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On Group Chromatic Number of Graphs

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Abstract. Let G be a graph and A an Abelian group. Denote by $F(G, A)$ the set of all functions from $E(G)$ to A. Denote by D an orientation of $E(G)$. For $f \in F(G, A)$, an (A, f)-coloring of G under the orientation D is a function $c : V(G) \mapsto A$ such that for every directed edge uv from u to v, $c(u) - c(v) \neq f(uv)$. G is *A-colorable* under the orientation D if for any function $f \in F(G, A)$, G has an (A, f)-coloring. It is known that A-colorability is independent of the choice of the orientation. The *group chromatic number* of a graph G is defined to be the least positive integer m for which G is A-colorable for any Abelian group A of order $\geq m$, and is denoted by $\chi_g(G)$. In this note we will prove the following results. (1) Let H_1 and H_2 be two subgraphs of G such that $V(H_1) \cap V(H_2) = \emptyset$ and $V(H_1) \cup V(H_2) = V(G)$. Then $\chi_g(G) \leq \min\{\max\{\chi_g(H_1), \max_{v \in V(H_2)} deg(v, G) + 1\}, \max\{\chi_g(H_2), \max_{u \in V(H_1)} deg(u, G) + 1\}\}$. We also show that this bound is best possible. (2) If G is a simple graph without a $K_{3,3}$-minor, then $\chi_g(G) \leq 5$.

1. Introduction

Graphs in this note are finite, loopless. Unless otherwise stated, terminology and notation not defined here are referred to Bondy and Murty [1]. Let G be a graph. Denote by $deg(v, G)$ the degree of v in a graph G and denote by $N(v, G)$ the set of all neighbors of v in a graph G. When G is a directed graph, denote by $E^+(v, G)$ the directed edges with their tail at the vertex v. $\chi(G)$ and $\delta(G)$ denote the chromatic number and the minimum degree of a graph G, respectively. Let G and H be two graphs. Denote $H \subseteq G$ if H is a subgraph of G. If H can be obtained from G by contracting some edges of G and deleting the resulting loops, then G is *contractible* to H. If G contains a subgraph which is contractible to Γ, then Γ is a *minor* of G. If G does not contain a minor Γ, We say G is Γ-*minor free.*

The *proper coloring* of a graph G is a mapping c from $V(G)$ to $\{1, 2, \dots, k\}$ where k is a positive integer such that $c(u) \neq c(v)$ if uv is an edge of G. Jagear, Linial, Payan and Tarsi [4] introduced group coloring as follows.

Let G be a graph and A an Abelian group. Denote by $|A|$ the cardinality of A and denote by $F(G, A)$ the set of all functions from $E(G)$ to A. Denote by D an orientation of $E(G)$. For $f \in F(G, A)$, an (A, f)-coloring of G under the orientation D is a function $c : V(G) \mapsto A$ such that for every directed edge uv from

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u to v, $c(u) - c(v) \neq f(uv)$. G is *A-colorable* under the orientation D if for any function $f \in F(G, A)$, G has an (A, f)-coloring. It is known [4] that A-colorability is independent of the choice of the orientation. The *group chromatic number* of a graph G is defined to be the least positive integer m for which G is A-colorable for any Abelian group A with $|A| \geq m$ under a given orientation D, and is denoted by $\chi_g(G)$.

Given each edge of G a weight, group coloring is to color all vertices of G such that the difference of two colors of two end-vertices of each edge of G does not equal the weight of this edge. For making the definition sense, Abelian group and orientation of G are required. When the weight of each edge is 0, the coloring is proper coloring. In this meaning, group coloring is a generalization of proper coloring.

Let $H \subseteq G$ be graphs, and A be an Abelian group. Given an $f \in F(G, A)$, if for an $(A, f|_{E(H)})$-coloring c_0 of H, there is an (A, f)-coloring c of G such that c is an extension of c_0. Then we say that c_0 is *extended to c*. If any $(A, f|_{E(H)})$-coloring c_0 of H can be extended to an (A, f)-coloring c, then we say that (G, H) is (A, f)*-extensible*. If for any $f \in F(G, A)$, (G, H) is (A, f)-extensible, then (G, H) is *A-extensible*.

General speaking, the group chromatic number is large than the chromatic number for a graph. As pointed in [6], for some graph G, $\chi_g(G) - \chi(G)$ can be arbitrarily large. Thus, some properties of proper coloring cannot hold for group coloring. Group coloring has been around many years. The group colorability of simple planar graphs was investigated by Jagear *et al* [4] and Lai and Zhang [7]. The group coloring of simple non-planar graphs was also studied by Lai and Zhang [7]. The group chromatic number of simple bipartite graphs was studied by Lai and Zhang [6]. In this note, we first establish an upper bound for the group chromatic number of a graph obtained from its two subgraphs. We show that this bound is best possible too. We also investigate the group chromatic number of a simple graph without a $K_{3,3}$-minor.

2. Main Results

Let G_1 and G_2 be subgraphs of G. The *union* $G_1 \cup G_2$ of G_1 and G_2 is the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If G_1 and G_2 are vertex-disjoint, denote the union by $G_1 + G_2$. The *join* $G \vee H$ of distinct graphs G and H is the graph obtained from $G + H$ by joining each vertex of G to each vertex of H. Some formulas in proper coloring can be generalized to similar ones in group coloring. For example, $\chi(G_1 + G_2) = \max\{\chi(G_1), \chi(G_2)\}$. We easily get $\chi_g(G_1 + G_2) = \max\{\chi_g(G_1), \chi_g(G_2)\}$. Others in proper coloring cannot be generalized to similar ones in group coloring. One knows [3] that $\chi(G \vee H) = \chi(G) + \chi(H)$. But this formula cannot hold in group coloring. For example, $\chi(K_{2,2}) = 2$ and $K_{2,2} = \overline{K_2} \vee \overline{K_2}$. It was shown [6] that $\chi_g(K_{2,2}) = 3$. We first establish the following lemma.

Lemma 2.1. *Let H_1, H_2 be induced subgraphs of a graph G such that $V(H_1) \cap V(H_2) = \emptyset$ and $V(G) = V(H_1) \cup V(H_2)$. Let A be an Abelian group with $|A| \geq$*

$(\max_{v\in V(H_2)} deg(v, G)) + 1$. *Let* $f \in F(G, A)$*, and suppose that* H_1 *has an* $(A, f|_{E(H_1)})$*-coloring* c_0 *(under some orientation). Then* c_0 *can be extended to an* (A, f)*-coloring of* G*.*

Proof. We argue by contradiction and assume that G is a counterexample with $|V(H_2)|$ minimized. The lemma follows immediately if $|V(H_2)| = 0$ and so we assume that $|V(H_2)| > 0$. Let A be an Abelian group with $|A| \geq (\max_{v\in V(H_2)} deg(v, G)) + 1$ and let $w \in V(H_2)$. By the choice of H_2, c_0 can be extended to an $(A, f|_{E(G-w)})$-coloring c of $G - w$.

We may assume that G is oriented such that all edges incident with w are oriented from w (if there is an edge e oriented to w, we simply reverse the orientation of e and replace $f(e)$ by $-f(e)$). Let $N(w, G) = \{w_1, w_2, \ldots, w_t\}$ and $E^+(w, G) = \{e_1^1, e_1^2, \ldots, e_1^{i_1}, e_2^1, e_2^2, \ldots, e_2^{i_2}, \ldots e_t^1, e_t^2, \ldots, e_t^{i_t}\}$ where $e_j^1, e_j^2, \ldots e_j^{i_j}$ are all directed edges from w to w_j. It follows that $i_1 + i_2 + \cdots + i_t = deg(w, G) \leq |A| - 1$ and hence there is an $a \in A - \cup_{j=1}^{t} \cup_{l=1}^{l=i_j} \{c(w_j) + f(e_j^l)\}$. Define $c_1 : V(G) \mapsto A$ by

$$c_1(u) = \begin{cases} c(u), & \text{if } u \neq w, \\ a, & \text{if } u = w. \end{cases}$$

Thus c_0 is extended to an (A, f)-coloring of G. This contradiction establishes our theorem. □

From Lemma 2.1, we immediately get the following corollary.

Corollary 2.2. *Let* H_1 *and* H_2 *be two subgraphs of* G *such that* $V(H_1) \cap V(H_2) = \emptyset$ *and* $V(G) = V(H_1) \cup V(H_2)$*. Then*

$$\chi_g(G) \leq \min\{\max\{\chi_g(H_1), \max_{v\in V(H_2)} deg(v, G) + 1\}, \max\{\chi_g(H_2), \max_{u\in V(H_1)} deg(u, G) + 1\}\}. \tag{1}$$

The bound of Corollary 2.2 is sharp in the sense that there are infinite number of graphs which are join of their two subgraphs and their group chromatic number equals the value given by (1). For example, let $m, n \geq 2$ be two integers. Observe $G = K_n \vee K_m$. It follows that $G = K_{n+m}$ and hence $\chi_g(G) = n + m$ which equals what is given by (1). Another example is $H = \overline{K_2} \vee \overline{K_2} = K_{2,2}$. As stated in the beginning of this section, $\chi_g(H) = 3$ is the value given by (1).

From Corollary 2.2, we obtain a upper bound of group chromatic number of join of two graphs.

Corollary 2.3. *If* H_1 *and* H_2 *be two graphs, then*

$$\chi_g(G \vee H) \leq \min\{|V(H_1)| + \Delta(H_2) + 1, |V(H_2)| + \Delta(H_1) + 1\}.$$

Brook's Theorem [2] tells us that $\chi(G) \leq \Delta(G)+1$ for a graph G. We ask whether the bound of Corollary 2.3 can be decreased to $\min\{|V(H_1)| + \chi_g(H_2), |V(H_2)| + \chi_g(H_1)\}$. If true, for $m, n \geq 2$, $G = K_n \vee K_m$ shows that the bound would be best possible.

One of famous theorems in graph theory is the Four Color Theorem. It states that four colors are enough to color every planar graph. One naturally consider what is the group chromatic number for planar graphs. Jeager, Linial, Payan, and Tarsi [4] proved that if G is a simple planar graph, then $\chi_g(G) \leq 6$. This bound was improved by Lai and Zhang [7], as follows.

Theorem 2.4. *Let G be a simple planar graph and let $H \subseteq G$ be a subgraph isomorphic to a K_2. Then for any Abelian group $|A| \geq 5$, (G, H) is A-extensible.*

Kuratowski [5] proved the next well-known theorem which is applied in the proof of Theorem 2.7.

Theorem 2.5. *A graph is planar if and only if it is K_5-minor free and $K_{3,3}$-minor free.*

Group colorability of non-planar graphs was also considered by Lai and Zhang [7]. They proved that the group chromatic number of a graph without K_5-minors is at most 5. We here observe other kind of non-planar graphs. This is the graphs without $K_{3,3}$-minors. The following theorem will play a key role in our proof of Theorem 2.7.

Theorem 2.6. (Wagner [8]) *Suppose that G is not planar with $|V(G)| \geq 6$. If G is 3-connected, then G contains a $K_{3,3}$ minor.*

We next prove a stronger version of theorem 2.8. The extension method is applied in the proof of the following theorem.

Theorem 2.7. *Let G be a connected simple graph without $K_{3,3}$-minors and let A be an Abelian group with $|A| \geq 5$. Suppose that H is a subgraph of G isomorphic to a K_2. Then (G, H) is A-extensible.*

Proof. Let $f \in F(G, A)$ and c_0 be an given $(A, f|_{E(H)})$-coloring. We shall prove by induction on $|V(G)|$ that c_0 can be extended to an (A, f)-coloring of G. By Theorem 2.4, we may assume that G is not planar with $|V(G) \geq 5$.

Assume first that $|V(G)| = 5$. Since G is not planar, by Theorem 2.5, $G \cong K_5$. Assume $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$ and that the edge $v_i v_j$ is oriented from v_i to v_j if $i < j$. Without loss of generality, we assume that $H = K_2$ is a subgraph of G induced by $\{v_1, v_2\}$ and that $c_0 : v_k \mapsto a_k$ for $k = 1, 2$ such that $a_1 - a_2 \neq f(v_1 v_2)$. By Lemma 2.1, c_0 can be extended to an (A, f)-coloring of G.

Assume then that $|V(G)| \geq 6$ and that the theorem holds for graphs with smaller values of $|V(G)|$. Let T be a minimum vertex cut of G. By Theorem 2.6, $1 \leq |T| \leq 2$.

Let G_1 and G_2 be two proper subgraphs of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = T$.

We define G_i' from G_i, $i = 1, 2$, as follows. If $|T| = 1$, let $G_i' = G_i$ for $i = 1, 2$. If $|T| = 2$, let $T = \{u, v\}$. When $uv \in E$, define $G_i' = G_i$ for $i = 1, 2$. When $uv \notin E$, define $G_i' = G_i + uv$ for $i = 1, 2$. Since T is a vertex cut of G, $V(G_i) - V(T) \neq \emptyset$, $i = 1, 2$. It implies that $|V(G_i')| < |V(G)|$, $i = 1, 2$. By the choice of T, G_i' is connected for $i = 1, 2$. We also define $f' : E(G_1') \cup E(G_2') \mapsto A$ as follows. If $e \in E(G)$, define $f'(e) = f(e)$; if $e = uv \notin E(G)$ but $uv \in E(G_i')$, define $f'(uv) = 0$.

It is easy to see that both G_1' and G_2' have no $K_{3,3}$-minor. We assume, without loss of generality, that $H \subset G_1$. Since $|V(G_1')| < |V(G)|$, by induction there is an $(A, f'|_{E(G_1')})$-coloring $c_1 : V(G_1') \mapsto A$ which extends c_0. If $|T| = 1$, let $T = \{w\}$. Suppose that $w' \in V(G_2)$ such that $w'w \in E(G)$ and $w'w$ is oriented from w to w'. Define $c(w') \in A_5 - \{c_1(w) - f(ww')\}$. By induction, the color $c_1(w)$ and $c_1(w')$ can be extended to an $(A, f'|_{E(G_2')})$-coloring $c_2 : V(G_2') \mapsto A$. If $|T| = 2$, by induction, the color $c_1(u)$ and $c_1(v)$ can be extended to an $(A, f'|_{E(G_2')})$-coloring $c_2 : V(G_2') \mapsto A$. Using c_1 and c_2, c_0 can be extended to an (A, f)-coloring $c : V(G) \mapsto A$. □

From Theorem 2.7, we obtain the following theorem immediately.

Theorem 2.8. *Let G be a simple graph without $K_{3,3}$ minors. Then $\chi_g(G) \leq 5$.*

The bound of Theorem 2.8 is as best as possible in the sense that $\chi_g(K_5) = 5$ since $\chi_g(K_5) \geq \chi(K_5) = 5$. The result of Theorem 2.8 is different from that of Lai and Zhang [7] since K_5 has no a $K_{3,3}$-minor but a K_5-minor. So far no one knows what is the group chromatic number for a simple planar graph. If this number is 4, that would be a generalization of the Four Color Theorem. Thus, we have the following question.

Question 2.9. *What is the group chromatic number of a simple planar graph?*

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References

1. Bondy, J.A., Murty, U.S.R.: Graph theory with applications, American Elsevier, New York, 1976
2. Brooks, R.L.: On coloring the nodes of a network. Proc. Cambridge Phil. Soc. **37**, 194–197 (1941)
3. Harary, F.: Graph Theory, Addison-Wesley, 1969
4. Jaeger, F., Linial, N., Payan, C., Tarsi, M.: Graph connectivity of graphs–a nonhomogeneous analogue of nowhere-zero flow properties. J. Combin. Theory, B. **56**, 165–182 (1992)
5. Kuratowski, K.: Sur la problème des courbes gauches en topologie. Fund. Math. **156**, 271–283 (1930)
6. Lai, H.-J., Zhang, X.: Group colorability of graphs. Ars Combinatorics **62**, 299–317 (2002)

7. Lai, H.-J., Zhang, X.: Group chromatic number of graphs without K_5-minors. Graphs and Combinatorics **18**, 147–154 (2002)
8. Wagner, K.: Über eine eigneschaft der ebenen komplexe. Math. Ann. **144**, 570–590 (1937)

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