Hamiltonian N_2-Locally Connected Claw-Free Graphs

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Abstract: A graph G is N_2-*locally connected* if for every vertex v in G, the edges not incident with v but having at least one end adjacent to v in G induce a connected graph. In 1990, Ryjáček conjectured that every 3-connected N_2-locally connected claw-free graph is Hamiltonian. This conjecture is proved in this note. © 2004 Wiley Periodicals, Inc. J Graph Theory 48: 142–146, 2005

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1. INTRODUCTION

We use [1] for terminology and notations not defined here, and consider finite simple graphs only. Let G be a graph. Denote by $d_G(v)$ the degree of a vertex

$v \in V(G)$. For a vertex v of G, the neighborhood of v, that is, the induced subgraph on the set of all vertices that are adjacent to v, will be called the neighborhood of the *first type* of v in G and denoted by $N_1(v, G)$, or briefly, $N_1(v)$ or $N_G(v)$. For notational convenience, we shall use $N_G(v)$ to denote both the induced subgraph and the set of vertices adjacent to v in G. We define the neighborhood of the *second type* of v in G (denoted by $N_2(v, G)$, or briefly, $N_2(v)$) as the subgraph of G induced by the edge subset $\{e = xy \in E(G) : v \notin \{x, y\}$ and $\{x, y\} \cap N(v) \neq \emptyset\}$. We say that a vertex v is *locally connected* if $N(v)$ is connected; and G is *locally connected* if every vertex of G is locally connected. Analogously, a vertex v is *N_2-locally connected* if its second type neighborhood $N_2(v)$ is connected; and G is called *N_2-locally connected* if every vertex of G is N_2-locally connected. It follows from the definitions that every locally connected graph is N_2-locally connected. A graph G is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. The following theorems give the hamiltonicity of a locally and N_2-locally connected graph.

Theorem 1.1 (Oberly and Sumner, [6]). *Every connected locally connected claw-free graph on at least three vertices is hamiltonian.*

Theorem 1.2 (Ryjáček, [8]). *Let G be a connected, N_2-locally connected claw-free graph without vertices of degree* 1, *which does not contain an induced subgraph H isomorphic to either G_1 or G_2* (Fig. 1) *such that $N_1(x, G)$ of every vertex x of degree* 4 *in H is disconnected. Then G is Hamiltonian.*

We say that G is *vertex pancyclic* if it contains cycles of every length through every vertex.

Theorem 1.3 (Li, [4]). *Let G be a connected, N_2-locally connected claw-free graph with $\delta(G) \geq 3$, which does not contain an induced subgraph H isomorphic to either G_1 or G_2 (Fig.* 1*). Then G is vertex pancyclic.*

In another recent paper [5], Li suggested a new relaxation of the locally connectedness condition for Hamiltonian claw-free graphs. The main purpose of this note is to prove the following theorem, conjectured by Ryjáček in [8].

Theorem 1.4. *Every* 3*-connected N_2-locally connected claw-free graph is Hamiltonian.*

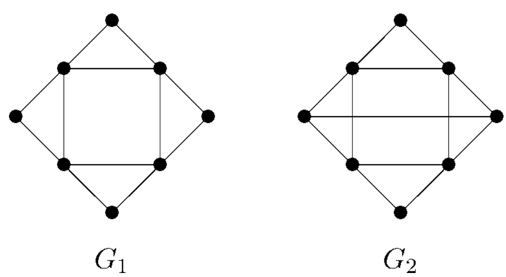

FIGURE 1.

2. PROOF OF THE MAIN THEOREM

Our approach is to firstly apply the line graph closure (invented by Ryjáček in [8]) to convert the problem into a line graph problem. Then apply techniques in supereulerian graphs to solve the corresponding line graph problem.

The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have a vertex in common. Let G be the line graph $L(H)$ of a graph H. If $L(H)$ is k-connected, then H is *essentially* k-edge-connected, which means that the only edge-cut sets of H having less than k edges are the sets of edges incident with some vertex of H.

In [9], Ryjáček defined the closure $cl(G)$ of a claw-free graph G by recursively completing the neighborhood of any locally connected vertex of G, as long as this is possible. The closure $cl(G)$ is a well-defined claw-free graph and its connectivity is at least equal to the connectivity of G. The *circumference* of G is the length of a longest cycle in G.

Theorem 2.1 (Ryjáček, [9]). *Let G be a claw-free graph and $cl(G)$ its closure. Then*

(i) *there is a triangle-free graph H such that $cl(G)$ is the line graph of H,*
(ii) *both graphs G and $cl(G)$ have the same circumference.*

Let $O(G)$ denote the set of all vertices in G with odd degree. A graph G is *eulerian* if $O(G) = \emptyset$ and G is connected. A spanning closed trail of G is called a *spanning Eulerian subgraph* of G. A subgraph H of G is *dominating* if $G - V(H)$ is edgeless. If a closed trail C of G satisfies $E(G - V(C)) = \emptyset$, then C is called a *dominating Eulerian subgraph*.

Theorem 2.2 (Harary and Nash-Williams, [2]). *The line graph $G = L(H)$ of a graph H is Hamiltonian if and only if H has a dominating Eulerian subgraph.*

Theorem 2.2 reveals the relationship between a dominating Eulerian subgraph in H and a Hamiltonian cycle in $L(H)$.

Theorem 2.3 below provides a sufficient condition for a graph to have a spanning Eulerian subgraph (therefore a dominating Eulerian subgraph), which is originally conjectured by Paulraja ([7]).

Theorem 2.3 (Lai, [3]). *Let G be a 2-connected graph with $\delta(G) \geq 3$. If every edge of G is in an m-cycle of $G (m \leq 4)$, then G has a spanning Eulerian subgraph.*

Lemma 2.4. *Let G be an N_2-locally connected graph and let x be a locally connected vertex of G such that $G[N_G(x)]$ is not complete. Let $N' = \{uv : u, v \in N_G(x), uv \notin E(G)\}$ and let G' be the graph with vertex set $V(G') = V(G)$ and with edge set $E(G') = E(G) \cup N'$. Then G' is N_2-locally connected.*

Proof. Let $w \in V(G')$. If $w = x$, then $N_2(w, G')$ is connected since $N_{G'}(x)$ is complete. So we may assume that $w \neq x$. Since G is N_2-locally connected, $N_2(w, G)$ is connected. If $E(N_2(w, G')) - E(N_2(w, G)) = \emptyset$, then $E(N_2(w, G')) = E(N_2(w, G))$ and $N_2(w, G')$ is connected. Thus we assume that $E(N_2(w, G')) - E(N_2(w, G)) \neq \emptyset$. Let $e = uv \in E(N_2(w, G')) - E(N_2(w, G))$. Since $e = uv \in E(N_2(w, G'))$, we have $w \notin \{u, v\}$, and so $uv \in E(G')$. Without loss of generality, we assume that $wu \in E(G')$.

Case 1. $uv \in E(G)$.

By $e = uv \notin E(N_2(w, G))$, we have $wu, wv \notin E(G)$. Since $wu \in E(G')$ by the assumption, $w, u \in N_G(x)$. So $xu \in E(N_2(w, G))$. Therefore adding a new edge uv to $N_2(w, G)$ does not change its connectivity, and so $N_2(w, G')$ is connected.

Case 2. $uv \notin E(G)$.

Since $uv \in E(G')$, we have $u, v \in N_G(x)$. If $w \in N_G(x)$, then $xu, xv \in E(N_2(w, G))$. Thus adding a new edge uv to $N_2(w, G)$ does not change its connectivity, and so $N_2(w, G')$ is connected. If $w \notin N_G(x)$, then we have $wu \in E(G)$ since $wu \in E(G')$(otherwise, $w \in N_G(x)$, a contradiction). Thus $xu \in E(N_2(w, G))$. So adding a new edge uv to $N_2(w, G)$ does not change its connectivity, and therefore $N_2(w, G')$ is connected. ■

***Proof of Theorem* 1.4.** By Theorem 2.1(ii), the graph G is Hamiltonian if and only if its closure $cl(G)$ is Hamiltonian. By Lemma 2.4 and as $cl(G)$ is both 3-connected and claw free, the graph $cl(G)$ is also a 3-connected N_2-locally connected claw-free graph. By Theorem 2.1, we may assume that for a triangle-free graph H, $G = cl(G) = L(H)$.

An edge $e = uv$ is called a *pendant edge* if either $d_G(u) = 1$ or $d_G(v) = 1$.

Claim 1. Let $e = uv \in E(H)$. If e is not a pendant edge, then e is in some 4-cycle of H.

Proof. Since H is triangle free, we have $N_H(u) \cap N_H(v) = \emptyset$. Let $v_e \in V(G)$ corresponds to the edge $e \in E(H)$ in terms of the line graph. Since e is not a pendant edge and G is claw free, $N_G(v_e)$ is the union of two disjoint cliques. Suppose they are L_1, L_2. Since G is 3-connected, there exits at least one path $w_1 w_2 \cdots w_n$ which is edge disjoint with $G[V(L_1) \cup V(L_2) \cup \{v_e\}]$ in $G - v_e$ with $w_1 \in V(L_1), w_n \in V(L_2)$. Since G is N_2-locally connected, we have that $n = 3$. Thus $v_e w_1 w_2 w_3 v_e$ is an induced 4-cycle of G, which corresponds to a 4-cycle in H containing e. ■

Let $\widetilde{H}$ be the graph obtained from H by deleting the vertices of degree 1 or 2 and replacing each path xyz in H with $d_H(y) = 2$ by an edge xy. Then it is straightforward to verify the following claim.

Claim 2. *If $\widetilde{H}$ has a spanning Eulerian subgraph, then H has a dominating Eulerian subgraph.*

Since G is 3-connected, $\widetilde{H}$ is 3-edge-connected. Let B be an arbitrary block of $\widetilde{H}$. Since $\widetilde{H}$ is 3-edge-connected, $\delta(B) \geq 3$. By Claim 1, every edge of B lies in a cycle of B of length at most 4. By Theorem 2.3 and since B is 2-connected, B has a spanning Eulerian subgraph. Since every block of $\widetilde{H}$ has a spanning Eulerian subgraph, $\widetilde{H}$ has a spanning Eulerian subgraph. By Claim 2, H has a dominating Eulerian subgraph. By Theorem 2.2, $cl(G)$ is Hamiltonian. ■

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