Nowhere Zero 4-Flow in Regular Matroids

Hong-Jian Lai,[1] Xiangwen Li,[2] and Hoifung Poon[1]

[1]DEPARTMENT OF MATHEMATICS
WEST VIRGINIA UNIVERSITY
MORGANTOWN, WEST VIRGINIA 26506-6310
E-mail: hjlai@math.wvu.edu

[2]DEPARTMENT OF MATHEMATICS
SOUTH CHINA UNIVERSITY OF TECHNOLOGY
GUANGZHOU, P. R. CHINA

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Abstract: Jensen and Toft [8] conjectured that every 2-edge-connected graph without a K_5-minor has a nowhere zero 4-flow. Walton and Welsh [19] proved that if a coloopless regular matroid M does not have a minor in $\{M(K_{3,3}), M^*(K_5)\}$, then M admits a nowhere zero 4-flow. In this note, we prove that if a coloopless regular matroid M does not have a minor in $\{M(K_5), M^*(K_5)\}$, then M admits a nowhere zero 4-flow. Our result implies the Jensen and Toft conjecture. © 2005 Wiley Periodicals, Inc. J Graph Theory 49: 196–204, 2005

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1. INTRODUCTION

We shall assume familiarity with graph theory and matroid theory. For terms that are not defined in this note, see Bondy and Murty [4] for graphs, and Oxley [10] or Welsh [20] for matroids.

Throughout this paper, $\mathbf{Z}, \mathbf{Z}^+$, and $\mathbf{Z}_n$ denote the additive group of the integers, the set of all positive integers, and the cyclic group of order n, respectively. To be consistent with the matroid terminology, a nontrivial 2-regular connected graph will be called a *circuit*, and a disjoint union of circuits a *cycle*. Note that as we allow empty unions, the empty set is also a cycle (in both graphs and matroids). For matroids $N_1, N_2, \ldots, N_k$, let $EX(N_1, N_2, \ldots, N_k)$ denote the collection of matroids such that a matroid $M \in EX(N_1, N_2, \ldots, N_k)$ if and only if M does not have a minor isomorphic to any one in $\{N_1, N_2, \ldots, N_k\}$. The Fano matroid F_7 is the vector matroid over GF(2) of the following matrix A:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Flow was initially defined for graphs. For a discussion on flow and flow conjectures, see Jaeger [7] or Zhang [21]. The definition of flow has a natural extension to regular matroids. Let M be a regular matroid and D_M be its incidence matrix of circuits against elements. An *orientation* $(w(D_M), w(D_{M^*}))$ is an assignment of $+, -$ signs to the "1" entries of D_M and D_{M^*}, respectively, so that the resulting matrices $w(D_M)$ and $w(D_{M^*})$ satisfy

$$w(D_M)w(D_{M^*})^T = 0.$$

Let A be an abelian group. For an element $a \in A$, and for integers $+1, -1, 0$, we adopt the convention to write $(+1) \cdot a = a$, $(-1) \cdot a = -a$, and $0 \cdot a = 0$. Let $F^*(M, A) = \{f : E(M) \mapsto A \backslash \{0\}\}$ denote the set of all functions from $E(M)$ into $A \backslash \{0\}$. A map $f \in F^*(M, A)$ can be viewed as an $|E(M)|$-dimensional column vector. For a regular matroid M with an orientation $(w(D_M), w(D_{M^*}))$, a map $f \in F^*(M, A)$ satisfying

$$w(D_{M^*}) \cdot f = 0$$

is a *nowhere zero A-flow* (A-NZF for short) of M. When $A = \mathbf{Z}$, a $\mathbf{Z}$-NZF f of M is called a *nowhere zero k-flow* (k-NZF for short) of M if $\forall e \in E(M)$, $0 < |f(e)| < k$.

For positive integers k and m, an *m-cycle k-cover* of a matroid M is a family of cycles $C_1, C_2, \ldots, C_m$ of M such that every element of $E(M)$ lies in exactly k members of these C_i's. It has been observed that a graph G admits a 4-NZF if and only if G has a 3-cycle 2-cover (for example, see Zhang [21]). We shall show later in this section that this is also true for regular matroids.

Proposition 1.1. *Let M be a regular matroid. Then M admits a* 4*-NZF if and only if M has a* 3*-cycle* 2*-cover.*

Tutte proposed the famous 4-flow conjecture as follows:

Conjecture 1.2 (Tutte [16] and [17], Matthews [9])**.** *Let G be a* 2*-edge-connected graph. If G does not have a Peterson graph minor, then G admits a* 4*-NZF.*

Jensen and Toft presented a weaker form of Conjecture 1.2 in 1995.

Conjecture 1.3 (Jensen and Toft [8]). *Every* 2*-edge-connected graph without a* K_5*-minor has a* 4*-NZF.*

The main objective of this paper is to prove Conjecture 1.3 by proving a stronger result in the matroid context as follows:

Theorem 1.4. *If* M *is a coloopless regular matroid such that* $M \in EX(M(K_5), M^*(K_5))$*, then* M *admits a* 4*-NZF.*

Corollary 1.5. *Conjecture 1.3 holds affirmatively.*

Proof. Let G be a 2-edge-connected graph without a K_5-minor. Then $M(G)$ is graphic and thus is regular. Since $M^*(K_5)$ is not graphic, neither $M(K_5)$ nor $M^*(K_5)$ is a minor of $M(G)$. By Theorem 1.4, $M(G)$ has a 4-NZF. ■

The definition of flow has no natural extension to binary matroids, whereas cycle cover is defined for general matroids. In view of Proposition 1.1 and the excluded-minor characterization of regular matroids, Theorem 1.4 is equivalent to saying that if $M \in EX(F_7, F_7^*, M(K_5), M^*(K_5))$ is a coloopless binary matroid, then M has a 3-cycle 2-cover. In Section 3, we will show that this result can be extended in the following form.

Corollary 1.6. *Let* M *be a coloopless binary matroid. If* $M \in EX(F_7^*, M(K_5), M^*(K_5))$*,* M *has a* 3*-cycle* 2*-cover.*

As the matroid F_7^* does not have a 3-cycle 2-cover, Corollary 1.6 does not hold if F_7^* is not excluded.

In the remainder of this section, we will introduce the relevant definitions and briefly review the relevant results. In Section 2, we extract a decomposition theorem for regular matroids without $M(K_5)$ or $M^*(K_5)$ minors from the well-known decomposition theorems of Seymour [13] and Wagner [18]. In Section 3, this theorem will be employed to prove Theorem 1.4 and Corollary 1.6.

Theorem 1.7 (Tutte [15], Brylawski [6], Arrowsmith and Jaeger [3]). *Let* M *be a regular matroid and* A *be an abelian group of order* k*. Then* M *has an* A*-NZF if and only if* M *has a* k*-NZF.*

Proof of Proposition 1.1. Let M be a regular matroid. If M has a 3-cycle 2-cover $\{C_1, C_2, C_3\}$. Then $E(M) = C_1 \cup C_2$. Let $f_i : C_i \mapsto \{1\} \in \mathbf{Z}_2$, for $i \in \{1, 2\}$. Since C_i is a cycle, $f = (f_1, f_2) \in F^*(M, \mathbf{Z}_2 \times \mathbf{Z}_2)$ is a $(\mathbf{Z}_2 \times \mathbf{Z}_2)$-NZF. By Theorem 1.7, M has a 4-NZF. By reversing this argument, we can also construct a 3-cycle 2-cover from a $(\mathbf{Z}_2 \times \mathbf{Z}_2)$-NZF of M. ■

The Four-Color theorem can be stated in terms of nowhere zero flows as follows:

Theorem 1.8. (Appel and Haken [1], Appel, Haken, and Hoch [2], Robertson, Sanders, Seymour, and Thomas [11]). *Every* 2*-edge-connected planar graph admits a* 4*-NZF.*

Applying the Four-Color theorem, and the duality between colorings and nowhere zero flows, a result by Walton and Welsh implies the following:

Theorem 1.9 (Walton and Welsh [19]). *If M is a coloopless regular matroid such that $M \in EX(M(K_{3,3}), M^*(K_5))$, then M admits a* 4*-NZF.*

Recently Robertson et al. proved Conjecture 1.2 for cubic graphs.

Theorem 1.10 (Robertson, Sanders, Seymour, and Thomas, [12]). *Every* 2*-edge-connected cubic graph without a minor isomorphic to the Petersen graph admits a* 4*-NZF.*

2. DECOMPOSITION OF REGULAR MATROIDS IN $EX(M(K_5), M^*(K_5))$

In this paper, we use $\triangle$ to denote both a set operator and a matroid operator. Given two sets X and Y, the symmetric difference of X and Y is defined as

$$X \triangle Y = (X \cup Y) - (X \cap Y).$$

Now suppose that M_1, M_2 are binary matroids on E_1 and E_2, respectively. We follow Seymour [13] and define a new binary matroid $M_1 \triangle M_2$ to be the matroid with ground set equal to $E_1 \triangle E_2$ and with its set of cycles equal to

$$\{C_1 \triangle C_2 \subseteq E_1 \triangle E_2 : \ C_i \text{ is a cycle of } M_i, i = 1, 2\}.$$

Three special cases of this operation are introduced by Seymour ([13] and [14]) as follows.

(i) If $E_1 \cap E_2 = \emptyset$ and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a *1-sum* of M_1 and M_2.
(ii) If $|E_1 \cap E_2| = 1$ and $E_1 \cap E_2 = \{z\}$, say, if z is not a loop or coloop of M_1 or M_2, and if $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a *2-sum* of M_1 and M_2.
(iii) If $|E_1 \cap E_2| = 3$ and $E_1 \cap E_2 = Z$, say, if Z is a circuit of M_1 and M_2, and Z includes no cocircuit of either M_1 or M_2, and if $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a *3-sum* of M_1 and M_2.

For $i = 1, 2, 3$, an i-sum of M_1, M_2 is denoted as $M_1 \oplus_i M_2$. The 1-sum $M_1 \oplus_1 M_2$ is also written as $M_1 \oplus M_2$. Let R_{10} denote the vector matroid of the following matrix over $GF(2)$:

$$R_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

It is known that R_{10}^* is isomorphic to R_{10}. Based on the notion of matroid sums, Seymour proved the following decomposition theorem for regular matroids.

Theorem 2.1 (Seymour [13]). *Let M be a regular matroid. One of the following must hold:*

(i) *M is graphic.*
(ii) *M is cographic.*
(iii) $M \cong R_{10}$.
(iv) *For some $i \in \{1,2,3\}$, $M = M_1 \oplus_i M_2$ is the i-sum of two matroids M_1 and M_2, each of which is isomorphic to a proper minor of M.*

Theorem 2.2 (Seymour, Proposition (2.9) of [13]). *Let M be a binary matroid. Then each of the following holds.*

(i) *Let (X_1, X_2) be an exact 3-separation of a binary matroid M with $|X_1|, |X_2| \geq 4$, and let Z be a 3-element set that is disjoint from $E(M)$. Then, there are binary matroids M_1 and M_2 on $X_1 \cup Z$ and $X_2 \cup Z$, respectively, such that $M = M_1 \oplus_3 M_2$.*
(ii) *If M is a 3-sum of M_1 and M_2, then $(E(M_1)\backslash E(M_2), E(M_2)\backslash E(M_1))$ is an exact 3-separation of M, and $\min\{|E(M_1)\backslash E(M_2)|, |E(M_2)\backslash E(M_1)|\} > 3$.*

Lemma 2.3. *Let M be a 3-connected binary matroid such that M is a 3-sum of two matroids M_1 and M_2. Then M^* is also a 3-sum.*

Proof. Define $X_1 = E(M_1)\backslash E(M_2), X_2 = E(M_2)\backslash E(M_1)$. By Theorem 2.2(ii), (X_1, X_2) is an exact 3-separation of M such that $\min\{|X_1|, |X_2|\} \geq 4$. Note that (X_1, X_2) is also an exact 3-separation of M^*, and that M^* is a 3-connected binary matroid. By Theorem 2.2(i), M^* must also be a 3-sum. ■

If a matroid M is isomorphic to the cycle matroid of a planar graph, then M is called a *planar matroid*. Thus, a matroid M is planar if and only if M^* is planar. Let H_8 denote the graph depicted in Figure 1 below.

Wagner's original statement of his decomposition theorem is in pure graph theory terms. A matroidal version is given as follows (see Seymour [13] and [14]).

Theorem 2.4 (Wagner [23]). *Let M be a graphic matroid that does not contain a minor isomorphic to $M(K_5)$. One of the following must hold:*

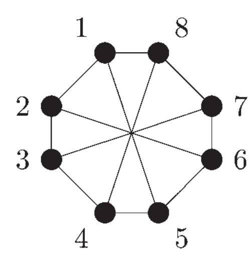

FIGURE 1. The graph H_8.

(i) *M is a planar matroid.*
(ii) $M \cong M(H_8)$.
(iii) $M \cong M(K_{3,3})$.
(iv) *For some $i \in \{1, 2, 3\}$, $M = M_1 \oplus_i M_2$ is the i-sum of two matroids M_1 and M_2, such that both M_1 and M_2 are proper minors of M.*

Proposition 2.5 (Propositions 4.2.11, 8.3.1 and 12.4.16 of [10])**.** *Each of the following holds:*

(i) *The matroid M is not 2-connected, if and only if for some proper non-empty subset T of $E(M)$, $M = (M|T) \oplus (M|(E\backslash T))$. Note that $M|T$ and $M|(E\backslash T)$ are both proper minors of M.*
(ii) *The matroid M is 2-connected but not 3-connected, if and only if $M = M_1 \oplus_2 M_2$ for some matroids M_1 and M_2, each of which is isomorphic to a proper minor of M.*
(iii) *If M is a 3-connected binary matroid and a 3-sum of M_1 and M_2, then M_1 and M_2 are isomorphic to proper minors of M.*

Let $\mathcal{G}$ denote the family of matroids such that a matroid $M \in \mathcal{G}$, if and only if M is a planar matroid or M is isomorphic to a member in the collection $\{M(H_8)$, $M^*(H_8)$, $M(K_{3,3})$, $M^*(K_{3,3})$, $R_{10}\}$. By definition, a matroid $M \in \mathcal{G}$, if and only if $M^* \in \mathcal{G}$.

Theorem 2.6. *Let M be a regular matroid that does not have a minor isomorphic to $M(K_5)$ or $M^*(K_5)$. Then one of the following must hold:*

(i) $M \in \mathcal{G}$.
(ii) *For some $i \in \{1, 2, 3\}$, $M = M_1 \oplus_i M_2$ is the i-sum of two matroids M_1 and M_2, such that both M_1 and M_2 are proper minors of M.*

Proof. Let M be a regular matroid without a minor isomorphic to $M(K_5)$ or $M^*(K_5)$, and such that $M \notin \mathcal{G}$. Since $\mathcal{G}$ is closed under taking duals, $M^* \notin \mathcal{G}$. We shall show that M satisfies Theorem 2.6 (ii).

If M is not 2-connected, by Proposition 2.5 (i), M must be a 1-sum of two proper minors of M. If M is 2-connected but not 3-connected, by Proposition 2.5 (ii), M must be a 2-sum of two proper minors of M. In either case, Theorem 2.6 (ii) must hold.

Therefore, we assume that M is 3-connected. Since M is regular, one of the conclusions of Theorem 2.1 must hold. Since $R_{10} \in \mathcal{G}$, M cannot be R_{10}.

If M is graphic, then by Theorem 2.4 and since M is 3-connected, either $M \in \mathcal{G}$ or M is a 3-sum of two proper minors M_1 and M_2, and so Theorem 2.6 (ii) must hold.

Now suppose that M is a 3-connected cographic matroid. Since $M^* \notin \mathcal{G}$, by Theorem 2.4, M^* must be a 3-sum of two proper minors. By Lemma 2.3, M must also be a 3-sum of two matroids M_1 and M_2. By Proposition 2.5 (iii), M_1 and M_2 are isomorphic to proper minors of M, and hence Theorem 2.6 (ii) must hold.

Now if M is neither graphic nor cographic, Theorem 2.6 (ii) follows from Theorem 2.1 (iv), and this completes the proof for Theorem 2.6. ∎

3. THE PROOFS OF THEOREM 1.4 AND COROLLARY 1.6

In view of Proposition 1.1, we will prove Theorem 1.4 by showing that M has a 3-cycle 2-cover given the assumption of the theorem. We first establish some lemmas.

Proposition 3.1. *Each of the following holds:*

(i) *Each of* $M(H_8)$, $M^*(H_8)$, $M(K_{3,3})$, $M^*(K_{3,3})$, R_{10}, F_7 *has a* 3*-cycle* 2*-cover.*
(ii) F_7^* *cannot have a* 3*-cycle* 2*-cover.*

These results follow from the known facts about tangential 2-blocks. See, for example, the discussion on Tutte's tangential 2-block conjecture in [5]. The results can also be verified directly in a straightforward way.

Proposition 3.2 (Seymour [13], also see Oxley [10], Exercise 6 in Section 12.4)**.** *Suppose that* M, M_1, M_2 *are binary matroids. If* $M = M_1 \triangle M_2$, *then* $M^* = M_1^* \triangle M_2^*$.

Lemma 3.3. *Suppose that* M, M_1, M_2 *are binary matroids and that each of* M_1 *and* M_2 *has a* 3*-cycle* 2*-cover. Then each of the following holds:*

(i) *If* $M = M_1 \oplus M_2$ *is a* 1*-sum of* M_1 *and* M_2*, then* M *also has a* 3*-cycle* 2*-cover.*
(ii) *If* $M = M_1 \oplus_2 M_2$ *is a* 2*-sum of* M_1 *and* M_2*, then* M *also has a* 3*-cycle* 2*-cover.*

Proof. (i) Suppose that $M = M_1 \oplus M_2$. For $k = 1, 2$, we assume that M_k has a 3-cycle 2-cover, denoted as $C_{k,1}, C_{k,2}, C_{k,3}$. It follows that $\{C_{1,i} \cup C_{2,i} : i = 1, 2, 3\}$ is a 3-cycle 2-cover of M.

(ii) Now assume that $M = M_1 \oplus_2 M_2$. Denote $E(M_1) \cap E(M_2) = \{e\}$. For each $k \in \{1, 2\}$, assume that M_k has a 3-cycle 2-cover, denoted as $C_{k,1}, C_{k,2}, C_{k,3}$. Note that by the definition of a 2-cover, e appears exactly twice in each set of 3 cycles. Without loss of generality, we may assume that $e \in C_{k,i}$, $k, i = 1, 2$. Now it is easy to verify that $\{C_{1,i} \triangle C_{2,i} : i = 1, 2, 3\}$ is a 3-cycle 2-cover of M. ∎

Let $\mathcal{F}_4$ denote the set of all matroids that have a 3-cycle 2-cover. Let $\mathcal{F}$ denote the set of loopless and coloopless regular matroids, which have no minors isomorphic to $M(K_5)$ or $M^*(K_5)$. Note that $\mathcal{F}$ is closed under taking duals and isomorphism, and that if $M \in \mathcal{F}$ and if N is a loopless and coloopless minor of N, then $N \in \mathcal{F}$ also. Thus, for some $i \in \{1, 2, 3\}$, if $M \in \mathcal{F}$ is an i-sum of M_1 and M_2, each of which is a proper minor of M, then neither M_1 nor M_2 has any loop or coloop and therefore M_1 and M_2 are also in $\mathcal{F}$.

Let M be a matroid satisfying the hypothesis of Theorem 1.4, and let C be the union of all its loops. Then C is a cycle of M and $M \backslash C$ is a matroid in $\mathcal{F}$, and M has a 3-cycle 2-cover if and only if $M \backslash C$ has a 3-cycle 2-cover. Therefore, to prove Theorem 1.4, it suffices to prove that $\mathcal{F} \subseteq \mathcal{F}_4$.

Proof of Theorem 1.4. Let $M \in \mathcal{F}$. If $M \in \mathcal{G}$, by Proposition 1.1, Theorem 1.8 and Proposition 3.1, $M \in \mathcal{F}_4$. We argue by induction on $|E(M)|$, and assume that $M \notin \mathcal{G}$ and that for every $N \in \mathcal{F}$ with $|N| < |M|$, $N \in \mathcal{F}_4$.

Case 1. M is not 3-connected.

By Proposition 2.5 (i) and (ii), for $i \in \{1,2\}$, $M = M_1 \oplus_i M_2$ for some proper minors M_1, M_2. Hence by induction and by Lemma 3.3, $M \in \mathcal{F}_4$.

Case 2. M is 3-connected.

Since $\mathcal{G}$ is closed under taking dual, $M^* \notin \mathcal{G}$. By Theorem 2.6, M^* must be a 1-, 2-, or 3-sum of two proper minors of M^*. Since M, and so M^* are 3-connected, it follows that $M^* = M_1 \oplus_3 M_2$ is a 3-sum of some proper minor M_1 and M_2 of M^*. Denote $E(M_1) \cap E(M_2) = \{e_1, e_2, e_3\} = Z$. Then Z is a circuit of M_1 and M_2.

By Proposition 3.2, $M = M_1^* \triangle M_2^*$. Note that M_1^* and M_2^* are proper minors of M, therefore $M_1^*, M_2^* \in \mathcal{F}$. For $k = 1,2$, by induction, M_k^* has a 3-cycle 2-cover, denoted as $C_{k,1}, C_{k,2}, C_{k,3}$. Since Z is a cocircuit in binary matroids M_1^* and M_2^*, $|C_{k,i} \cap Z|$ must be even for any $k \in \{1,2\}$, $i \in \{1,2,3\}$. Therefore, $|C_{k,i} \cap Z| \in \{0,2\}$. As for each $k \in \{1,2\}$, $\{C_{k,1}, C_{k,2}, C_{k,3}\}$ is a 2-cover of M_k, the following must hold:

$$\{C_{k,1} \cap Z, C_{k,2} \cap Z, C_{k,3} \cap Z\} = \{\{e_1,e_2\},\{e_2,e_3\},\{e_1,e_3\}\}.$$

Without loss of generality, we may assume that $C_{k,i} \cap Z = Z - \{e_i\}$ for $k \in \{1,2\}$ and $i \in \{1,2,3\}$. Now it is easy to see that $\{C_{1,i} \triangle C_{2,i} : i = 1,2,3\}$ is a 3-cycle 2-cover of M, and so $M \in \mathcal{F}_4$.

This proves Case 3 and thus completes the proof of Theorem 1.4. ■

For binary matroids without F_7^* minor, Seymour has established the following decomposition theorem.

Theorem 3.4 (Seymour [17]). *Every binary matroid without F_7^* minor may be obtained by means of proper* 1*-sums or* 2*-sums from regular matroids and copies of F_7.*

Proof of Corollary 1.6. This follows from Proposition 3.1, Lemma 3.3, Theorem 3.4, and Theorem 1.4. ■

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