Graphs and Combinatorics (2005) 21:77–88
Digital Object Identifier (DOI) 10.1007/s00373-004-0579-7

Graphs and Combinatorics

Spanning Trails Connecting Given Edges

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Abstract. Suppose that $\mathcal{F}$ is the set of connected graphs such that a graph $G \in \mathcal{F}$ if and only if G satisfies both (F1) if X is an edge cut of G with $|X| \leq 3$, then there exists a vertex v of degree $|X|$ such that X consists of all the edges incident with v in G, and (F2) for every v of degree 3, v lies in a k-cycle of G, where $2 \leq k \leq 3$.

In this paper, we show that if $G \in \mathcal{F}$ and $\kappa'(G) \geq 3$, then for every pair of edges $e, f \in E(G)$, G has a trail with initial edge e and final edge f which contains all vertices of G. This result extends several former results.

1. Introduction

Graphs in this paper are finite, undirected, and may contain multiple edges but no loops. We call a graph *simple* if it contains no multiple edges. Undefined terms and notation are from [1]. As in [1], the edge-connectivity of a graph G is denoted by $\kappa'(G)$. For a vertex $v \in V(G)$, $d_G(v)$ denotes the degree of v in G. We use $H \subseteq G$ ($H \subset G$) to denote the fact that H is a subgraph of G (proper subgraph of G). If $X \subseteq E(G)$ is an edge subset, then $G[X]$ denotes the subgraph of G induced by X. If $H \subset G$, then for an edge subset $X \subseteq E(G) - E(H)$, we write $H + X$ for $G[E(H) \cup X]$. When $X = \{e\}$, we also use $H + e$ for $H + \{e\}$.

Let $X \subseteq E(G)$. The *contraction* G/X is obtained from G by contracting each edge of X and deleting the resulting loops. If $H \subseteq G$, we write G/H for $G/E(H)$. Note that even if G is a simple graph, contracting some edges of G may result in a graph with multiple edges. Note that any subset $X \subseteq E(G/H)$ can also be viewed as a subset in $E(G)$. A connected graph with at least two vertices is called a *nontrivial* graph.

The concept of collapsibility was introduced by Catlin [4], as follows. Let $O(G)$ denote the set of odd degree vertices of G. For a subset $R \subset V(G)$ with $|R|$ even, a subgraph Γ of G is called an *R-subgraph* if $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph G is *collapsible* if for any even subset R of $V(G)$, G has an R-subgraph.

A graph G is *eulerian* if $O(G) = \emptyset$ and G is connected. A graph G is *supereulerian* if G has a spanning eulerian subgraph. In particular, K_1 is both eulerian and supereulerian. Pulleyblank indicated that determining whether a graph G is supereulerian, even within the family of planar graphs, is NP-complete ([14]). For the literature of supereulerian graphs, see the survey of Catlin [3] and its update [8].

A subgraph H of a graph G is *dominating* if $G - V(H)$ is edgeless. A *dominating eulerian subgraph* is also called a DES. For an integer $i \geq 1$, define

$$D_i(G) = \{v \in V(G) : d(v) = i\}.$$

The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. Harary and Nash-Williams [10] established a close relationship between dominating eulerian subgraphs in graphs and Hamilton cycles in $L(G)$. Although Harary and Nash-Williams proved their theorem for simple graphs, this result is also true for graphs.

Theorem 1.1. (Harary and Nash-Williams [10]) *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a DES.*

A graph G is hamiltonian connected if for every pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-path. We view a trail of G as a vertex-edge alternating sequence

$$v_0, e_1, v_1, e_2, \ldots, e_k, v_k \tag{1}$$

such that all the e_i's are distinct and such that for each $i = 1, 2, \ldots, k$, e_i is incident with both v_{i-1} and v_i. The vertices in $v_1, v_2, \ldots, v_{k-1}$ are *internal vertices* of trail in (1).

For edges $e', e'' \in E(G)$, an (e', e'')-trail of G is a trail of G whose first edge is e' and whose last edge is e''. (Thus the trail in (1) is an (e_1, e_k)-trail). A *dominating (e', e'')-trail* of G is an (e', e'')-trail T of G such that every edge of G is incident with an internal vertex of T and a *spanning (e', e'')-trail* of G is a dominating (e', e'')-trail T of G such that $V(T) = V(G)$. By a similar argument in the proof of Theorem 1.1, one can obtain the following theorem for hamiltonian connected line graphs.

Theorem 1.2. *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is Hamiltonian connected if and only if for any pair of edges $e', e'' \in E(G)$, G has a dominating (e', e'')-trail.*

The following conjecture is due to Thomassen [15].

Conjecture 1.3. *Every 4-connected line graph is hamiltonian.*

This conjecture has been around for many years. Zhan [17, 18] proved that $L(G)$ is hamiltonian connected if $\kappa'(G) \geq 4$ and that $L(G)$ is hamiltonian connected if $L(G)$ is 7-connected. Note that every 4-edge-connected graph has 2 edge-disjoint spanning trees. Catlin and Lai [7] improved Zhan's result and proved that when G is a graph with 2 edge-disjoint spanning trees, then $L(G)$ is hamiltonian connected if and only if $L(G)$ is 3-connected.

Let G be a nontrivial graph (that is, $E(G) \neq \emptyset$,) that is not a path. Define $L^0(G) = G$, and for integer $k > 0$, define the k-th iterated line graph $L^k(G) = L(L^{k-1}(G))$ (if $L^{k-1}(G)$ is nontrivial). Chen et al. [9] proved that if $L^2(G)$ is 4-connected, then $L^2(G)$ is hamiltonian.

To further improve these known results, we continue the investigation on 3-edge-connected graphs which would have a hamiltonian connected line graph, and we also ask whether every 4-connected $L^2(G)$ is hamiltonian connected. The purpose of this paper is to seek partially answers to these questions. The techniques we use in this paper are different from those by others. But the authors think there is no hope to prove Conjecture 1.3 along the lines of the current techniques and methods.

We say that an edge $e \in E(G)$ is *subdivided* when it is replaced by a path of length 2 whose internal vertex, denoted $v(e)$, has degree 2 in the resulting graph. The process of taking an edge e and replacing it by that length 2 path is called *subdividing* e. For a graph G and edges $e', e'' \in E(G)$, let $G(e')$ denote the graph obtained from G by subdividing e', and let $G(e', e'')$ denote the graph obtained from G by subdividing both e' and e''. Thus,

$$V(G(e', e'')) - V(G) = \{v(e'), v(e'')\}.$$

From the definitions, one immediately has the following observation.

Lemma 1.4. *For a graph G and edges $(e', e'') \in E(G)$, if $G(e', e'')$ has a spanning $(v(e'), v(e''))$-trail, then G has a spanning (e', e'')-trail.*

Note that if G has a spanning (e', e'')-trail this does not necessarily imply that $G(e', e'')$ has a spanning $(v(e'), v(e''))$-trail. To approach the hamiltonian connectivity of line graph $L(G)$, one has to face the vertices of degree at most 3 in G. Thus, we consider the following class of graphs.

Let $\mathcal{F}$ denote the set of connected graphs such that a graph $G \in \mathcal{F}$ if and only if each of the following holds.

(F1) If X is an edge cut of G with $|X| \leq 3$, then there exists a vertex $v \in D_{|X|}(G)$ such that X consists of all the edges incident with v in G, and

(F2) For every $v \in D_3(G)$, v lies in a k-cycle of G, where $2 \leq k \leq 3$.

The following theorem is our main result in this paper.

Theorem 1.5. *Let $G \in \mathcal{F}$. If $\kappa'(G) \geq 3$, then for every pair of edges $e', e'' \in E(G)$ we have*

(i) *$G(e', e'')$ is collapsible and*

(ii) *$G(e', e'')$ has a spanning $(v(e'), v(e''))$-trail.*

The results of Theorem 1.5 are sharp in the sense that there exist infinite families of graphs showing that the conditions of Theorem 1.5 cannot be relaxed. Let G be a graph obtained from cycle $C_{2n} = v_1v_2 \ldots v_{2n}v_1$ by doubling edge v_iv_{i+1}

where $i = 1, 3, \ldots, 2n-1$. It follows that $\kappa'(G) = 2$ and that for edges $e_1 = v_2v_3$ and $e_2 = v_4v_5$, $G(e_1, e_2)$ has no a spanning $(v(e_1), v(e_2))$-trail. Thus, the condition of the edge-connectivity in Theorem 1.5 cannot be relaxed. We take two adjacent edges e', e'' from the Petersen graph P_{10}. Since P_{10} is cubic, there is a spanning (e', e'')-trail in P_{10} if and only if P_{10} has a Hamilton path from e' to e''. Of course, P_{10} has no such Hamilton path. Thus, $P_{10}(e', e'')$ has no spanning $(v(e'), v(e''))$-trail. Moreover, for an integer $m \geq 5$, let $G(m)$ denote the graph obtained from the Petersen graph P_{10} by replacing each vertex of P_{10} with a complete graph K_m. Then $\kappa'(G(m)) \geq 3$ but $G(m)$ is not in $\mathcal{F}$. For any pair of adjacent edges e', e'' in $G(m)$, $G(m)(e', e'')$ does not have a spanning $(v(e'), v(e''))$-trail. Hence, the condition $G \in \mathcal{F}$ in Theorem 1.5 cannot be relaxed either.

Theorem 1.5 improves several known results as the following corollaries. We shall prove them in section 4.

Corollary 1.6. *$L(G)$ is hamiltonian connected if one of the following holds.*

(1) *G is a graph such that the set of neighbors of each vertex of degree 3 in G is not an independent set and such that $L(G)$ is 4-connected.*
(2) (Kriesell [12]) *G is a $K_{1,3}$-free graph and $L(G)$ is 4-connected.*
(3) (Zhan [17]) $\kappa'(G) \geq 4$.

Corollary 1.7. *If $L^2(G)$ is 4-connected, then $L^2(G)$* is *hamiltonian connected.*

Let C_4 denote a 4-cycle in K_5. The graph $K_5 - E(C_4)$ is called an *hourglass*. A graph G is *hourglass free* if G does not have an induced subgraph isomorphic to $K_5 - E(C_4)$.

Corollary 1.8. (Broersma, Kriesell and Ryjáček [2]) *Every 4-connected hourglass free line graph is hamiltonian connected.*

Theorem 1.5 is stronger than Corollaries 1.6, 1.7 and 1.8 in the sense that there exists an infinite family of graphs such that the hamiltonian connectivity of their line graphs is assured by Theorem 1.5 but not by any of Corollaries 1.6, 1.7 and 1.8. We construct a family of graphs as follows. Let $n \geq 4$ be an even integer and define $G(n)$ as follows. $V(G(n)) = \{x_1, \ldots, x_{2n+1}, y_1, \ldots, y_{2n+1}, z_1, \ldots, z_n\}$ and $E(G(n)) = \cup\{x_iy_i : 1 \leq i \leq 2n+1\} \cup \{x_iy_{i-1} : 1 \leq i \leq 2n+1\} \cup\{x_iy_{i+1} : 1 \leq i \leq 2n+1\} \cup \{z_ix_{2i-1} : 1 \leq i \leq n$ and i is odd $\} \cup \{z_ix_{2i+1}$: $1 \leq i \leq n$ and i is odd $\cup$ $\{z_i x_{2i}$: $1 \leq i \leq n$ and i is odd $\} \cup \{z_iy_{2i-1} : 1 \leq i \leq n$ and i is even$\} \cup \{z_iy_{2i+1} : 1 \leq i \leq n$ and i is even$\} \cup \{z_iy_{2i}$: $1 \leq i \leq n$ and i is even$\}$ $\cup\{x_ix_{i+1} : 1 \leq i \leq 2n$ and i is odd$\}$ $\cup\{y_iy_{i+1} : 1 \leq i \leq 2n$ and i is even$\}$ where the indices are expressed modulo $2n+1$. By Lemma 1 and Theorem 1.5, for every pair of edges $e', e'' \in E(G(n))$, $G(n)(e', e'')$ has a spanning $(v(e'), v(e''))$-trail. Thus, $L(G(n))$ is hamiltonian connected. On the other hand, $G(n)$ has a subgraph $K_{1,3}$ induced by vertices z_1, x_3, y_2 and y_4 and $\kappa'(G(n)) = 3$. $L(G(n))$ contains an hourglass subgraph induced by edges $z_1x_3, x_1y_2, x_2y_2, x_3y_2$ and x_3y_4. Thus, we cannot apply any of Zhan [17], Kriesell [12] and Broersma, Kriesell and Ryjáček [2]'s theorems to $G(n)$.

The proof of Theorem 1.5 depends on edge-disjoint spanning trees. The classic theorem for the existence of k edge-disjoint spanning trees was proved by Nash-Williams [13], and Tutte [16] independently. However, Catlin obtained a stronger theorem as follows.

Theorem 1.9. (Catlin [5]) *Let G be a graph and let $k \geq 1$ be an integer. The following are equivalent.*

(i) $\kappa'(G) \geq 2k$.
(ii) *For any edge subset $X \subset E(G)$ with $|X| \leq k$, $G - X$ has at least k edge-disjoint spanning trees.*

In Section 2, we discuss Catlin's reduction method which will be needed in our proof of the main result. In Section 3, we prove Theorem 1.5. The last section is devoted to the generalizations of Theorem 1.5 and to applications of the main results.

2. Catlin's Reduction Method

Catlin showed in [4] that every vertex of G lies in a unique maximal collapsible subgraph of G. The *reduction* of G is obtained from G by contracting all maximal collapsible subgraphs. A graph G is *reduced* if G has no nontrivial collapsible subgraphs. A *nontrivial vertex* in the reduction of G is a vertex which is the contraction image of a nontrivial connected subgraph of G. Note that if G has an $O(G)$-subgraph Γ, then $G - E(\Gamma)$ is a spanning eulerian subgraph of G. Therefore, every collapsible graph is supereulerian. We summarize some results on Catlin's reduction method and other related facts as follows.

Theorem 2.1. *Let G be a graph and let H be a collapsible subgraph of G. Let v_H denote the vertex onto which H is contracted in G/H. Each of the following holds.*

(i) *(Catlin, Theorem 3 of [4]) G is collapsible (supereulerian, respectively) if and only if G/H is collapsible (supereulerian, respectively). In particular, G is supereulerian if and only if the reduction of G is supereulerian; and G is collapsible if and only if the reduction of G is K_1.*
(ii) *If G is collapsible, then for any pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-trail.*
(iii) *For vertices $u, v \in V(G/H) - \{v_H\}$, if G/H has a spanning (u, v)-trail, then G has a spanning (u, v)-trail.*
(iv) *2-cycles and 3-cycles are collapsible.*

Proof. (ii) Let $R = (O(G) \cup \{u, v\}) - (O(G) \cap \{u, v\})$. Then $|R|$ is even. Let Γ_R be an R-subgraph of G. Note that $G - E(\Gamma_R)$ is connected and that u and v are the only two vertices of odd degree in $G - E(\Gamma_R)$. Thus $G - E(\Gamma_R)$ is a spanning (u, v)-trail of G.

(iii) It follows from 2.1(i).

(iv) It follows from the definition of collapsibility immediately. □

Jaeger in [11] showed that if G has two edge-disjoint spanning trees, then G is supereulerian. This result was later improved by Catlin (Theorem 7 in [4]). Defining $F(G)$ to be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees, Catlin [4] and Catlin *et al.* [6] improved Jaeger's result. We put these former results in the following theorem.

Theorem 2.2. *Let G be a graph. Each of the following holds.*

(i) *(Jaeger [11]) If* $F(G) = 0$, *then G is supereulerian.*
(ii) *(Catlin, Theorem 7 in [4]) If* $F(G) \leq 1$ *and if G is connected, then G is collapsible if and only if G is not contractible to a* K_2.
(iii) *(Catlin, Han and Lai, Theorem 1.5 in [6]) If* $F(G) \leq 2$ *and if G is connected, then G is collapsible if and only if the reduction of G is neither a* K_2 *nor a* $K_{2,s}$ *for some integer* $s \geq 1$

In order to apply Theorem 2.2 in our proofs, we also need the following observations.

Lemma 2.3. *Let G be a graph. Each of the following holds.*

(i) *For any* $e \in E(G)$, $F(G(e)) \leq F(G) + 1$.
(ii) $F(G) \leq F(G/e) + 1$.

Proof. (i) Suppose that X is a set of edges none of which is in G such that $G + X$ has two edge-disjoint spanning trees T_1 and T_2. Assume that $e = v_1v_2$. Then at most one of them, say T_1, contains e and hence T_2 does not contain e. Therefore, one needs at most one more edge ($v_1v(e)$, for example) to X so that $G + (X \cup \{v_1v(e)\})$ has 2 edge-disjoint spanning trees.

(ii) Let X be a set of additional edges such that $G/e + X$ has 2 edge-disjoint spanning trees. Let e' be an edge not in G but parallel to e. Then $(G + X) + e'$ will have 2 edge-disjoint spanning trees. □

3. Proof of Theorem 1.5

In order to prove Theorem 1.5, we first prove some lemmas.

Let $G \in \mathcal{F}$ be a 3-edge-connected graph. For each $v \in D_3(G)$, fix a cycle C_v such that $v \in V(C_v)$ and such that $2 \leq |V(C_v)| \leq 3$. Let

$$W(G) = \bigcup_{v \in D_3(G)} C_v \tag{2}$$

We have the following observations.

Lemma 3.1. *Suppose that $G \in \mathcal{F}$ is a 3-edge-connected graph and $G/W(G) \not\cong K_1$. Then $G/W(G)$ is 4-edge-connected.*

Proof. Let $X \subset E(G/W(G))$ be an edge cut. Since $G/W(G) \not\cong K_1$, $X \neq \emptyset$. Note that X is also an edge cut of G and $X \subseteq E(G) - W$. If $|X| \leq 3$, then since $\kappa'(G) \geq 3$, one has $|X| = 3$. By (F1), there exists vertex $v \in D_3(G)$ such that X consists of the three edges incident with v in G. By (F2), G has a cycle C_v containing two edges in X such that $E(C_v) \subseteq W(G)$, contrary to the fact that $X \cap W(G) = \emptyset$. Hence one must have $|X| \geq 4$. □

Lemma 3.2. *Suppose $G \in \mathcal{F}$.If $e \in E(G)$ has no end vertex of degree 1, then $G/e \in \mathcal{F}$.*

Proof. By the definition of contraction, G/e is connected. If $X \subset E(G/e)$ is an edge cut, then X is also an edge cut of G, and so G/e satisfies (F1). Suppose that v_e is the contraction image of e. If $v_e \in D_3(G/e)$, let v_1, v_2 and v_3 be three neighbors of v_e and let $e = uv$. Without loss of generality, we assume that v_1 and v_2 are both neighbors of v in G. It follows that $v \in D_3(G)$. Since $G \in \mathcal{F}$, by (F2) v lies in a k-cycle C of G, where $2 \leq k \leq 3$. When C contains both vv_1 and vv_2, C is also a cycle of G/e and hence v_e lies in C. When C contains uv, C also contains one of edges v_1v and v_2v. By (F2) and $v_e \in D_3(G/e)$, C is of length 3. Thus, v_e lies in a 2-cycle or 3-cycle. We conclude that G/e also satisfies (F2). □

Lemma 3.3. *Let G be a graph. If $\kappa'(G) \geq 4$, then for any $e', e'' \in E(G)$,*

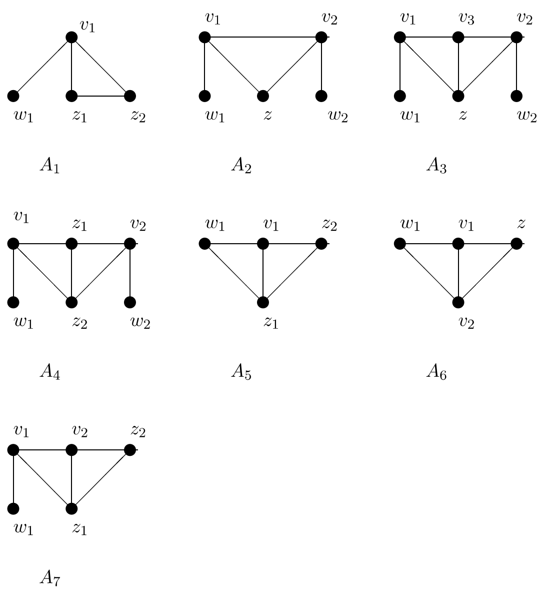

Fig. 1. Possible local structures of G

(i) *$G(e', e'')$ has 2 edge-disjoint spanning trees, and*
(ii) *$G(e', e'')$ has a spanning $(v(e'), v(e''))$-trail.*

Proof. By Theorem 1.9, $G - \{e', e''\}$ has two edge-disjoint spanning trees, and so $G(e', e'')$ also has 2 edge-disjoint spanning trees. This proves Lemma 3.3 (i).

If $G(e', e'')$ has 2 edge-disjoint spanning trees, then by Theorem 2.2 (ii) or (iii), $G(e', e'')$ is collapsible, and so Lemma 3.3 (ii) follows from Theorem 2.1 (ii). □

The induced subgraphs in Fig. 1 will be used in the proofs of Lemma 3.4 and Theorem 1.5. We assume that $v_i \in D_3(G)$ but $w_j, z_k, z \notin D_3(G), i = 1, 2, 3,\ j = 1, 2;$ $k = 1, 2$ in these subgraphs.

Lemma 3.4. *Suppose $G \in \mathcal{F}$ with $\kappa'(G) \geq 3$ and $|V(G)| \geq 5$. Assume that one of the following holds:*

(i) *G contains two local structures which are isomorphic to $A_i, A_j \in \{A_1, \ldots, A_7\}$, or*
(ii) *Gcontains a local structure which is isomorphic to $A_i \in \{A_1, \ldots, A_7\}$.*

Then $F(G) = 0$.

Proof. Suppose G has a local structure $H \cong A_i$ $(1 \leq i \leq 7)$. Define

$$X_H = \begin{cases} G[\{z_1, z_2, v_1\}] & \text{if } H \cong A_1 \text{ or } A_5, \\ G[\{v_1, v_2, z\}] & \text{if } H \cong A_2 \text{ or } A_6, \\ G[\{v_1, v_2, v_3, z\}] & \text{if } H \cong A_3, \\ G[\{v_1, v_2, z_1, z_2\}] & \text{if } H \cong A_4 \text{ or } A_7. \end{cases}$$

and $G' = G/X_H - w_1 v_1$.

Claim 1. *If $F(G') = 0$, then $F(G) = 0$.*

Proof. Let v_x be the vertex in G' which is the contraction image of X_H. Suppose that T_1', T_2' are 2 edge-disjoint spanning trees of G'. We view T_1', T_2' as edge induced subgraphs in G. Then T_1' and T_2' are edge disjoint forests with $|V(G)| - \epsilon_i$ vertices, where

$$\epsilon_i = \begin{cases} 3 & \text{if } H \cong A_i, i = 1, 2, 5, 6, \\ 4 & \text{if } H \cong A_j, j = 3, 4, 7. \end{cases}$$

If $H \cong A_i, i = 2, 3, 4$, then we may assume that the edge $v_2 w_2 \notin E(T_2')$. Let

$$T_1 = \begin{cases} G[E(T_1') \cup \{z v_1, v_1 v_2\}] & \text{if } H \cong A_2, \\ G[E(T_1') \cup \{z v_1, z v_3, v_2 v_3\}] & \text{if } H \cong A_3, \\ G[E(T_1') \cup \{v_1 z_1, v_1 z_2, z_1 v_2\}] & \text{if } H \cong A_4, \end{cases}$$

and

$$T_2 = \begin{cases} G[E(T_2') \cup \{v_1w_1, zv_2\}] & \text{if } H \cong A_2, \\ G[E(T_2') \cup \{v_1w_1, v_1v_3, v_2z\}] & \text{if } H \cong A_3, \\ G[E(T_2') \cup \{v_1w_1, z_1z_2, z_2v_2\}] & \text{if } H \cong A_4. \end{cases}$$

Then each of T_1 and T_2 is a connected subgraph of G with $|V(G)|$ vertices and $|V(G)| - 1$ edges. Therefore T_1 and T_2 are two edge-disjoint spanning trees of G.

If $H \cong A_i, i = 1, 5$, define $T_1 = G[E(T_1') \cup \{v_1z_1, v_1z_2\}]$ and $T_2 = G[E(T_2') \cup \{z_1z_2, v_1w_1\}]$. If $H \cong A_6$, define $T_1 = G[E(T_1') \cup \{v_1z, v_1v_2\}]$ and $T_2 = G[E(T_2') \cup \{w_1v_1, zv_2\}]$. If $H \cong A_7$, let $T_1 = G[E(T_1') \cup \{z_2v_2, z_1v_2, z_1v_1\}]$ and $T_2 = G[E(T_2') \cup \{w_1v_1, v_1v_2, z_1z_2\}]$. Then both T_1 and T_2 are connected, acyclic spanning subgraphs of G. Thus G has 2 edge-disjoint spanning trees. We complete the proof of our claim 1. □

Now we are ready to complete the proof of Lemma 3.4. Any subgraph of G which is isomorphic to A_i, for some $i \in \{1, \dots, 7\}$, is called a *special subgraph*. If H is a special subgraph, the edge $e = w_1v_1$ is the *distinguished edge* of H. If G has only one special subgraph H, then by Lemma 3.1, $\kappa'(G/X_H) \geq 4$. By Theorem 1.9 with $k = 2$, $(G/X_H - w_1v_1)$ has 2 edge-disjoint spanning trees. By Claim 1, $F(G) = 0$.

Hence G must have exactly 2 special subgraphs H_1 and H_2, with e_{H_1} and e_{H_2} as their distinguished edges, respectively. By Lemma 3.1, $\kappa'(G/(X_{H_1} \cup X_{H_2})) \geq 4$. By Theorem 1.9, $G/(X_{H_1} \cup X_{H_2}) - \{e_{H_1}, e_{H_2}\}$ has 2 edge-disjoint spanning trees. By repeated applications of Claim 1, $F(G) = 0$. □

Proof of Theorem 1.5. (i) Let e', e'' be a pair of edges in G. We argue by induction on $|V(G)|$ to prove Theorem 1.5, which is trivial when $|V(G)| \leq 4$. Thus we assume that $|V(G)| \geq 5$. If G has a nontrivial collapsible subgraph H such that each of e' and e'' has at most one end vertex in $V(H)$, then one can argue by Theorem 2.1 (i) and apply induction on $G(e', e'')/H$ to obtain that $G(e', e'')$ is collapsible.

Hence we assume that for any nontrivial collapsible subgraph H of G,

$$\text{at least one of } e' \text{ and } e'' \text{ has both end vertices in } V(H) \tag{3}$$

If $\{e', e''\} \cap W(G) = \emptyset$, then by Lemma 3.3 (i) and by Lemma 3.1, $G(e', e'')/W(G)$ has 2-edge-disjoint spanning trees. By Theorem 2.2 (ii), $G(e', e'')/W(G)$ is collapsible. By Theorem 2.1 (i) and (iv) $G(e', e'')$ is collapsible.

Hence we assume that $\{e', e''\} \cap W(G) \neq \emptyset$. By (3), we may assume that every cycle C in $W(G)$ must contains at least one of e' and e''.

We claim that $F(G) = 0$. It follows that $F(G) = 0$ if C is a 2-cycle in $W(G)$ and if $F(G/C) = 0$. Thus we may further assume that $W(G)$ contains no 2-cycles. Let $C_1 \in W(G)$ containing e'. Suppose that $E(C_1) \cap E(C) = \emptyset$ for every $C \in W(G)$. Since $G \in \mathcal{F}$, each 3-cycle of G can have at most 2 vertices in $D_3(G)$. If C_1 contains only one vertex of D_3, then G has a local structure which is isomorphic to A_1. If C_1 contains two vertices of D_3, then G has a local structure which is isomorphic to A_2.

Now we assume that there is another 3-cycle $C_2 \in W(G)$ such that $E(C_1) \cap E(C_2) \neq \emptyset$. Note that every two distinct 3-cycles have at most two common vertices. Since $E(C_1) \cap E(C_2) \neq \emptyset$, C_1 and C_2 have exactly two common vertices. If $V(C_1) \cap V(C_2) = \{v, z\}$ with $v \in D_3$ but $z \notin D_3$, then G has a local structure which is isomorphic to A_3 or A_5 or A_7. If $V(C_1) \cap V(C_2) = \{v_1, v_2\} \subseteq D_3$, then G has a local structure which is isomorphic to A_6. If $V(C_1) \cap V(C_2) = \{z_1, z_2\}$ and $\{z_1, z_2\} \cap D_3 = \emptyset$, then G has a local structure which is isomorphic to A_4. By Lemma 3.4, $F(G) = 0$.

Now we show that $G(e', e'')$ is collapsible. By Lemma 2.3 (i), $F(G(e', e'')) \leq 2$. It follows by Theorem 2.2 (iii) that the reduction of $G(e', e'')$ is either a K_1, or a K_2, or a $K_{2,t}$ for some $t \geq 1$.

If the reduction of $G(e', e'')$ is K_1, then $G(e', e'')$ is collapsible. Thus we assume that the reduction of $G(e', e'')$ is not K_1 to derive a contradiction.

By Lemma 3.1, $G/W(G)$ is 4-edge-connected. Thus $G(e', e'')$ cannot have a cut edge and hence the reduction of $G(e', e'')$ must be a $K_{2,t}$. Since $\kappa'(G) \geq 3$, it follows that the reduction of $G(e', e'')$ contains only two vertices of degree 2. So the reduction of $G(e', e'')$ is $K_{2,t}$ for some $t \leq 2$. It follows that the reduction of $G(e', e'')$ must be a $K_{2,2}$, and so we denote the reduction of $G(e', e'')$ by C_4. Since $G/W(G)$ is 4-edge-connected, two nonadjacent vertices of this C_4 must be $\{v(e'), v(e'')\}$. It follows that $\{e', e''\}$ is an edge cut of G, contrary to the assumption that $\kappa'(G) \geq 3$.

(ii) It follows from (i) and Theorem 2.1(ii). □

4. Generalizations and Applications

For the purpose of applications to hamiltonian line graphs, the requirement that $\kappa'(G) \geq 3$ in Theorem 1.5 can be relaxed.

Let G be a graph. For each $v \in D_2(G)$, fix exactly one edge e_v that is incident with v in G, and let $W'(G) = \cup\{e_v : v \in D_2(G)\}$. Define $\tilde{G} = G/W'(G)$. Also, define $W''(G) = E(G) - E(G - D_1(G))$ which denotes the set of edges that are incident with a vertex in $D_1(G)$.

Lemma 4.1. *Let G be a graph such that $G - D_1(G)$ is 2-edge-connected and such that $D_2(G)$ is an independent set. Then any spanning trail of $\tilde{G} - D_1(\tilde{G})$ is a dominating trail of G.*

Proof. Let L denote a spanning trail of $\tilde{G} - D_1(\tilde{G})$. Note that $D_1(\tilde{G}) = D_1(G)$. Therefore, any vertex $v \in V(G) - V(L)$ must be a vertex in $D_1(G) \cup D_2(G)$. If $v \in D_1(G)$, then since $G - D_1(G)$ is 2-edge-connected, v must be incident to a vertex in $V(\tilde{G} - D_1(\tilde{G})) = V(L)$; if $v \in D_2(G)$, then since $D_2(G)$ is an independent set in G and since $G - D_1(G)$ is 2-edge-connected, v must be incident with a vertex in $V(L)$ as well. It follows that $G - V(L)$ is edgeless and so L is a dominating trail of G. □

Theorem 4.2. *Let $G \in \mathcal{F}$ be a graph such that $\kappa'(\tilde{G} - D_1(\tilde{G})) \geq 3$. Then for any $e', e'' \in E(G)$, G has a dominating (e', e'')-trail.*

Proof. If e', e'' are two edges incident with a vertex v of degree 2 of G, let $e' = xv, e'' = vy$. We assume that $xv \in W'(G)$. Then $e = vy \in E(\tilde{G} - D_1(\tilde{G}))$. We can think that e', e'' are obtained by subdividing edge xy. By Theorem 1.5 (i) $(\tilde{G} - D_1(\tilde{G}))(e)$ is collapsible. Thus $(\tilde{G} - D_1(\tilde{G}))(e)$ is supereulerian. By Lemma 4.1, G has a dominating (e', e'')-trail. So suppose e', e'' are not incident with the same vertex of degree 2 in G. By the definition of $W'(G)$, we can choose $W'(G)$ such that $\{e', e''\} \cap W'(G) = \emptyset$. We first assume that $e', e'' \in E(\tilde{G} - D_1(\tilde{G}))$. By Theorem 1.5, $(\tilde{G} - D_1(\tilde{G}))(e', e'')$ has a spanning $(v(e'), v(e''))$-trail, and so by Lemma 4.1 and by Lemma 1.4 G has a dominating (e', e'')-trail. We then assume that $e' \in W''(G)$. Let v' denote the vertex in $D_1(G)$ incident with e'. Note that either $e'' \in W''(G)$ or $e'' \in E(G - D_1(G))$.

Suppose first that $e'' \in W''(G)$ and let v'' be the vertex in $D_1(G)$ incident with e''. By Lemma 3.1, $(\tilde{G} - D_1(\tilde{G}))/W(G)$ is 4-edge-connected; and so by Theorem 2.2, $(\tilde{G} - D_1(\tilde{G}))/W(G)$ is collapsible. By Theorem 2.1(i) and (v), $\tilde{G} - D_1(\tilde{G})$ is also collapsible, and so by Theorem 2.1(ii), $\tilde{G} - D_1(\tilde{G})$ has a spanning (v', v'')-trail. It follows by Lemma 4.1 that G has a dominating (e', e'')-trail.

Hence $e'' \in E(G - D_1(G))$. Let $u = v(e'')$. By Lemma 3.1, $(\tilde{G} - D_1(\tilde{G}))/W(G)$ is 4-edge-connected; and so by Lemma 2.3 and by Theorem 2.2 (ii), $(\tilde{G} - D_1(\tilde{G}))(e'')/W(G)$ is collapsible. By Theorem 2.1(i) and (v), $(\tilde{G} - D_1(\tilde{G}))(e'')$ is also collapsible, and so by Theorem 2.1(ii), $(\tilde{G} - D_1(\tilde{G}))(e'')$ has a spanning (v', v'')-trail. It follows by Lemma 4.1 that G has a dominating (e', e'')-trail. □

The following lemma is straightforward.

Lemma 4.3. *Let G be a graph such that $L(G)$ is 4-connected. Then each of the following holds.*

(i) *G satisfies (F1).*
(ii) $\kappa'(\tilde{G} - D_1(\tilde{G})) \geq 3$.

Proof of Corollary 1.6. (1) Since the set of neighbors of each vertex of degree 3 is not an independent set, G satisfies (F2). By Lemma 4.3, $G \in \mathcal{F}$ and $\kappa'(\tilde{G} - D_1(\tilde{G})) \geq 3$. By Theorems 4.2 and 1.2, $L(G)$ is hamiltonian connected.

(2) Since G is $K_{1,3}$-free, the set of neighbors of each vertex of degree 3 is not an independent set. (2) immediately follows from (1).

(3) Since $\kappa'(G) \geq 4$, there is no vertex of degree 3. To the contrary, suppose that $X \subset V(L(G))$ is a vertex cut with $|X| \leq 3$. Then $L(G) - X$ has at least two components such that the number of vertices of each component is at least 1. The edge set of G corresponding to X is an edge cut of G. It implies that G has an edge cut of cardinality at most 3. This contradiction shows that $\kappa'(L(G)) \geq 4$. Therefore, (3) follows from (1). □

Proof of Corollary 1.7. It is well-known that a line graph does not have a $K_{1,3}$ as an induced subgraph. Thus Corollary 1.7 follows from Corollary 1.6. □

Proof of Corollary 1.8. We may assume that $L(G)$ is not a complete graph. By Lemma 4.3, G satisfies (F1) and $\kappa'(\tilde{G} - D_1(\tilde{G})) \geq 3$. We shall show that G also satisfies (F2).

To the contrary, suppose that there exists vertex $v \in D_3(G)$ with $v_1, v_2, v_3 \in V(G)$ being three distinct vertices adjacent to v in G, such that v_1, v_2, v_3 are mutually nonadjacent, and such that $v_1 \notin D_1(G)$. By $\kappa(L(G)) \geq 4$, $d_G(v_1) \geq 3$. Let v_1u, v_1u' be two edges of G such that $v \notin \{u, u'\}$. Obviously, $\{u, u_1\} \cap \{v_2, v_3\} = \emptyset$. It follows that the subgraph in $L(G)$ induced by the edges $\{vv_1, vv_2, vv_3, v_1u, v_1u'\}$ is an hourglass. This contradiction proves that G satisfies (F2). Thus, $G \in \mathcal{F}$.

Therefore, Corollary 1.8 follows from Theorem 4.2 and Lemma 1.2. □

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Received: November 14, 2002
Final version received: May 27, 2004