H be a claw-free graph and $cl(H)$ its closure. Then*:

(i) *$cl(H)$ is well defined, and $\kappa(cl(H)) \geqslant \kappa(H)$,*
(ii) *there is a triangle-free graph G such that $cl(H) = L(G)$,*
(iii) *both graphs H and $cl(H)$ have the same circumference.*

As a corollary of Theorem 1.2, a claw-free graph H is Hamiltonian if and only if $cl(H)$ is Hamiltonian. H is said to be *closed* if $H = cl(H)$.

Many works have been done to give sufficient conditions for a claw-free graph H to be Hamiltonian in terms of its minimum degree $\delta(H)$. These conditions depend on the connectivity $\kappa(H)$. If $\kappa(H) = 4$, Matthews and Sumner [13] conjectured that H is Hamiltonian and this conjecture is still open. When $\kappa(H) = 2$, Kuipers and Veldman [10], and independently Favaron et al. [6], proved that if H is a 2-connected claw-free graph with sufficiently large order ν, and if $\delta(H) \geqslant (\nu + c)/6$ (where c is a constant), then H is Hamiltonian except a member of ten well-defined families of graphs. Recently, the degree conditions [9] were further strengthened for 2-connected claw-free graphs. Kovářík et al. [9] proved that if G is a 2-connected claw-free graph of order $\nu \geqslant 153$ with $\delta(G) \geqslant (\nu + 39)/8$, then either G is Hamiltonian or the closure of G is in the five classes of graphs. When $\kappa(H) = 3$, the following have been proved and proposed.

Theorem 1.3. (Kuipers and Veldman [10]) *If H is a 3-connected claw-free simple graph with sufficiently large order ν, and if $\delta(H) \geqslant (\nu + 29)/8$, then H is Hamiltonian.*

Theorem 1.4. (Favaron and Fraisse [7]) *If H is a 3-connected claw-free simple graph with order ν, and if $\delta(H) \geqslant (\nu + 37)/10$, then H is Hamiltonian.*

Conjecture 1.5. (Kuipers and Veldman [10], see also [7]) *Let H be a 3-connected claw-free simple graph of order ν with $\delta(H) \geqslant (\nu + 6)/10$. If ν is sufficiently large, then H is Hamiltonian.*

The main purpose of this paper is to prove Conjecture 1.5. In fact, we proved a somewhat stronger result.

Theorem 1.6. *If H is a 3-connected claw-free simple graph with $\nu \geqslant 196$, and if $\delta(H) \geqslant (\nu + 5)/10$, then either H is Hamiltonian, or $\delta(H) = (\nu + 5)/10$ and $cl(H)$ is the line graph of G obtained from the Petersen graph P_{10} by adding $(\nu - 15)/10$ pendant edges at each vertex of P_{10}.*

2. Mechanism

In [2] Catlin defined collapsible graphs. Given a subset $R \subseteq V(G)$ with $|R|$ is even, a subgraph Γ of G is an *R-subgraph* if both $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph G is *collapsible* if for any even subset R of $V(G)$, G has an R-subgraph. Catlin showed in [2] that every vertex of G lies in a unique maximal collapsible subgraph of G. The *reduction* of G, denoted by G', is obtained from G by contracting all maximal collapsible subgraphs of G. A graph G is *reduced* if G has no nontrivial collapsible subgraphs, or equivalently, if $G = G'$, the reduction of G. A *nontrivial vertex* in G' is a vertex that is the contraction image of a nontrivial connected subgraph of G. Note that if G has an $O(G)$-subgraph Γ, then $G - E(\Gamma)$ is a spanning Eulerian subgraph of G. Therefore, every collapsible graph has a spanning Eulerian subgraph.

Theorem 2.1. (Catlin [2]) *Let G be a connected graph.*

(i) *If G is reduced, then G is a simple graph and has no cycle of length less than four.*
(ii) *G is reduced if and only if G has no nontrivial collapsible subgraphs.*
(iii) *Let G' be the reduction of G. Then G is collapsible if and only if $G' = K_1$.*

Defining $F(G)$ to be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees, we present some of the former results in the following theorems.

Theorem 2.2. *Let G be a graph. Then the following statements hold.*

(i) (Catlin [2]) *If $F(G) \leqslant 1$ and if G is connected, then G is collapsible if and only if the reduction of G is not a K_2.*
(ii) (Catlin [3]) *If G is reduced, then $F(G) = 2|V(G)| - |E(G)| - 2$.*

Theorem 2.3. (Catlin [3]) *Let $K_{3,3} - e$ denote the graph obtained from $K_{3,3}$ by removing an edge. Then $K_{3,3} - e$, K_n $(n \geqslant 3)$ and C_2 are collapsible.*

Theorem 2.4. (Chen [4]) *Let G be a reduced graph with $|V(G)| \leqslant 11$ vertices, and $\kappa'(G) \geqslant 3$. Then G is either K_1 or the Petersen graph.*

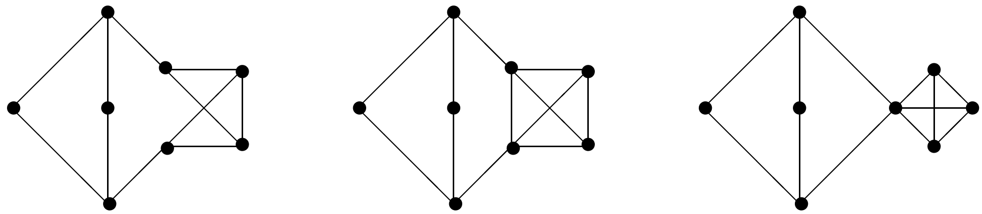

Fig. 1.

Lemma 2.5. (Lai et al. [12]) *Let G be a connected simple graph with $|V(G)| \leqslant 8$ vertices and with $D_1(G) = \emptyset$, $|D_2(G)| \leqslant 2$. Then either G is one of three graphs in Fig.* 1*, or the reduction of G is K_1 or K_2.*

Let G be a graph and let $S \subseteq V(G)$ be a vertex subset. An Eulerian subgraph H of G is called an *S-Eulerian subgraph* if $S \subseteq V(H)$. Let $K_{2,3}$, $K_{2,5}$, W_3', W_4', L_1, L_2 and L_3 be the labelled graphs defined in Figs. 2–4, and let $\mathcal{F} = \{K_{2,3}, K_{2,5}, W_3', W_4', L_1, L_2, L_3\}$. Using the labels in Figs. 2–4, for each $L \in \mathcal{F}$, we define $B(L)$, the bad set of L, to be the vertex subset of $V(L)$ that are labeled with the b_i's.

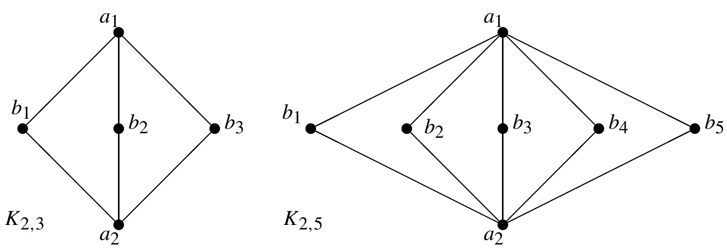

Fig. 2. The graphs $K_{2,3}$ and $K_{2,5}$.

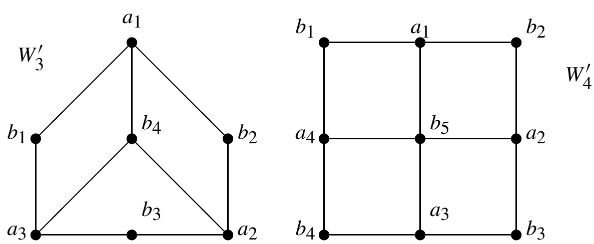

Fig. 3. The graphs W_3' and W_4'.

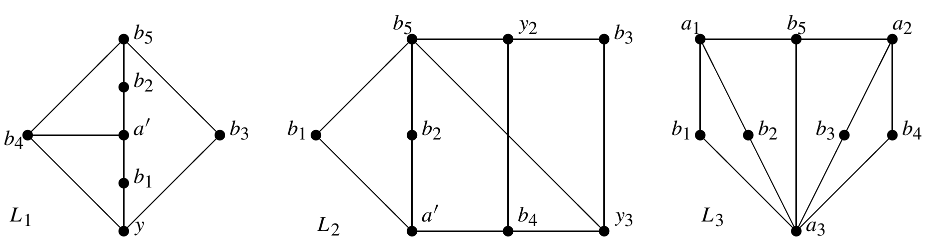

Fig. 4. The graphs L_1, L_2 and L_3.

Theorem 2.6. (Lai [11]) *Let G be a 2-edge-connected graph and let $S \subseteq V(G)$ with $|S| \leqslant 5$. If $G - S$ is edgeless, and if G does not have an S-Eulerian subgraph, then G is contractible to a member $L \in \mathcal{F}$ such that S intersects the preimage of every vertex in $B(L)$.*

Lemma 2.7. *Suppose that G does not contain $K_4 - e$ as its subgraph. Then the following statements hold.*

(i) *If $|V(G)| = 3$, then $|E(G)| \leqslant 3$.*
(ii) *If $|V(G)| = 4$, then $|E(G)| \leqslant 4$.*
(iii) *If $|V(G)| = 5$, then $|E(G)| \leqslant 6$.*
(iv) *If $|V(G)| = 6$, then $|E(G)| \leqslant 9$.*
(v) *If $|V(G)| = 7$, then $|E(G)| \leqslant 12$.*

Proof. If $|V(G)| = 3$, then $|E(G)| \leqslant 3$. If $|V(G)| = 4$, then $|E(G)| \leqslant 4$ since G does not contain $K_4 - e$ as its subgraph. Thus let $5 \leqslant |V(G)| \leqslant 7$. If G has more edges, then $|E(G)| > |V(G)|^2/4$ and, by Turán's theorem, G contains a triangle T. Denote $R = G - T$. Then $2 \leqslant |V(R)| \leqslant 4$, and $|N_T(y)| \leqslant 1$ for any $y \in V(R)$ (otherwise we have a $K_4 - e$), which implies that $|[T, R]_G| \leqslant |V(R)|$. So we have

$$\begin{aligned}\left|E(G)\right| &= \left|E(T)\right| + \left|[T, R]_G\right| + \left|E(R)\right| \leqslant \left|V(T)\right| + \left|V(R)\right| + \left|E(R)\right| \\ &= \left|V(G)\right| + \left|E(R)\right|.\end{aligned}$$

If $|V(R)| = 2$, then clearly $|E(R)| \leqslant 1$ and for $3 \leqslant |V(R)| \leqslant 4$ we have $|E(R)| \leqslant |V(R)|$ by (i) or (ii), respectively. Hence the lemma follows. □

Lemma 2.8. *Suppose that G is a 2-edge-connected graph with at most 10 vertices, and that G does not contain $K_4 - e$ as a subgraph. If $|E(G)| \geqslant 17$, then G is collapsible.*

Proof. Note that if H is a simple collapsible subgraph of G with $|V(H)| = 4$, then H must contain $K_4 - e$ as a subgraph. We have the following:

$$\text{If } H \text{ is a simple collapsible subgraph of } G, \quad \text{then} \quad \left|V(H)\right| \geqslant 3 \quad \text{and} \quad \left|V(H)\right| \neq 4. \tag{1}$$

Let G' be the reduction of G. Note that G is collapsible if and only if $G' = K_1$. Suppose, by contradiction, that $G' \neq K_1$. Then $\kappa'(G') \geqslant 2$ and $4 \leqslant |V(G')| \leqslant 10$. By Theorem 2.2(i), $F(G') \geqslant 2$. Let $V(G') = \{v_1, v_2, \ldots, v_s\}$ and $H_i = PI(v_i)$ $(i = 1, 2, \ldots, s)$ with $|V(H_1)| \geqslant |V(H_2)| \geqslant \cdots \geqslant |V(H_s)|$. As $|V(G')| \geqslant 4$, $|V(H_1)| \leqslant 7$. If $V(G) = V(G')$, then $|E(G')| \geqslant 17$, and so $F(G') = 2|V(G')| - |E(G')| - 2 \leqslant 2 \cdot 10 - 17 - 2 = 1$, a contradiction.

If $6 \leqslant |V(H_1)| \leqslant 7$, then $|V(H_2)| = \cdots = |V(H_s)| = 1$ by (1). Thus

$$\left|V(G')\right| = \left|V(G)\right| - \left|V(H_1)\right| + 1 \leqslant \begin{cases} 10 - 6 + 1 = 5, & \text{if } |V(H_1)| = 6, \\ 10 - 7 + 1 = 4, & \text{if } |V(H_1)| = 7. \end{cases}$$

By Lemma 2.7, we have

$$\left|E(G')\right| \geqslant 17 - \left|E(H_1)\right| \geqslant \begin{cases} 17 - 9 = 8, & \text{if } |V(H_1)| = 6, \\ 17 - 12 = 5, & \text{if } |V(H_1)| = 7. \end{cases}$$

Then, $|E(G')| > |V(G')|^2/4$. By the Turán's theorem, G' contains a triangle, a contradiction.

If $|V(H_1)| = 5$, then $|V(H_3)| = \cdots = |V(H_s)| = 1$ and $|V(H_2)| = 1$ or 3. Thus

$$\left|V(G')\right| = \left|V(G)\right| - \left|V(H_1)\right| - \left|V(H_2)\right| + 2 \leqslant \begin{cases} 6, & \text{if } |V(H_2)| = 1, \\ 4, & \text{if } |V(H_2)| = 3. \end{cases}$$

By Lemma 2.7, we have

$$E(G') \geqslant 17 - \left|E(H_1)\right| - \left|E(H_2)\right| \geqslant \begin{cases} 17 - 6 = 11, & \text{if } |V(H_2)| = 1, \\ 17 - 6 - 3 = 8, & \text{if } |V(H_2)| = 3. \end{cases}$$

Thus, $|E(G')| > |V(G')|^2/4$. By the Turán's theorem, G' contains a triangle, a contradiction.

If $|V(H_1)| = 3$, let $|V(H_1)| = \cdots = |V(H_t)| = 3$ and $|V(H_{t+1})| = \cdots = |V(H_s)| = 1$. Then $|E(G')| \geqslant 17 - 3t$ and $V(G') \leqslant 10 - 2t$. Thus $F(G') = 2|V(G')| - |E(G')| - 2 \leqslant 2(10 - 2t) - (17 - 3t) - 2 = 1 - t \leqslant 1$, a contradiction. □

Lemma 2.9. *If G is collapsible, then for any pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-trail.*

Proof. Let $R = (O(G) \cup \{u, v\}) \setminus (O(G) \cap \{u, v\})$. Then $|R|$ is even. Let Γ_R be an R-subgraph of G. Then $G - E(\Gamma_R)$ is a spanning (u, v)-trail of G. □

3. Proof of Theorem 1.6

The proof of Theorem 1.6 needs the following theorem and lemma.

Theorem 3.1. (Chen et al. [5]) *Let G be a 3-edge-connected graph and let $S \subseteq V(G)$ be a vertex subset such that $|S| \leqslant 12$. Then either G has an Eulerian subgraph C such that $S \subseteq V(C)$, or G can be contracted to the Petersen graph in such a way that the preimage of each vertex of the Petersen graph contains at least one vertex in S.*

Lemma 3.2. (Favaron and Fraisse [7]) *Let S be a set of vertices of a graph G contained in an Eulerian subgraph of G and let C be a maximal Eulerian subgraph of G containing S. Assume that some component A of $G - V(C)$ is not an isolated vertex and is related to C by at least r edges. Then*:

(i) *G contains a matching T of $r + 1$ edges such that at most $2r$ edges of G are adjacent to two distinct edges of T.*
(ii) *The number $m(G)$ of edges of G is related to the minimum degree $\delta(H)$ of the line graph H of G by $m(G) \geqslant (r + 1)\delta(H) - r + 1$.*

Portion of the proof of Theorem 1.6 (the treatment to deal with Claims 1 and 2) is a modification of Favaron and Fraisse's proof for Theorem 1 in [7], with Theorem 3.1 being utilized in our proof.

Proof of Theorem 1.6. By Theorem 1.2, the graph H is Hamiltonian if and only if its closure $cl(H)$ is Hamiltonian. As $\nu(cl(H)) = \nu(H)$, $\delta(cl(H)) \geqslant \delta(H)$, and $cl(H)$ is 3-connected, the graph $cl(H)$ satisfies the same hypotheses as H. Hence it suffices to prove Theorem 1.6 for closed claw-free graphs.

By Theorem 1.2, we may assume that H is the line graph of a triangle-free graph G (i.e., $H = L(G)$), and suppose that H is 3-connected and satisfies $\delta(H) \geqslant (\nu(H)+5)/10$. Assume by contradiction that neither of the conclusions of Theorem 1.6 holds. By Theorem 1.1, G does not contain a dominating Eulerian graph.

Let $B = \{v \in V(G) \mid d_G(v) = 1, 2\}$. Since H is 3-connected, the sum of degrees of the two ends of each edge in G is at least 5 and thus the set B is independent. Let $X_0 = N_G(B)$. We name the vertices of X_0 as $x_1, x_2, \ldots, x_p$ in the following way. Assume the vertices $x_1, \ldots, x_i$ are already defined or else put $i = 0$. Let y_{i+1} denote a vertex of B which is adjacent to some vertex of $X_0 - \{x_1, \ldots, x_i\}$. Either y_{i+1} has exactly one neighbor in $X_0 - \{x_1, \ldots, x_i\}$ and we name it x_{i+1}, or y_{i+1} has exactly two neighbors in $X_0 - \{x_1, \ldots, x_i\}$ and we name them x_{i+1} and x_{i+2} and put $y_{i+2} = y_{i+1}$. Let $Y_0 = \{y_1, \ldots, y_p\}$. We note that if $1 \leqslant i < j \leqslant p$, then $y_i y_j \notin E(G)$ and $y_i x_j \notin E(G)$, except for the edges $y_i x_{i+1}$ when $y_i = y_{i+1}$; and that the components of the subgraph induced by the edges $x_i y_i$, $1 \leqslant i \leqslant p$, are paths of length 1 or 2.

Consider now a matching M of G formed by $q - p$ edges $x_i y_i$ of G, $p+1 \leqslant i \leqslant q$, considered in this order and such that

(i) the sets $X_0, Y_0, X = \{x_{p+1}, \ldots, x_q\}$ and $Y = \{y_{p+1}, \ldots, y_q\}$ are pairwise disjoint,
(ii) for $p+1 \leqslant i < j \leqslant q$, $y_i y_j, y_i x_j \notin E(G)$.

We choose this matching as large as possible subject to the conditions (i) and (ii). Note that by the definition of X_0 and Y_0, the whole set B is disjoint from $X \cup Y$ and that property (ii) holds for any i and j with $1 \leqslant i < j \leqslant q$.

Let J be the set of indices j between $p+1$ and q such that y_j is adjacent to some vertex $z \notin X_0 \cup Y_0 \cup X \cup Y$ with $y_k z \notin E(G)$ for $1 \leqslant k < j$. For each $j \in J$ we choose such a vertex z_j and we put $I = \{p+1, \ldots, q\} - J$. Let $X_I = \{x_i \in X \mid i \in I\}$, $X_J = \{x_i \in X \mid i \in J\}$, $Y_I = \{y_i \in Y \mid i \in I\}$ and $Y_J = \{y_i \in Y \mid i \in J\}$.

Claim 1. (Favaron and Fraisse [7]) *The set $S = X_0 \cup X_I \cup Y_J$ is not contained in any Eulerian subgraph of G.*

Proof. Suppose Claim 1 is false and let C be a maximal Eulerian subgraph of G containing $S = X_0 \cup X_I \cup Y_J$ and $R = V(G) - V(C)$. By the assumption that G has no dominating Eulerian subgraph, at least one component A of $G[R]$ is not a single vertex. This component A is disjoint from Y_0 since the vertices of Y_0 are isolated in $G[R]$.

Suppose first that every vertex of A has a neighbor in C. Then, if uv is an edge of A and if s denotes the number of edges between A and C, $s \geqslant d_C(u) + d_C(v) + |A| - 2$. Since G is triangle-free, $d_A(u) + d_A(v) \leqslant |A|$ and thus $d_G(u) + d_G(v) = d_C(u) + d_C(v) + d_A(u) + d_A(v) \leqslant d_C(u) + d_C(v) + |A|$. Hence $s \geqslant d_G(u) + d_G(v) - 2 \geqslant \delta(H)$. Apply Lemma 3.2 with $r = \delta(H)$ to conclude that the number of edges of G satisfies $m(G) \geqslant \delta^2(H) + 1$. Since $\delta(H) \geqslant (\nu(H)+5)/10$, then $m(G) = \nu(H) \leqslant 10\delta(H) - 5$, and so $\delta^2(H) + 1 \leqslant 10\delta(H) - 5$, contrary to the hypothesis that $\nu(H) \geqslant 196$.

Therefore A contains a vertex z such that $N_G(z) \subseteq A$. Then $z \notin X_0 \cup Y_0 \cup X \cup Y$ and the neighbors of z are all in $Y_I \cup X_J \cup (R - (Y_0 \cup Y_I \cup X_J))$.

If z has a neighbor in Y_I, let i be the least index such that $y_i \in Y_i$ and $zy_i \in E(G)$. Since z has no neighbor in Y_J, $zy_k \notin E(G)$ for all $k < i$, in contradiction to the definition of I. Hence z has no neighbor in Y_I, and thus in Y.

If z has a neighbor in X_J, let x_j be the vertex of $N_G(z) \cap X_J$ with the largest index. Consider the ordered sets $X' = \{x_{p+1}, \ldots, x_{j-1}, x_j, z_j, x_{j+1}, \ldots, x_q\}$ and $Y' = \{y_{p+1}, \ldots, y_{j-1}, z, y_j, y_{j+1}, \ldots, y_q\}$. Then the vertex z is adjacent neither to any x_k with $k > j$ (by the definition of x_j), nor to any vertex of Y (as said above). The vertex z_j is not adjacent to any vertex y_k with $k < j$ by the choice of z_j. If $zz_j \notin E(G)$, then the sets X' and Y' define a matching M' which satisfies (i) and (ii), and thus which contradicts the maximality of M. If $zz_j \in E(G)$, then the Eulerian subgraph $G[(E(C) - E(C')) \cup (E(C') - E(C))]$, with $C' = y_j z_j z x_j y_j$, satisfies $V(C) \cap V(C') = \{y_j\}$ since z has no neighbor in C, and thus contradicts the maximality of C. Hence $N_G(z) \cap X_J = \emptyset$ and z has no neighbor in X.

Finally if z has a neighbor t in $R - (Y_0 \cup Y_I \cup X_J)$, then the matching M'' corresponding to the ordered sets $X'' = \{t, x_{p+1}, \ldots, x_q\}$ and $Y'' = \{z, y_{p+1}, \ldots, y_q\}$ satisfies the conditions (i) and (ii) since z has no neighbor in $X \cup Y$. This contradicts the maximality of M and achieves the proof of Claim 1. □

Claim 2. (Favaron and Fraisse [7]) *G must be contracted to the Petersen graph.*

Proof. By contradiction. Suppose that G cannot be contracted to the Petersen graph. Let G^1 be the graph or multigraph obtained from G by deleting the vertices of degree 1 or 2 and replacing each path ayb where $d_G(y) = 2$ by the edge ab. Since G is essentially 3-edge-connected, G^1 is 3-edge-connected. Moreover, for each Eulerian subgraph C of G^1, there is a corresponding Eulerian subgraph of G containing $V(C)$. Since $S \cap B = \emptyset$, the set S is contained in $V(G^1)$. Since S is not contained in any Eulerian subgraph of G by Claim 1, S is not contained in any Eulerian subgraph of G^1. By Theorem 3.1, $|S| \geqslant 13$. Let $F = \{x_i y_i \mid 1 \leqslant i \leqslant 13\}$, $P = \{x_i \mid 1 \leqslant i \leqslant 13\}$ and $Q = \{y_i \mid 1 \leqslant i \leqslant 13\}$. We suppose that F consists of l paths of length 2 with $0 \leqslant l \leqslant 6$ and $13 - 2l$ edges of a matching. Then $|P| = 13$ and $|Q| = 13 - l$. We know that Q is independent, that $y_i x_j \notin E(G) - F$ for any $y_i \in Q$ and $x_j \in P$ with $1 \leqslant i < j \leqslant 13$, and that G is triangle-free. Hence, two different edges of F are joined by at most one edge of G which is of type $x_i x_j$ or $x_i y_j$ with $1 \leqslant i < j \leqslant 13$. More precisely, we can give an upper bound on the number μ of edges of G which are adjacent to two different edges of F. For a given value of l, this number can be maximum if the l paths of F occur with smaller indices than those of the $13 - 2l$ edges of the matching. This is due to the fact that the l vertices y_i belonging to paths of length 2 have degree 2 and thus they cannot be adjacent by an edge not in F to any vertex x_i with $i < j$. When this condition is fulfilled, there are at most l^2 edges between the vertices $x_1, x_2, \ldots, x_{2l}$ (since the number of edges of a triangle-free graph of order $2l$ is at most $(2l)^2/4$), $2l(13 - 2l)$ edges of type $x_i y_j$ between the sets $\{x_1, x_2, \ldots, x_{2l}\}$ and $\{y_{2l+1}, y_{2l+2}, \ldots, y_{13}\}$, and $(13 - 2l)(13 - 2l - 1)/2$ edges of type $x_i x_j$ or $x_i y_j$ with $i < j$ between the vertices of the set $\{x_{2l+1}, \ldots, x_{13}, y_{2l+1}, \ldots, y_{13}\}$. Then

$$\mu \leqslant l^2 + 2l(13 - 2l) + \frac{(13 - 2l)(13 - 2l - 1)}{2} = l - l^2 + 78.$$

Counting the edges of $G - F$ adjacent to some edge of F, we find at least $(13 - 2l)\delta(H)$ edges adjacent to an edge of a matching of F and $2l(\delta(H) - 1)$ edges adjacent to an edge of a path of length 2 (since each vertex y_i on such a path has degree 2 in G). At most $l - l^2 + 78$ of these edges have their two endvertices in $P \cup Q$ and are thus counted twice. Hence $m(G) \geqslant (13 - 2l)\delta(H) + 2l(\delta(H) - 1) - (l - l^2 + 78) + 13$, that is $\nu(H) = m(G) \geqslant 13\delta(H) + l^2 - 3l - 65 \geqslant 13\delta(H) - 67 \geqslant 10\delta(H) - 4$ since l is an integer between 0 and 6 and $\nu(H) \geqslant 196$. This contradicts the hypothesis that $\delta(H) \geqslant (\nu(H) + 5)/10$, and so Claim 2 must hold. □

By Claim 2, G can be contracted to the Petersen graph P_{10}. Let $v_1, v_2, \ldots, v_{10}$ be the ten vertices of the Petersen graph P_{10}, and W_i be the preimage of v_i $(i = 1, 2, \ldots, 10)$. Denote $\mathcal{SV} = \{v \in V(G) \mid d_G(v) \geqslant 12\}$. Since $d_G(u) + d_G(v) - 2 \geqslant \delta(H) \geqslant 21$ for every edge $e = uv \in E(G)$, we have either $d_G(u) \geqslant 12$ or $d_G(v) \geqslant 12$. So we have

$$\text{for every edge } e = uv \in E(G), \quad \text{either} \quad u \in \mathcal{SV} \quad \text{or} \quad v \in \mathcal{SV}. \tag{2}$$

Moreover, if $u, v \notin \mathcal{SV}$, then $uv \notin E(G)$. By the hypothesis of Theorem 1.6 that H is 3-connected, we have

$$G \text{ is essentially 3-edge-connected.} \tag{3}$$

Let $W \in \{W_i \mid 1 \leqslant i \leqslant 10\}$. Note that G is contracted to P_{10}. Then $|N_W(V(G) - V(W))| = 3$. If for any two vertices $w_1, w_2 \in N_W(V(G) - V(W))$, there is a dominating (w_1, w_2)-trail in W, then say W is *dominatiable*.

Claim 3. *Let W' be a graph obtained from W by deleting the vertices of degree* 1*. If $E(W') \neq \emptyset$, then W' is* 2*-edge-connected. Therefore W' contains some cycle.*

Proof. Since G is contracted to the P_{10} and W is the preimage of some vertex v_i, we may assume that $[V(W), V(G) - V(W)]_G = \{e_1, e_2, e_3\}$, where e_1, e_2, e_3 are edges adjacent to v_i in P_{10}. Suppose that W' contains a cut-edge $e = z_1 z_2$. Then e is also a cut-edge of W. Let (U_1, V_1) be the partition of $V(W)$ such that $[U_1, V_1]_W = \{e\}$ and $z_1 \in U_1$ and $z_2 \in V_1$. Since $z_1, z_2 \in V(W')$, we have $d_W(z_1) \geqslant 2$ and $d_W(z_2) \geqslant 2$. Thus $E(G[U_1]) \neq \emptyset$ and $E(G[V_1]) \neq \emptyset$. Note that $[V(W), V(G) - V(W)]_G = \{e_1, e_2, e_3\}$. We may assume that the number of edges joining U_1 and $V(G) - V(W)$ is 1, say e_1. Then $\{e_1, e\}$ is an essential edge-cut in G, contrary to (3). So Claim 3 holds. □

Claim 4. *If $\alpha'(W) = 1$, then $W = K_{1,p}$ for some $p \geqslant 1$. Therefore all three edges in $[V(W), V(G) - V(W)]_G$ must be incident with the vertex of $K_{1,p}$ with degree p, and so H_1 is dominatiable.*

Proof. Since W is a connected triangle-free graph and $\alpha'(W) = 1$, G is acyclic. By Claim 3 and $\alpha'(W) = 1$, $W = K_{1,p}$ for some $p \geqslant 1$. □

Claim 5. *Suppose that $\alpha'(W) = t \in \{2, 3, 4, 5\}$ and $\{u_1 a_1, u_2 a_2, \ldots, u_t a_t\}$ is a matching in W. Suppose that $u_i \in \mathcal{SV}$ $(i = 1, 2, \ldots, t)$. Then $V(W) \cap \mathcal{SV} = \{u_1, u_2, \ldots, u_t\}$ and $E(W - \{u_1, u_2, \ldots, u_t\}) = \emptyset$.*

Proof. Let $A = \{u_1, \ldots, u_t, a_1, \ldots, a_t\}$, $A_1 = A - u_i$ and $A_2 = A - a_i$. As $\alpha'(W) = t$, $E(W - A) = \emptyset$. Note that G is triangle-free and $\mathcal{SV} = \{v \in V(G) \mid d_G(v) \geqslant 12\}$. For each $z \in V(W) - A$, $d_W(z) \leqslant 5$ and so $d_G(z) \leqslant 8$. Thus $z \notin \mathcal{SV}$.

Since G does not contain a triangle and $\alpha'(W) = t \leqslant 5$, by $d_G(u_i) \geqslant 12$, we have $N_W(u_i) - A_1 \neq \emptyset$. Thus $N_W(a_i) \subseteq A_2$ (otherwise, $\{u_1 a_1, \ldots, u_{i-1} a_{i-1}, u_{i+1} a_{i+1}, \ldots, u_t a_t, u_i u, a_i a\}$ is a matching of W, where $u \in N_W(u_i) - A_1$ and $a \in N_W(a_i) - A_2$, contrary to the assumption that $\alpha'(W) = t$). Since G is triangle-free, we have $d_W(a_i) \leqslant 5$, and so $d_G(a_i) \leqslant 8$. Thus $a_i \notin \mathcal{SV}$. Therefore $\mathcal{SV} \cap V(W) = \{u_1, u_2, \ldots, u_t\}$, and $E(W - \{u_1, u_2, \ldots, u_t\}) = \emptyset$. □

Claim 6. *If $\alpha'(W) = t \in \{2, 3, 4\}$, then W is dominatiable.*

Proof. Suppose that $\alpha'(W) = t$ and $\{u_1a_1, \ldots, u_ta_t\}$ is a matching in W. Without loss of generality, we assume that $u_i \in \mathcal{SV}$ $(i = 1, 2, \ldots, t)$ by (2). By Claim 5, $\mathcal{SV} \cap V(W) = \{u_1, u_2, \ldots, u_t\}$, and $E(W - \{u_1, u_2, \ldots, u_t\}) = \emptyset$. Let $w_1, w_2, w_3 \in N_W(V(G) - V(W))$ and $w_1z_1, w_2z_2, z_3w_3 \in [V(W), V(G) - V(W)]_G$. If $w_1 = w_2$ and $d_W(w_1) = 1$, then $\{z_3w_3, w_1x\}$ is an essential edge-cut in G for some $x \in N_W(w_1)$, contrary to (3). So we have $d_W(w_1) \geqslant 2$ if $w_1 = w_2$.

Suppose, by contradiction, that W does not have a dominating (w_1, w_2)-trail. If $w_1 \neq w_2$, we let $K_1 = W + \{w_1w, w_2w\}$, where w is a new vertex; if $w_1 = w_2$, we let $K_1 = W$ and $w = w_1$. Let $K = K_1 - D_1(K_1)$. Then $u_1, \ldots, u_t, w \in V(K)$, and K is 2-edge-connected by Claim 3. Let $S = \{u_1, \ldots, u_t\} \cup \{w\}$. Then $K - S$ is edgeless, and K does not have an S-Eulerian subgraph. By Theorem 2.6, K is contracted to a member $L \in \mathcal{F}$ (see Figs. 2–4) such that S intersects the preimage of every vertex in $B(L)$. Note that for each $L \in \mathcal{F}$, $d_L(b_i) = 2$ $(i = 1, 2, 3)$ and the set of degree 2 vertices is independent. Without loss of generality, we assume that the preimages of b_1, b_2 do not contain w.

Note that $[V(W), V(G) - V(W)]_G = \{w_1z_1, w_2z_2, z_3w_3\}$. Suppose that $w \in V(L)$. Then $w_1, w_2 \in V(L)$. If $w_1 \neq w_2$, then $d_L(w) = 2$. Thus $w_1, w_2 \notin \{b_1, b_2\}$. If $w_1 = w_2$, then $w_1 = w_2 = w$. Thus $w_1, w_2 \notin \{b_1, b_2\}$ still hold. Since either $w_3 \notin V(PI(b_1))$ or $w_3 \notin V(PI(b_2))$, we may assume that $w_3 \notin V(PI(b_1))$. Thus $[V(PI(b_1)), V(G) - V(W)]_G = \emptyset$ and the set of two edges adjacent to $V(PI(b_1))$ is an essential edge-cut of G, contrary to (3). So $w \notin V(L)$. We assume that the preimage of some $b_i (\notin \{b_1, b_2\})$ contains w. Thus $w_1, w_2 \notin V(PI(b_i))$ $(i = 1, 2)$. Therefore either $|[V(PI(b_1)), V(G) - V(W)]_G| = 0$ or $|[V(PI(b_2)), V(G) - V(W)]_G| = 0$. Without loss of generality, we assume that $|[V(PI(b_1)), V(G) - V(W)]_G| = 0$. Then the set of two edges adjacent to $V(PI(b_1))$ is an essential edge-cut of G, contrary to (3). □

Claim 7. *If $\alpha'(W) = t \geqslant 1$, then $|E(W)| \geqslant t\delta(H) + 2t - t^2 - 3$.*

Proof. Let $\{u_1v_1, \ldots, u_tv_t\}$ be a matching in W. Then $E(W - \{u_1, \ldots, u_t, v_1, \ldots, v_t\}) = \emptyset$, and for any pair of u_iv_i, u_jv_j $(i \neq j)$, $|[\{u_i, v_i\}, \{u_j, v_j\}]_W| \leqslant 2$ since W does not contain a triangle. Since for $\sum_{v \in V(W)} d_W(v)$, the edges of u_iv_i and the edges in $[\{u_i, v_i\}, \{u_j, v_j\}]_W$ are counted twice, and since $|[V(W), V(G) - V(W)]_G| = 3$, we have

$$\begin{aligned}\left|E(W)\right| &= \sum_{v \in V(W)} d_W(v) - \left|\{u_1v_1, u_2v_2, \ldots, u_tv_t\}\right| - \sum_{i \neq j}\left|\left[\{u_i, v_i\}, \{u_j, v_j\}\right]_W\right| \\ &\geqslant \left(\sum_{v \in V(W)} d_G(v) - 3\right) - t - 2\binom{t}{2}.\end{aligned}$$

Since $\delta(H) \leqslant d_G(u_i) + d_G(v_i) - 2$ for each u_iv_i, we have

$$\left|E(W)\right| \geqslant t\big(\delta(H) + 2\big) - 3 - t - 2\binom{t}{2} = t\delta(H) + 2t - t^2 - 3. \qquad \square$$

Now we finish the proof of Theorem 1.6. Let $|\{v_i \mid v_i \text{ is a trivial vertex in } P_{10}\}| = s$. By (2), the set of all trivial vertices in P_{10} is independent. Since $\alpha(P_{10}) = 4$, we have $0 \leqslant s \leqslant 4$. If $s = 0$, then each v_i is a nontrivial vertex. Thus $|E(W_i)| \geqslant \delta(H) - 2$ by Claim 7. Therefore

$$m(G) = \sum_{i=1}^{10}\left|E(W_i)\right| + 15 \geqslant 10\big(\delta(H) - 2\big) + 15 = 10\delta(H) - 5.$$

By the hypothesis of Theorem 1.6, we have

$$\delta(H)=\frac{\nu(H)+5}{10},\qquad \left|E(W_i)\right|=\delta(H)-2,$$

$\alpha'(W_i)=1$ and $W_i=K_{1,p}$, where $p=\delta(H)-2=(\nu(H)-15)/10$.

If $s\geqslant 1$, without loss of generality, we assume that v_1 is trivial. Since $P_{10}-v_1$ has a spanning cycle, there exists a W_i, say W_{10}, such that $\alpha'(W_{10})\geqslant 5$ by Claims 4 and 6. If $s\leqslant 3$, then

$$\begin{aligned} m(G)&=\sum_{i=1}^{10}\left|E(W_i)\right|+15\geqslant(10-s-1)\bigl(\delta(H)-2\bigr)+\bigl(5\delta(H)-18\bigr)+15\\ &\geqslant 6\bigl(\delta(H)-2\bigr)+5\delta(H)-3=11\delta(H)-15\geqslant 10\delta(H)-4. \end{aligned}$$

Thus $\delta(H)\leqslant(\nu(H)+4)/10$, a contradiction. So $s=4$. By Claims 3, 6 and $\delta(H)\geqslant(\nu(H)+5)/10$, $\alpha'(W_{10)})=5$. If there exists some W_j $(j\neq 10)$ such that $\alpha'(W_j)\geqslant 2$, then

$$\begin{aligned} m(G)&=\sum_{i=1}^{10}\left|E(W_i)\right|+15\geqslant\left|E(W_{10})\right|+\left|E(W_j)\right|+4\bigl(\delta(H)-2\bigr)+15\\ &=\bigl(5\delta(H)-18\bigr)+\bigl(2\delta(H)-3\bigr)+4\delta(H)+7=11\delta(H)-17\geqslant 10\delta(H)+4, \end{aligned}$$

a contradiction. So the number of W_i with $\alpha'(W_i)=1$ is 5. Without loss of generality, we assume that $\alpha'(W_i)=1$ $(i=5,6,7,8,9)$ and $\alpha'(W_{10})=5$. Let $\{e_1f_1,e_2f_2,e_3f_3,e_4f_4,e_5f_5\}$ be a matching of W_{10} and $B=\{e_1,\dots,e_5,f_1,\dots,f_5\}$ and $Z=W_{10}[B]$. By (2), we assume that $e_i\in\mathcal{SV}$ $(i=1,2,\dots,5)$. By Claim 5, $\mathcal{SV}\cap V(W_{10})=\{e_1,e_2,\dots,e_5\}$, and $E(W_{10}-\{e_1,e_2,\dots,e_5\})=\emptyset$.

If $|E(Z)|\leqslant 16$, then

$$\left|E(W_{10})\right|=\sum_{v\in B}d_G(v)-\left|E(Z)\right|-3\geqslant 5\bigl(\delta(H)+2\bigr)-16-3=5\delta(H)-9.$$

Thus

$$\begin{aligned} m(G)&=\sum_{i=5}^{9}\left|E(W_i)\right|+\left|E(W_{10})\right|+15\geqslant 5\bigl(\delta(H)-2\bigr)+\bigl(5\delta(H)-9\bigr)+15\\ &=10\delta(H)-4, \end{aligned}$$

and so $\delta(H)\leqslant(\nu(H)+4)/10$, a contradiction. So we have

$$\left|E(Z)\right|\geqslant 17. \tag{4}$$

If Z is collapsible, then $W_{10}-D_1(W_{10})$ is collapsible by Theorem 2.3. Thus for any pair of vertices $u,v\in W_{10}-D_1(W_{10})$, $W_{10}-D_1(W_{10})$ has a spanning (u,v)-trail by Lemma 2.9. Then for any pair of vertices $u,v\in V(W_{10})$, W_{10} has a dominating (u,v)-trail, and so W_{10} is dominatiable. Since each W_i $(i=1,2,3,4)$ is a trivial graph, since each W_i $(i=5,6,\dots,9)$ is dominatiable, and since $P_{10}-v_1$ has a spanning cycle, G has a dominating Eulerian subgraph, a contradiction. So Z is not collapsible. Moreover,

$$W_{10}-D_1(W_{10})\text{ is not collapsible.} \tag{5}$$

Therefore Z is not 2-edge-connected by Lemma 2.8.

Let $K\subseteq Z$ with $|V(K)|=8$. Suppose that $|E(K)|\geqslant 14$. Then K is 2-edge-connected by Lemma 2.7. If $|D_2(K)|\geqslant 2$, then $|E(K)|\leqslant 2+2+9=13$ by Lemma 2.7(iv), a contradiction.

So $|D_2(K)| \leqslant 1$. By Lemma 2.5 and by the fact that G is triangle-free, K is collapsible. By Claim 3 and Theorem 2.3, $M_{10} - D_1(M_{10})$ is collapsible, contrary to (5). So

$$\left|E(K)\right| \leqslant 13. \tag{6}$$

Suppose that Z is not connected and Z_1 is a component of Z. Then $|V(Z_1)| \in \{2, 4, 6, 8\}$. By Lemma 2.7(ii), (iv) and (4), $|V(Z_1)|$ is either 2 or 8. We may assume that $|V(Z_1)| = 2$ and $Z_2 = Z - V(Z_1)$. Then $|E(Z_1)| = 1$, $|V(Z_2)| = 8$ and $|E(Z_2)| \geqslant 16$, contrary to (6). So Z is connected. Let X be a cut-edge of Z and Z_3, Z_4 be components of $Z - X$ with $|V(Z_3)| \leqslant |V(Z_4)|$. By Lemma 2.7 and (4), $|V(Z_3)|$ is either 1 or 2. If $|V(Z_3)| = 2$, then $|E(Z_4)| \geqslant 17-2 = 15$, contrary to (6). So $|V(Z_3)| = 1$, $|V(Z_4)| = 9$, $|[V(Z_3), V(Z_4)]_Z| = 1$ and $|E(Z_4)| \geqslant 16$.

By (6) and Lemma 2.7, Z_4 is 3-edge-connected. Let Z'_4 be the reduction of Z_4. Then Z'_4 is still 3-edge-connected and $|V(Z'_4)| \leqslant 9$. Thus $Z'_4 = K_1$ by Theorem 2.4, that is, Z_4 is collapsible. By Claim 3 and Theorem 2.3, $W_{10} - D_1(W_{10})$ is collapsible, contrary to (5). □

Acknowledgments

The authors thank the referees for their careful reading of the paper and their useful suggestions.

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