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Eulerian subgraphs and Hamilton-connected line graphs

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Abstract

Let $C(l, k)$ denote a class of 2-edge-connected graphs of order n such that a graph $G \in C(l, k)$ if and only if for every edge cut $S \subseteq E(G)$ with $|S| \leqslant 3$, each component of $G - S$ has order at least $(n-k)/l$. We prove the following: (1) If $G \in C(6, 0)$, then G is supereulerian if and only if G cannot be contracted to $K_{2,3}$, $K_{2,5}$ or $K_{2,3}(e)$, where $e \in E(K_{2,3})$ and $K_{2,3}(e)$ stands for a graph obtained from $K_{2,3}$ by replacing e by a path of length 2. (2) If $G \in C(6, 0)$ and $n \geqslant 7$, then $L(G)$ is Hamilton-connected if and only if $\kappa(L(G)) \geqslant 3$. Former results by Catlin and Li, and by Broersma and Xiong are extended.

Keywords: Supereulerian; Collapsible; Reduction; Contraction; Hamilton-connected; Line graph

1. Introduction

We use [1] for terminology and notations not defined here and consider finite, undirected graphs. We allow graphs to have multiple edges but not loops.

Let G be a graph. We use $\kappa(G)$, $\kappa'(G)$ to denote the connectivity and the edge-connectivity of G, respectively. For each $i = 0, 1, 2, \ldots$, denote $D_i(G) = \{v \in V(G) \mid d_G(v) = i\}$. For $X \subseteq E(G)$, the *contraction* G/X is obtained from G by contracting each edge of X and deleting the resulting loops. If $H \subseteq G$, we write G/H for $G/E(H)$. Let $O(G)$ denote the set of all vertices of G with odd degrees. An *eulerian graph* G is a connected graph with $O(G) = \emptyset$. A graph is *supereulerian* if it has a spanning eulerian subgraph. In particular, K_1 is both eulerian and supereulerian. Denote by $\mathcal{SL}$ the family of all supereulerian graphs.

For integers $k \geqslant 0$ and $l > 0$, let $C(l, k)$ denote a class of 2-edge-connected graphs of order n such that $G \in C(l, k)$ if and only if for every edge cut $S \subseteq E(G)$ with $|S| \leqslant 3$, each component of $G - S$ has order at least $(n-k)/l$.

Catlin and Li, and Broersma and Xiong proved the following results concerning when a graph in a certain family $C(l, k)$ is supereulerian.

Theorem 1.1. (Catlin and Li [6]). *If $G \in C(5, 0)$, then $G \in \mathcal{SL}$ if and only if G is not contractible to $K_{2,3}$.*

Theorem 1.2. (Broersma and Xiong [2]). *If $G \in C(5, 2)$ and $n \geqslant 13$, then $G \in \mathcal{SL}$ if and only if G is not contractible to $K_{2,3}$ or $K_{2,5}$.*

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Let $e \in E(K_{2,3})$, and let $K_{2,3}(e)$ stand for a graph obtained from $K_{2,3}$ by replacing e by a path of length 2. In this paper, we further study the distribution of the small degree vertices in the reduction of a graph (to be defined in Section 2), and sharpen Theorems 1.1 and 1.2, as shown in Theorem 1.3 and Corollary 1.1.

Theorem 1.3. *If $G \in C(6, 0)$, then $G \in \mathcal{SL}$ if and only if G is not contractible to $K_{2,3}$, $K_{2,5}$ or $K_{2,3}(e)$.*

Note that when $n \geqslant 6k + 1$, $C(5, k) \subseteq C(6, 0)$. Moreover, when $G \in C(5, k)$ with $n \geqslant 6k + 1$, G cannot be contracted to a $K_{2,3}(e)$. Therefore, we have

Corollary 1.1. *If $G \in C(5, k)$ and $n \geqslant 6k + 1$, then $G \in \mathcal{SL}$ if and only if G is not contractible to $K_{2,3}$ or $K_{2,5}$.*

The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

A subgraph H of a graph G is *dominating* if $G - V(H)$ is edgeless. Let $v_0, v_k \in V(G)$. A (v_0, v_k)- *trail* of G is a vertex-edge alternating sequence

$$v_0, e_1, v_1, e_2, \ldots, e_k, v_k$$

such that all the e_i's are distinct and for each $i = 1, 2, \ldots, k$, e_i is incident with both v_{i-1} and v_i. With the notation above, this (v_0, v_k)-trial is also called an *(e_1, e_k)-trail*. All the vertices in $v_1, v_2, \ldots, v_{k-1}$ are internal vertices of trail. A *dominating (e_1, e_k)-trail* T of G is an (e_1, e_k)-trail such that every edge of G is incident with an internal vertex of T. A *spanning (e_1, e_k)-trail* of G is a dominating (e_1, e_k)-trail such that $V(T) = V(G)$. There is a close relationship between dominating eulerian subgraphs in graphs and Hamilton cycles in $L(G)$.

Theorem 1.4. (Harary and Nash-Williams [9]). *Let G be a graph with $|E(G)| \geqslant 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

A graph G is Hamilton-connected if for $u, v \in V(G) (u \neq v)$, there exists a (u, v)-path containing all vertices of G. With a similar argument in the proof of Theorem 1.4, one can obtain a theorem for Hamilton-connected line graphs.

Theorem 1.5. *Let G be a graph with $|E(G)| \geqslant 3$. Then $L(G)$ is Hamilton-connected if and only if for any pair of edges $e_1, e_2 \in E(G)$, G has a dominating (e_1, e_2)-trail.*

We say that an edge $e \in E(G)$ is *subdivided* when it is replaced by a path of length 2 whose internal vertex, denote $v(e)$, has degree 2 in the resulting graph. The process of taking an edge e and replacing it by the length 2 path is called *subdividing e*. For a graph G and edges $e_1, e_2 \in E(G)$, let $G(e_1)$ denote the graph obtained from G by subdividing e_1, and let $G(e_1, e_2)$ denote the graph obtained from G by subdividing both e_1 and e_2. Thus,

$$V(G(e_1, e_2)) - V(G) = \{v(e_1), v(e_2)\}.$$

From the definitions, one immediately has the following observation.

Proposition 1.1. *Let G be a graph and $e_1, e_2 \in E(G)$. If $G(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail, then G has a spanning (e_1, e_2)-trail.*

We investigate the Hamilton-connectedness of line graphs of graphs in $C(l, k)$ and get the following:

Theorem 1.6. *If $G \in C(6, 0)$ and $n \geqslant 7$, then $L(G)$ is Hamilton-connected if and only if $\kappa(L(G)) \geqslant 3$.*

Corollary 1.2. *If $G \in C(5, k)$ and $n \geqslant \max\{6k + 1, 6\}$, then $L(G)$ is Hamilton-connected if and only if $\kappa(L(G)) \geqslant 3$.*

Before we present the proofs of these results, we have to define what we mean with the reduction of a graph G. In Section 2, we discuss Catlin's reduction method that will be needed in the proofs of Theorems 1.3 and 1.6. We present our proofs of Theorems 1.3 and 1.6 in Section 3. The last section is devoted to some applications of Theorems 1.3 and 1.6.

2. Catlin's reduction method

In [4] Catlin defined collapsible graphs. For $R \subseteq V(G)$, a subgraph Γ of G is called an *R-subgraph* if both $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph is *collapsible* if G has an R-subgraph for every even set $R \subseteq V(G)$. In particular, K_1 is collapsible. Let $\mathscr{CL}$ denote the family of all collapsible graphs. For a graph G and its connected subgraph H, G/H denotes the graph obtained from G by contracting H, i.e. by replacing H by a vertex v_H such that the numbers of edges in G/H joining any $v \in V(G) - V(H)$ to v_H in G/H equals the number of edges joining v in G to H. A graph G is contractible to a graph G' if G contains pairwise vertex-disjoint connected subgraphs

$$H_1, H_2, \ldots, H_k \quad \text{with} \quad \bigcup_{i=1}^{k} V(H_i) = V(G)$$

such that G' is obtained from G by successively contracting $H_1, H_2, \ldots, H_k$. The subgraph H_i of G is called the *pre-image* of the vertex v_{H_i} of G'. Catlin [4] showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_k$ such that $\bigcup_{i=1}^{k} V(H_i) = V(G)$. The *reduction* of G is the graph obtained from G by successively contracting $H_1, H_2, \ldots, H_k$. A graph is *reduced* if it is the reduction of some graph.

Theorem 2.1. (Catlin [4]). *Let G be a connected graph.*

(i) *If G has a spanning tree T such that each edge of T is in a collapsible subgraph of G, then G is collapsible.*
(ii) *If G is reduced, then G is a simple graph and has no cycle of length less than four.*
(iii) *G is reduced if and only if G has no nontrivial collapsible subgraphs.*
(iv) *Let G' be the reduction of G. Then $G \in \mathscr{SL}$ if and only if $G' \in \mathscr{SL}$, and $G \in \mathscr{CL}$ if and only if $G' = K_1$.*

Theorem 2.2. (Catlin [3]) *If G is isomorphic to $K_{3,3}$ minus an edge, then $G \in \mathscr{CL}$.*

Jaeger in [10] showed that if G has two edge-disjoint spanning trees, then G is supereulerian. Letting $F(G)$ be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees, Catlin [4] and Catlin et al. [5] improved Jaeger's result. We put these former results in the following theorem.

Theorem 2.3. *Let G be a graph. Each of the following holds.*

(i) (Jaeger [10]). *If $F(G) = 0$, then G is supereulerian.*
(ii) (Catlin [4]). *If $F(G) \leqslant 1$ and if G is connected, then G is collapsible if and only if G is not contractible to a K_2.*
(iii) (Catlin, Han and Lai [5]). *If $F(G) \leqslant 2$ and if G is connected, then either G is collapsible, or the reduction of G is a K_2 or a $K_{2,t}$ for some integer $t \geqslant 1$.*
(iv) (Catlin [3]). *If G is 2-edge-connected reduced graph with $|E(G)| > 0$, then $F(G) = 2|V(G)| - |E(G)| - 2$.*

Theorem 2.4. (Catlin, Han and Lai [5]). *Let G be a connected reduced graph. If $F(G) \leqslant 2$, then exactly one of following holds*:

(i) *$G \in \mathscr{SL}$,*
(ii) *G has a cut edge,*
(iii) *G is $K_{2,s}$ for some odd integer $s \geqslant 3$.*

Let s_1, s_2, s_3, m, l, t be natural numbers with $t \geqslant 2$ and $m, l \geqslant 1$. Let $M \cong K_{1,3}$ with center a and ends a_1, a_2, a_3. Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from M by adding s_i vertices with neighbors $\{a_i, a_{i+1}\}$, where $i \equiv 1, 2, 3 \pmod 3$. Let $K_{2,t}(u, u')$ be a $K_{2,t}$ with u, u' being the nonadjacent vertices of degree t. Let $K'_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to u' only. Hence u'' has degree 1 and u has degree t in $K'_{2,t}(u, u'')$. Let $K''_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to a vertex of degree 2 of $K_{2,t}$. Hence u'' has degree 1 and both u and u' have degree t in $K''_{2,t}(u, u'')$. We shall use $K'_{2,t}$ and $K''_{2,t}$ for a $K'_{2,t}(u, u', u'')$ and a $K''_{2,t}(u, u', u'')$, respectively. Let $S(m, l)$ be the graph obtained from a $K_{2,m}(u, u')$ and a $K'_{2,l}(w, w', w'')$ by identifying u with w, and w'' with u'; let $J(m, l)$ denote the graph obtained from a $K_{2,m+1}$ and a $K'_{2,l}(w, w', w'')$ by identifying w, w'' with the two ends of an edge in $K_{2,m+1}$, respectively; let $J'(m, l)$ denote the graph obtained from a $K_{2,m+2}$ and a $K'_{2,l}(w, w', w'')$ by

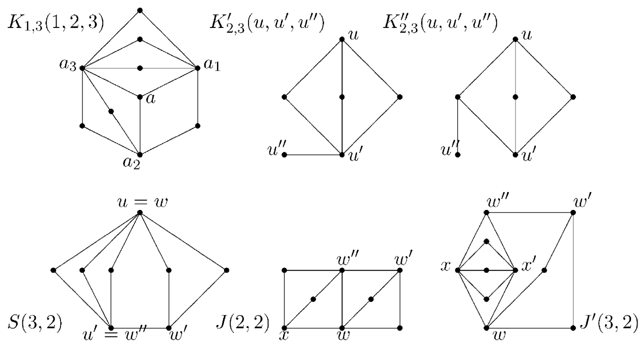

Fig. 1. Some graphs in $\mathscr{F}$ with small parameters.

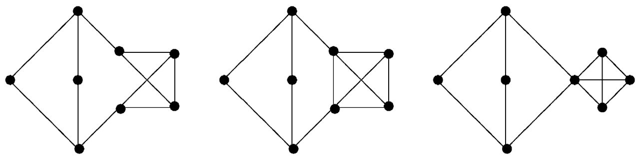

Fig. 2.

identifying w, w'' with two vertices of degree 2 in $K_{2,m+2}$, respectively. See Fig. 1 for examples of these graphs. Let

$$\mathscr{F} = \{K_1,\ K_2,\ K_{2,t},\ K'_{2,t},\ K''_{2,t},\ K_{1,3}(s,s',s''),\ S(m,l),\ J(m,l),\ J'(m,l),\ P\},$$

where t, s, s', s'', m, l are nonnegative integers, and P denotes the Petersen graph.

Theorem 2.5. (Chen and Lai [8]). *If G is connected reduced graph with* $|V(G)| \leqslant 11$ *and* $F(G) \leqslant 3$, *then* $G \in \mathscr{F}$.

Theorem 2.6. (Chen [7]). *Let G be a reduced graph with* $n \leqslant 11$ *vertices, and* $\kappa'(G) \geqslant 3$. *Then G is either* K_1 *or the Petersen graph.*

Lemma 2.1. *Let G be a connected simple graph with* $n \leqslant 8$ *vertices and with* $D_1(G) = \emptyset$, $|D_2(G)| \leqslant 2$. *Then either G is one of three graphs in Fig. 2, or the reduction of G is* K_1 *or* K_2.

Proof. By Theorem 2.6, we may assume $\kappa'(G) \in \{1, 2\}$. Firstly, suppose that G has a cut-edge e and that G_1 and G_2 are two components of $G - e$. Let $n_i = |V(G_i)|$, $1 \leqslant i \leqslant 2$ and assume, without loss of generality, $n_2 \geqslant n_1$. Since G is simple and $D_1(G) = \emptyset$, $n_1 \geqslant 3$ and equality holds if and only if $G_1 = K_3$. If both $n_2 = n_1 = 4$, then each of G_1 and G_2 must be K_4 or $K_4 - e$, where $e \in E(K_4)$. Since K_3 is collapsible and by Theorem 2.1(i), both G_1 and G_2 are collapsible, and so the reduction of G is K_2. Since $n_2 = 3$ will force $|D_2(G)| \geqslant 3$, we assume that $4 \leqslant n_2 \leqslant 5$ and $n_1 = 3$. If G_2 is not collapsible, then $G_2 \in \{C_4, C_5, K_{2,3}\}$ and $|D_2(G)| \geqslant 3$. So G_2 must be collapsible. Hence the reduction of G is K_2.

Now we assume that G is 2-edge-connected and G' is the reduction of G with $n' = |V(G')| \geqslant 2$. Then G' is 2-edge-connected and nontrivial. Let C_m be a longest cycle in G'. Then $m \geqslant 4$ by Theorem 2.1(ii).

If $n' = 8$ or 7, then $G = G'$. As $|D_2(G)| \leqslant 2$, we have $F(G') \leqslant 3$ by Theorem 2.3(iv). Apply Theorem 2.5 to G. Since every 2-edge-connected graph in $\mathscr{F}$ has at least 3 vertices of degree 2, we have $|D_2(G)| \geqslant 3$, contrary to the assumption that $|D_2(G)| \leqslant 2$. Thus we must have $n' \leqslant 6$. If $n' = 6$, then either $G = G'$ or the pre-image of a vertex in G' is a triangle and the pre-images of the other vertices in G' are themselves. Thus $|D_2(G')| \leqslant 2$. By Theorem 2.3(iv), we have $F(G') \leqslant 2$. Therefore $|D_2(G')| \geqslant 3$ by Theorem 2.3(iii), a contradiction. If $n' = 4$, then $G' = C_4$. Note that the size of the pre-image of each vertex is either 1 or at least 3. Thus $|D_2(G)| \geqslant 3$. It contradicts the hypothesis that $|D_2(G)| \leqslant 2$. So $n' = 5$. Note again that the size of the pre-image of each vertex is either 1 or at least 3. By $|D_2(G)| \leqslant 2$, $|D_3(G')| \neq 0$. Thus $F(G') \leqslant 2$. By Theorem 2.3(iii), $G' = K_{2,3}$. As $n \leqslant 8$ and $|D_2(G)| \leqslant 2$, the pre-image of a vertex having degree 2 in G' is either a K_4 or a K_4 minus an edge, and the pre-images of the other vertices in G' are themselves. Thus G is one of the graphs in Fig. 2. □

Lemma 2.2. *If G is collapsible, then for any pair of vertices* $u, v \in V(G)$, *G has a spanning* (u, v)*-trail.*

Proof. Let $R=(O(G)\cup\{u,v\})-(O(G)\cap\{u,v\})$. Then $|R|$ is even. Let Γ_R be an R-subgraph of G. Then $G-E(\Gamma_R)$ is a spanning (u,v)-trail of G.

3. Proofs of Theorems 1.3 and 1.6

Proof of Theorem 1.3. Let G' be the reduction of G. If $G'=K_1$, then $G\in\mathcal{SL}$ by Theorem 2.1(iv). Next we suppose that $G'\neq K_1$. Then G' is 2-edge-connected and nontrivial. Denote $d_i=|D_i(G')|\,(i\geqslant 2)$.

If $d_2+d_3\geqslant 7$, then we assume that $v_1,v_2,\ldots,v_7$ are the vertices of $V(G')$ in $D_2(G')\cup D_3(G')$, i.e. $d_{G'}(v_i)\leqslant 3$ for each i, and the corresponding pre-images are $H_1,H_2,\ldots,H_7$. Each H_i is joined to the rest of G by an edge cut consisting of $d_{G'}(v_i)\leqslant 3$ edges. By the hypothesis of Theorem 1.3, $|V(H_i)|\geqslant \frac{n}{6}$, and

$$n=|V(G)|\geqslant \sum_{i=1}^{7}|V(H_i)|\geqslant \frac{7n}{6},$$

a contradiction. Therefore we assume $d_2+d_3\leqslant 6$, and when $d_2+d_3=6$, $V(G')=D_2(G')\cup D_3(G')$. We break the proof into two cases.

Case 1. $F(G')\leqslant 2$.

By $\kappa'(G')\geqslant 2$ and by Theorem 2.4, $G'\in\mathcal{SL}$ or $G'=K_{2,s}$, where $s\geqslant 3$ is an odd integer. In the former case, $G\in\mathcal{SL}$ by Theorem 2.1(iv). In the latter, $s=3$ or $s=5$ by $d_2+d_3\leqslant 6$.

Case 2. $F(G')\geqslant 3$.

Note that $|V(G')|=\sum_{i\geqslant 2}d_i$, $2|E(G')|=\sum_{i\geqslant 2}id_i$. By Theorem 2.3(iv), we have the following:

$$2d_2+d_3\geqslant 10+\sum_{i\geqslant 5}(i-4)d_i. \tag{1}$$

Since $d_2+d_3\leqslant 6$, $d_2\geqslant 4$. We distinguish two cases to complete the proof.

Case 2.1. $d_2=4$.

By (1) and $d_2+d_3\leqslant 6$, $d_3=2$. Thus $V(G')=D_2(G')\cup D_3(G')$. Let $D_2(G')=\{u_1,u_2,u_3,u_4\}$ and $D_3(G')=\{v_1,v_2\}$. If $v_1v_2\in E(G')$, then $E(G')=\{u_1u_2,u_2v_2,v_2u_3,u_3u_4,u_4v_1,v_1u_1,v_1v_2\}$ by Theorem 2.1(ii). Thus $G'\in\mathcal{SL}$ and so $G\in\mathcal{SL}$. If $v_1v_2\notin E(G')$, then $G'=K_{2,3}(e)$, where $e\in E(K_{2,3})$.

Case 2.2. $d_2=5$ or 6.

If $d_2=5$, then $d_3=0$ and $d_i=0(i\geqslant 5)$ by (1). If $d_2=6$, then $d_i=0(i\geqslant 3)$. In both cases, $O(G)=\emptyset$, thus $G'\in\mathcal{SL}$ and so $G\in\mathcal{SL}$. □

Proof of Theorem 1.6. It is trivial that $\kappa(L(G))\geqslant 3$ if $L(G)$ is Hamilton-connected. So we only need to prove that $L(G)$ is Hamilton-connected when $\kappa(L(G))\geqslant 3$.

Let $e_1,e_2\in E(G)$. By Theorem 1.5, Proposition 1.1 and Lemma 2.2, we need to prove $G(e_1,e_2)\in\mathcal{CL}$. Let G' be the reduction of $G(e_1,e_2)$. By Theorem 2.1(iv), it suffices to prove that $G'=K_1$. Suppose that $G'\neq K_1$. Then G' is 2-edge-connected and nontrivial. Denote $d_i=|D_i(G')|(i\geqslant 2)$.

If $d_2\geqslant 3$, then there exists $v\in D_2(G')-\{v(e_1),v(e_2)\}$ such that $d_{G'}(v)=2$. Let H be the pre-image of v in $G(e_1,e_2)$. Then H is joined to the rest of $G(e_1,e_2)$, therefore of G, by an edge-cut consisting of $d_{G'}(v)=2$ edges. By the hypothesis of Theorem 1.6, $|V(H)|\geqslant\frac{n}{6}>1$. Thus $\kappa(L(G))\leqslant 2$, a contradiction. So $d_2\leqslant 2$.

If $d_2+d_3\geqslant 9$, then $|D_2(G')\cup D_3(G')-\{v(e_1),v(e_2)\}|\geqslant 7$. We assume that $v_1,v_2,\ldots,v_7$ are the vertices of $V(G')$ in $D_2(G')\cup D_3(G')-\{v(e_1),v(e_2)\}$, i.e. $d_{G'}(v_i)\leqslant 3$ for each i, and the corresponding pre-images are $H_1,H_2,\ldots,H_7$. Each H_i is joined to the rest of $G(e_1,e_2)$, therefore to the rest of G, by an edge cut consisting of $d_{G'}(v_i)\leqslant 3$ edges. By the hypothesis of Theorem 1.6, $|V(H_i)|\geqslant\frac{n}{6}$, and

$$n=|V(G)|\geqslant \sum_{i=1}^{7}|V(H_i)|\geqslant \frac{7n}{6},$$

a contradiction. So $d_2+d_3\leqslant 8$, and when $d_2+d_3=8$, $V(G')=D_2(G')\cup D_3(G')$.

Suppose that $F(G')\geqslant 3$, i.e., $2|V(G')|-|E(G')|\geqslant 5$ by Theorem 2.3(iv). Note that $|V(G')|=\sum_{i\geqslant 2}d_i$, $2|E(G')|=\sum_{i\geqslant 2}id_i$, we have the following:

$$2d_2+d_3\geqslant 10+\sum_{i\geqslant 5}(i-4)d_i. \tag{2}$$

By (2) and $d_2+d_3\leqslant 8$, $d_2=2$ and $d_3=6$. Thus $|V(G')|=8$. Since G' is a 2-edge-connected nontrivial reduced graph, G' is also a connected simple graph with $|V(G')|\leqslant 8$ and with $D_1(G')=\emptyset$, $|D_2(G')|=d_2\leqslant 2$. Therefore, by Lemma 2.1, either $G'\cong K_2$, contrary to the assumption that G' is 2-edge-connected; or G' is one of the three non reduced graphs displayed in Fig. 2, contrary to the assumption that G' is reduced. In either case, a contradiction obtains. Thus we must have $F(G')\leqslant 2$.

As G' is 2-edge-connected and $d_2\leqslant 2$, $G\neq K_2$, $K_{2,t}\,(t\geqslant 1)$, and so by Theorem 2.3(iii), $G'=K_1$, contrary to the assumption that G' is nontrivial. This completes the proof of Theorem 1.6. □

4. Applications

Theorem 1.3 and Corollary 1.2 have a number of applications, as shown below.

Theorem 4.1. (Zhan [11]). *Let G be a graph. If $\kappa'(G)\geqslant 4$, then $L(G)$ is Hamilton-connected.*

Proof. The case when $|V(G)|\leqslant 6$ can be easily verified. Assume that $|V(G)|\geqslant 7$. Since $\kappa'(G)\geqslant 4$, $G\in C(6,0)$ vacuously and we have $\kappa(L(G))\geqslant 4$. Therefore, by Theorem 1.6 $L(G)$ is Hamilton-connected. □

Theorem 4.2. *Let G be a 2-edge-connected simple graph. If $\delta(G)\geqslant 4$ and if*

$$\min\{\max\{d(x),d(y)\}|xy\in E(G)\}\geqslant \frac{n}{6}-1,$$

then $G\in\mathscr{SL}$ if and only if G is not contractible to $K_{2,3}$, $K_{2,5}$ or $K_{2,3}(e)$.

Proof. Let S be an edge cut of G with $|S|\leqslant 3$, and let G_1 and G_2 be the two components of $G-S$ with $|V(G_1)|\leqslant|V(G_2)|$. It is sufficient to prove that $|V(G_1)|\geqslant\frac{n}{6}$ by Theorem 1.3. Since $\delta(G)\geqslant 4$, G_1 has at least an edge, say uv, such that both of u, v are not incident with any edges of S. By the hypothesis of Theorem 4.2,

$$|V(G_1)|\geqslant \max\{d(u),d(v)\}+1\geqslant\frac{n}{6}. \qquad \square$$

Thus Theorem 4.2 follows from Theorem 1.3.

Corollary 4.1. *Let G be a 2-edge-connected simple graph with $n\geqslant 6k+1$ vertices. If $\delta(G)\geqslant 4$ and if*

$$\min\{\max\{d(x),d(y)\}\,|\,xy\in E(G)\}\geqslant\frac{n-k}{5}-1,$$

then $G\in\mathscr{SL}$ if and only if G is not contractible to $K_{2,3}$ or $K_{2,5}$.

Proof. By Theorem 4.2, it suffices to show that under the assumption of Theorem 4.3, G cannot be contracted to a $K_{2,3}(e)$. In fact, if G can be contracted to a $K_{2,3}(e)$, then the preimage of each vertex of the $K_{2,3}(e)$ has at least $(n-k)/5-1$ vertices. Since $|V(K_{2,3}(e))|=6$, G must have at least $6(n-k)/5$ vertices, and so

$$n\geqslant 6\left(\frac{n-k}{5}\right)=n+\frac{n}{5}-\frac{6k}{5},\text{ or } n\leqslant 6k,$$

contrary to the assumption that $n\geqslant 6k+1$. This completes the proof. □

Theorem 4.3 below follows Theorem 4.2 by taking $k=2$. Corollaries 4.2 and 4.3 also follow from 4.2 trivially.

Theorem 4.3. (Broersma and Xiong [2]). *Let G be a 2-edge-connected simple graph with $n\geqslant 13$ vertices. If $\delta(G)\geqslant 4$ and if*

$$\min\{\max\{d(x),d(y)\}\,|\,xy\in E(G)\}\geqslant\frac{n-2}{5}-1,$$

then $G\in\mathscr{SL}$ if and only if G is not contractible to $K_{2,3}$ or $K_{2,5}$.

Corollary 4.2. *G be a 2-edge-connected simple graph. If* $\delta(G) \geqslant 4$ *and if every edge* $uv \in E(G)$ *satisfies*

$$d(u) + d(v) \geqslant \frac{n}{3} - 2,$$

then $G \in \mathcal{SL}$ *if and only if G is not contractible to* $K_{2,3}$, $K_{2,5}$ *or* $K_{2,3}(e)$.

Corollary 4.3. *Let G be a 2-edge-connected simple graph of order* $n > 24$. *If*

$$\delta(G) \geqslant \frac{n}{6} - 1,$$

then $G \in \mathcal{SL}$ *if and only if G is not contractible to* $K_{2,3}$, $K_{2,5}$ *or* $K_{2,3}(e)$.

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