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# ON THE MAXIMAL CONNECTED COMPONENT OF HYPERCUBE WITH FAULTY VERTICES

XIAOFAN YANG[a,*], DAVID J. EVANS[b], BILL CHEN[c], GRAHAM M. MEGSON[d] and HONGJIAN LAI[e]

[a]*College of Computer Science, Chongqing University, Chongqing 400044, P.R. China*
[b]*Nottingham Trent University, School of Computing and Mathematics, Room N421a, Newton Building, Burton Street, Nottingham NG1 4BU, UK*
[c]*Center for Combinatorics, Nankai University, Tianjin 300071, P.R. China*
[d]*Department of Computer Science, School of Systems Engineering, University of Reading, P.O. Box 225, Whiteknights, Reading, Berkshire RG6 6AY, UK*
[e]*Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA*

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In evaluating an interconnection network, it is indispensable to estimate the size of the maximal connected components of the underlying graph when the network begins to lose processors. Hypercube is one of the most popular interconnection networks. This article addresses the maximal connected components of an $n$-dimensional cube with faulty processors. We first prove that an $n$-cube with a set $F$ of at most $2n - 3$ failing processors has a component of size $\geq 2^n - |F| - 1$. We then prove that an $n$-cube with a set $F$ of at most $3n - 6$ missing processors has a component of size $\geq 2^n - |F| - 2$.

*Keywords*: Interconnection network; Fault tolerance; Maximal connected component; Hypercube

*C.R. Categories*: B.4.3 (Hardware, Input/output and data communications, Interconnections, Topology)

## 1 INTRODUCTION

The *fault tolerance* of a distributed memory-parallel computing system (*i.e.* multicomputer system) means its capability of being functional in the presence of failures, which is commonly measured by the vertex connectivity of the underlying graph of the interconnection network in the system. However, the connectivity measure of fault tolerance does not take into consideration the sizes of the connected components of the damaged underlying graph. Usually, if the surviving graph contains a large connected component, it may be used as the functional subsystem, without incurring severe performance degradation [1]. Thus, in evaluating an interconnection network, it is indispensable to estimate the size of the maximal components of the underlying graph when the network begins to lose processors. Indeed, a number of measures of fault tolerance of networks, which are related to the size of the maximal components of the

* Corresponding author. Tel.: 0811-23-65342815; E-mail: yxf640126@sina.com

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underlying graph with missing vertices, were proposed [2, 3]. In general, an interconnection network that has larger maximal component in the presence of failures is desirable.

Hypercube is one of the most popular interconnection networks because it enjoys many attractive properties (including symmetry, recursive construction, logarithmic diameter and linear bisection width), which support many elegant and efficient parallel algorithms [4].

This article addresses the maximal component of cube structures with missing processors. We first prove that an $n$-cube with a set $F$ of at most $2n-3$ failing processors has a component of size $\geq 2^n - |F| - 1$. We then prove that an $n$-cube with a set $F$ of at most $3n-6$ missing processors has a component of size $\geq 2^n - |F| - 2$.

## 2 PRELIMINARIES

In this article, we use a graph $G = (V(G), E(G))$ to represent an interconnection network, in which the vertices represent the processors and the edges represent the communication links between the processors. Let $\kappa(G)$ and $\delta(G)$ denote the *vertex connectivity* and the *minimum degree* of the graph $G$, respectively. It is well known that $\kappa(G) \leq \delta(G)$. For a set $F$ of vertices in graph $G$, let $G[F]$ denote the subgraph of $G$ induced by $F$, and let $G - F$ denote the graph obtained from $G$ by deleting $F$. For fundamental graph-theoretical terminology the reader is referred to Ref. [5].

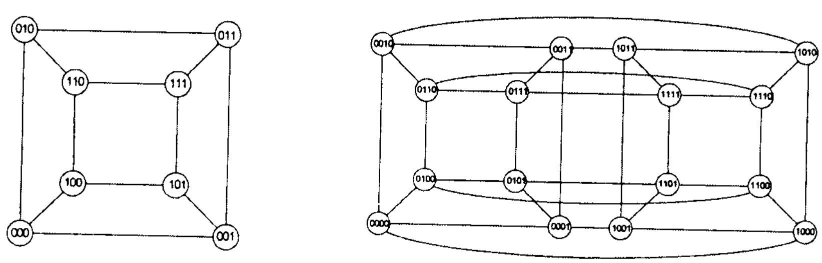
For a set $F$ of vertices in graph $G$, let

$$N(F) = \{v \in V(G) - F: \text{ there exists } u \in F \text{ such that } (u, v) \in E(G)\}.$$

Let $N[F] = N(F) \cup F$. For brevity, $N(\{u\})$ and $N[\{u\}]$ are written as $N(u)$ and $N[u]$, respectively.

Let $\{0, 1\}^n$ denote the set of all 0–1 binary strings of length $n$. An $n$-cube, denoted by $Q_n$, is a graph with $V(Q_n) = \{0, 1\}^n$. Two vertices are adjacent if and only if their labels differ in exactly one bit position. Two examples of hypercube are shown in Figure 1. It is well known that $Q_n$ is $n$-regular and bipartite (hence, $Q_n$ has no cycle of odd length). Moreover, let $S_0$ (respectively, $S_1$) denote the set of vertices in $Q_n$ each of which takes value 0 (respectively, 1) on the most significant bit position. Let $0Q_{n-1} = Q_n[S_0]$ and $1Q_{n-1} = Q_n[S_1]$. Then both $0Q_{n-1}$ and $1Q_{n-1}$ are isomorphic to $Q_{n-1}$, and every vertex of $0Q_{n-1}$ is adjacent to exactly one vertex of $1Q_{n-1}$, and vice versa.

THEOREM 2.1 [1] *Let $F$ be a set of at most $2n-3$ vertices in $Q_n (n \geq 3)$. If $N(u) \not\subseteq F$ for each vertex $u$ in $Q_n$, then $Q_n - F$ is connected.*




(a) $Q_3$ (b) $Q_4$

FIGURE 1 Two examples of hypercube.

THEOREM 2.2 [6] *Let $u$ be a vertex in $Q_n (n \geq 5)$. Then $\kappa(Q_n - N[u]) = n - 2$.*

THEOREM 2.3 [6] *Let $F$ be a set of at most $3n - 6$ vertices in $Q_n (n \geq 3)$. If $N(u) \not\subset F$ for any vertex $u$ in $Q_n$ and $N(\{u, v\}) \not\subset F$ for any two adjacent vertices $u$ and $v$ in $Q_n$, then $Q_n - F$ is connected.*

## 3 MAIN RESULTS

It is well known [7] that $\kappa(Q_n) = n$. Thereby, we immediately obtain the following theorem.

THEOREM 3.1 *Let $F$ be a set of at most $n - 1$ vertices in $Q_n (n \geq 3)$. Then*

$$m(Q_n - F) = V(Q_n) - |F|.$$

In what follows, we focus our attention on the estimation of $m(Q_n - F)$ for larger $F$. First, we present the following interesting result.

LEMMA 3.2 *Let $\{u, v\}$ be a pair of adjacent vertices in $Q_n (n \geq 3)$. Then*

$$\kappa(Q_n - (N[u] \cup N[v])) = n - 2.$$

*Proof* Clearly,

$$\kappa(Q_n - (N[u] \cup N[v])) \leq \delta(Q_n - (N[u] \cup N[v])) = n - 2.$$

So it suffices to prove $\kappa(Q_n - (N[u] \cup N[v])) \geq n - 2$. This inequality is clearly true for the case $n = 3$. Now, assume $n \geq 4$ and let $\{u, v\}$ be an arbitrary pair of adjacent vertices in $Q_n$. In view of the symmetry of hypercube, we may assume $u \in V(0Q_{n-1})$ and $v \in V(1Q_{n-1})$. For brevity, we set

$$N^*(u) = N(u) - \{v\}, \quad N^*[u] = N[u] - \{v\},$$
$$N^*(v) = N(v) - \{u\}, \quad N^*[v] = N[v] - \{u\}.$$

It remains to show that for any set $F$ of $n - 3$ vertices in the graph $Q_n - (N[u] \cup N[v])$, the graph $Q_n - (N[u] \cup N[v]) - F$ is connected. To this end, we set

$$F_0 = F \cap V(0Q_{n-1}), \quad F_1 = F \cap V(1Q_{n-1}).$$

Next, we need to examine three possibilities.

*Case 1* $|F_0| = n - 3$ (Fig. 2(a)). Then $|F_1| = 0$. By Theorem 2.2, $1Q_{n-1} - N^*[v]$ is $(n - 3)$-connected, which implies that it is connected. Furthermore, each vertex in $0Q_{n-1} - N^*[u] - F_0$ has a neighbour in $1Q_{n-1} - N^*[v]$. As a result, $Q_n - (N[u] \cup N[v]) - F$ is connected.

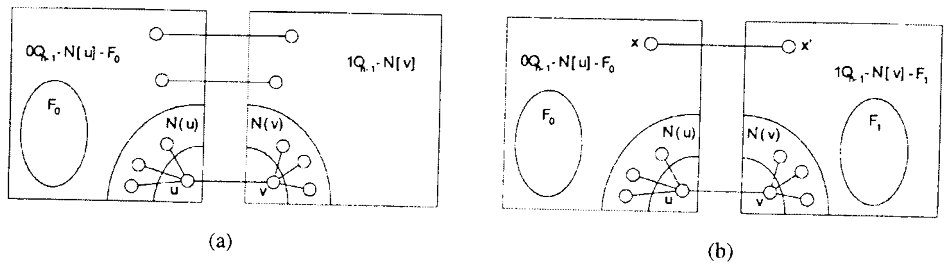



FIGURE 2 Schematic illustrations of the proof of Lemma 3.2.

*Case 2* $|F_0| = 0$. Then $|F_1| = n - 3$. Similar to Case 1, we can prove that $Q_n - (N[u] \cup N[v]) - F$ is connected.

*Case 3* $|F_0| \le n-4$ and $|F_1| \le n - 4$ (Fig. 2(b)). By applying Theorem 2.2 to $0Q_{n-1} - N^*[u]$ and $1Q_{n-1} - N^*[v]$, respectively, we derive that they are both $(n-3)$-connected. So $0Q_{n-1} - N^*[u] - F_0$ and $1Q_{n-1} - N^*[v] - F_1$ are both connected. Observe that there are $(2^{n-1} - n)$ edges in $Q_n$ each of which has one endpoint in $0Q_{n-1} - N^*[u]$ and the other endpoint in $1Q_{n-1} - N^*[v]$. Since $2^{n-1} - n > n - 3 = |F|$, there must exist an edge $(x, x')$ with one endpoint in $0Q_{n-1} - N^*[u] - F_0$ and the other endpoint in $1Q_{n-1} - N^*[v] - F_1$. Hence, $Q_n - (N[u] \cup N[v]) - F$ is connected.

Combining the above discussions, we conclude that $\kappa(Q_n - (N[u] \cup N[v])) \ge n - 2$. ■

Now, we establish the first main result of this article, which is stated as follows.

THEOREM 3.3 *Let $F$ be a set of at most $2n-3$ vertices in $Q_n (n \ge 3)$. Then*

$$m(Q_n - F) \ge V(Q_n) - |F| - 1.$$

Moreover, this inequality is optimal in the sense that there is a set $F$ of $2n - 2$ vertices in $Q_n$ such that

$$m(Q_n - F) = V(Q_n) - |F| - 2.$$

*Proof* Let $F$ be a set of at most $2n - 3$ vertices in $Q_n$. By Theorem 2.1, there are two possibilities.

*Case 1* $Q_n - F$ is connected. Then $m(Q_n - F) = V(Q_n) - |F|$.

*Case 2* There is a vertex $u$ in $Q_n$ such that $N(u) \subseteq F$. Then

$$|F - N[u]| \le |F - N(u)| = |F| - |N(u)| \le (2n-3) - n = n - 3.$$

By Theorem 2.2, the graph $Q_n - N[u]$ is $(n-2)$-connected. So the graph

$$Q_n - N[u] - (F - N[u]) = Q_n - F - \{u\}$$

is connected, which is the maximal component of the graph $Q_n - F$. As a result,

$$m(Q_n - F) = V(Q_n) - |F| - 1.$$

This completes the proof of the first assertion. Now, we consider a pair $\{u, v\}$ of adjacent vertices in $Q_n$. Let $F = N(\{u, v\})$. Clearly, $|F| = 2n - 2$. By Lemma 3.2, the graph

$$Q_n - F - \{u, v\} = Q_n - (N[u] \cup N[v])$$

is $(n-2)$-connected and hence is connected, which is the maximal component of $Q_n - F$. Thus,

$$m(Q_n - F) = V(Q_n) - |F| - 2.$$

Next, we deal with $m(Q_n - F)$ for the case $|F| \leq 3n - 6$. To this end, we first prove three lemmas.

LEMMA 3.4 *Let $F$ be a set of at most $3n - 6$ vertices in $Q_n (n \geq 3)$ such that there exists a vertex $u$ in $Q_n$ satisfying $N(u) \subseteq F$. Then one of the following two results holds.*

(i) *$Q_n - F - \{u\}$ is connected.*
(ii) *There exists a vertex $v \neq u$ in $Q_n$ such that $N(v) \subseteq F$ and $Q_n - F - \{u, v\}$ is connected.*

*Proof* We argue by induction on $n$. Clearly, (i) holds for the case $n = 3$. Suppose the assertion is true for $n = k - 1 (k \geq 4)$. Now assume $F$ is a set of at most $3k - 6$ vertices in $Q_k$ such that there exists a vertex $u$ in $Q_k$ satisfying $N(u) \subseteq F$. Without loss of generality, we may assume $u \in V(0Q_{k-1})$ and let $u'$ denote the neighbour of $u$ in $1Q_{k-1}$. For brevity, let

$$N^*(u) = N(u) - \{u'\}, \quad N^*[u] = N[u] - \{u'\}.$$

Furthermore, let

$$F_0 = F \cap V(0Q_{k-1}), \quad F_1 = F \cap V(1Q_{k-1}).$$

Then $F_0 \supseteq N^*(u)$ and $F_1 \supseteq \{u'\}$. Thus, $|F_0| \geq k - 1$ and $|F_1| \geq 1$. Next, we deal with four possible cases sequentially.

*Case 1* $k - 1 \leq |F_0| \leq 2k - 5$. Then

$$|F_0 - N^*[u]| \leq |F_0 - N^*(u)| = |F_0| - |N^*(u)| \leq (2k - 5) - (k - 1) \leq k - 4.$$

By Theorem 2.2, $0Q_{k-1} - N^*[u]$ is $(k-3)$-connected. So the graph

$$0Q_{k-1} - N^*[u] - (F_0 - N^*[u]) = 0Q_{k-1} - F_0 - \{u\}$$

is connected. On the other hand,

$$|F_1| = |F| - |F_0| \leq (3k - 6) - (k - 1) = 2k - 5 = 2(k - 1) - 3.$$

By applying Theorem 2.1 to $1Q_{k-1}$ and $F_1$, we derive that one of the following two subcases must occur.

*Subcase 1.1* The graph $1Q_{k-1} - F_1$ is connected. Observe that there are $2^{k-1} - k$ edges in $Q_k$ each of which has one endpoint in $0Q_{k-1} - N^*[u]$ and the other endpoint in $1Q_{k-1} - \{u'\}$, and that there are at most

$$|F - N(u)| = |F| - |N(u)| \leq (3k - 6) - k = 2k - 6$$

vertices in the set of such endpoints that belong to $F$. Since $2^{k-1} - k > 2k - 6$, there exists an edge whose two endpoints belong to $0Q_{k-1} - F_0 - \{u\}$ and $1Q_{k-1} - F_1$, respectively. So the two connected graphs $0Q_{k-1} - F_0 - \{u\}$ and $1Q_{k-1} - F_1$ are connected to each other via this edge. That is, the graph $Q_k - F - \{u\}$ is connected, or, equivalently, assertion (i) holds.

*Subcase 1.2* There exists a vertex $v$ in $1Q_{k-1}$ such that $N(v) - \{v'\} \subseteq F_1$, where $v'$ is the neighbour of $v$ in $0Q_{k-1}$. Let

$$N^*(v) = N(v) - \{v'\}, \quad N^*[v] = N[v] - \{v'\}.$$

Then

$$|F_1 - N^*[v]| \leq |F_1 - N^*(v)| = |F_1| - |N^*(v)| \leq (2k-5) - (k-1) = k-4.$$

By Theorem 2.2, $1Q_{k-1} - N^*[v]$ is $(k-3)$-connected. Thus, the graph

$$1Q_{k-1} - N^*[v] - (F_1 - N^*[v]) = 1Q_{k-1} - F_1 - \{v\}$$

is connected. We proceed by investigating three subcases.

*Subcase 1.2.1* $d(u, v) = 1$ (Fig. 3(a)). Then each vertex in $N^*[u]$ is connected to a vertex in $N^*[v]$. So there are $2^{k-1} - k$ edges in $Q_k$ each of which has one endpoint in $0Q_{k-1} - N^*[u]$ and the other endpoint in $1Q_{k-1} - N^*[v]$, and there are at most

$$\begin{aligned}|F - (N(u) \cup N(v))| &= |F| - |N(u) \cup N(v)| \\ &= |F| - |N(u)| - |N(v)| + |N(u) \cap N(v)| \leq (3k-6) - 2k = k-6\end{aligned}$$

vertices in the set of such endpoints that belong to $F$. Since $2^{k-1} - k > k - 6$, there exists an edge $(x, x')$ whose two endpoints belong to $0Q_{k-1} - F_0$ and $1Q_{k-1} - F_1$, respectively. Hence, $Q_k - F - \{u\} = Q_k - F$ is connected. That is, assertion (i) holds.

*Subcase 1.2.2* $d(u, v) = 2$ (Fig. 3(b)). Then $N(u) \cap N(v) = \{u', v'\}$ and $N(v) \subseteq F$. So there are at least $2^{k-1} - (2k-2)$ edges in $Q_k$ each of which has one endpoint in $0Q_{k-1} - N[u]$ and the other endpoint in $1Q_{k-1} - N[v]$, and there are at most

$$\begin{aligned}|F - (N(u) \cup N(v))| &= |F| - |N(u) \cup N(v)| \\ &= |F| - |N(u)| - |N(v)| + |N(u) \cap N(v)| \leq (3k-6) - 2k + 2 = k-4\end{aligned}$$

vertices in the set of such endpoints that belong to $F$. Since $2^{k-1} - (2k-2) > k-4$, there exists an edge $(x, x')$ whose two endpoints belong to $0Q_{k-1} - F_0 - \{u\}$ and $1Q_{k-1} - F_1 - \{v\}$, respectively. Hence, $Q_k - F - \{u, v\}$ is connected. That is, assertion (ii) holds.

*Subcase 1.2.3* $d(u, v) \geq 3$. Then $|N(u) \cap N(v)| = 0$. Observe that there are at least $2^{k-1} - 2k$ edges in $Q_k$ each of which has one endpoint in $0Q_{k-1} - N[u]$ and the other endpoint in $1Q_{k-1} - N[v]$, and there are at most

$$\begin{aligned}|F - (N(u) \cup N(v))| &= |F| - |N(u) \cup N(v)| \\ &= |F| - |N(u)| - |N(v)| \leq (3k-6) - 2k = k-6\end{aligned}$$

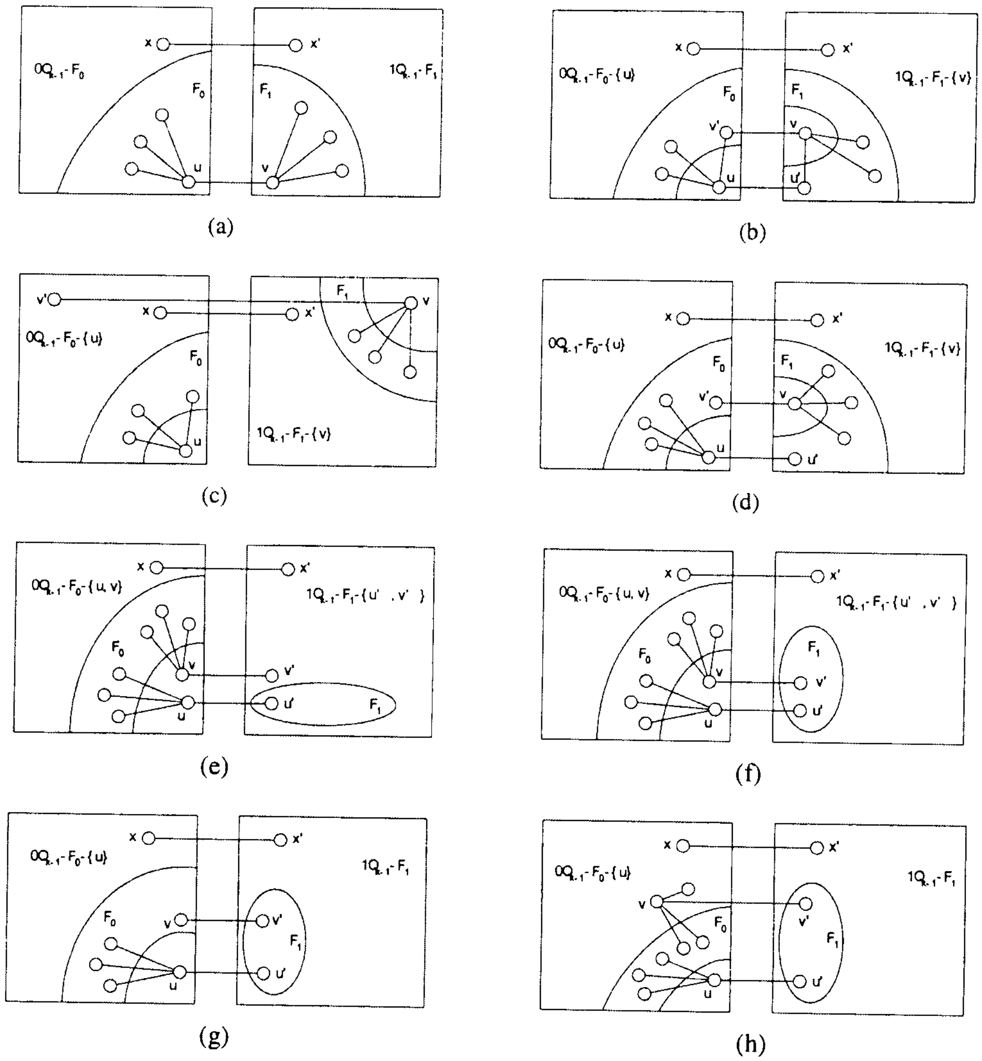


FIGURE 3 Schematic illustrations of the proof of Lemma 3.4.

vertices in the set of such endpoints that belong to $F$. Since $2^{k-1} - 2k > k - 6$, there exists an edge $(x, x')$ whose two endpoints belong to $0Q_{k-1} - F_0 - \{u\}$ and $1Q_{k-1} - F_1 - \{v\}$, respectively. Hence, $Q_k - F - \{u, v\}$ is connected. If $v' \notin F$, then $Q_k - F - \{u\}$ is connected. That is, assertion (i) holds (Fig. 3(c)). If $v' \in F$, then $N(v) \subseteq F$. That is, assertion (ii) holds (Fig. 3(d)).

*Case 2* $2k - 4 \leq |F_0| \leq 3k - 9 = 3(k-1) - 6$. Then

$$|F_1| = |F| - |F_0| \leq (3k - 6) - (2k - 4) = k - 2.$$

Since $1Q_{k-1}$ is $(k-1)$-connected, the graph $1Q_{k-1} - F_1$ is connected. By applying the inductive hypothesis to $0Q_{k-1}$ and $F_0$, we derive that one of the following two subcases must occur.

*Subcase 2.1* The graph $0Q_{k-1} - F_0$ is connected. Similarly to Subcase 1.1, we can prove that the graph $Q_k - F - \{u\}$ is connected, *i.e.* the assertion (i) holds.

*Subcase 2.2* There exists a vertex $v$ in $0Q_{k-1}$ such that $N(v) - \{v'\} \subseteq F_0$, where $v'$ is the neighbour of $v$ in $1Q_{k-1}$, and $0Q_{k-1} - F_0 - \{u, v\}$ is connected. Let

$$N^*(v) = N(v) - \{v'\}, \quad N^*[v] = N[v] - \{v'\}.$$

Clearly, $|N(u) \cap N^*(v)| \leq 2$. Observe that there are at least $2^{k-1} - 2k$ edges in $Q_k$ each of which has one endpoint in $0Q_{k-1} - (N^*[u] \cup N^*[v])$ and the other endpoint in $1Q_{k-1} - \{u'\}$, and that there are at most

$$\begin{aligned}|F - (N(u) \cup N^*(v))| &= |F| - |N(u) \cup N^*(v)| \\ &= |F| - |N(u)| - |N^*(v)| + |N(u) \cap N^*(v)| \leq (3k-6) - k - (k-1) + 2 = k - 3\end{aligned}$$

vertices in the set of such endpoints that belong to $F$. Since $2^{k-1} - 2k > k - 3$, there exists an edge whose two endpoints belong to $0Q_{k-1} - F_0 - \{u, v\}$ and $1Q_{k-1} - F_1$, respectively. Hence, $Q_k - F - \{u, v\}$ is connected. If $v' \notin F$, then $Q_k - F - \{u\}$ is connected. That is, assertion (i) holds (Fig. 3(e)). If $v' \in F$, then $N(v) \subseteq F$. That is, assertion (ii) holds (Fig. 3(f)).

*Case 3* $|F_0| = 3k - 7$. Then $|F_1| = 1$. So $F_1 = \{u'\}$. Clearly, $1Q_{k-1} - F_1$ is connected, and each vertex in $0Q_{k-1} - F_0 - \{u\}$ is connected to a vertex in $1Q_{k-1} - F_1$ via an edge. Thus, the graph $Q_k - F - \{u\}$ is connected. That is, assertion (i) holds.

*Case 4* $|F_0| = 3k - 8$. Then $|F_1| \leq 2$. If $|F_1| = 1$, similar to Case 3, we can prove that assertion (i) holds. Now assume $F_1 = \{u', v'\}$ and let $v$ denote the neighbour of $v'$ in $0Q_{k-1}$. Clearly, $1Q_{k-1} - F_1$ is connected, and each vertex in $0Q_{k-1} - F_0 - \{u, v\}$ is connected to a vertex in $1Q_{k-1} - F_1 - \{v'\}$ via an edge. Thus, the graph $Q_k - F - \{u, v\}$ is connected. We now consider three subcases.

*Subcase 4.1* $v \in F$. Then $Q_k - F - \{u\}$ is connected. That is, assertion (i) holds (Fig. 3(g)).

*Subcase 4.2* $v \notin F$ and $N(v) - \{v'\} \not\subseteq F$. Then $Q_k - F - \{u\}$ is connected. That is, assertion (i) holds (Fig. 3(h)).

*Subcase 4.3* $v \notin F$ and $N(v) - \{v'\} \subseteq F$. Then assertion (ii) holds (Fig. 3(f)).
This completes our inductive proof.

LEMMA 3.5 *Let $F$ be a set of at most $3n - 6$ vertices in $Q_n (n \geq 3)$ such that there exists a pair $\{u, v\}$ of adjacent vertices in $Q_n$ satisfying $N(\{u, v\}) \subseteq F$. Then the graph $Q_n - F - \{u, v\}$ is connected.*

*Proof* Observe that

$$|F \cap (N[u] \cup N[v])| \geq |N(\{u, v\})| = 2n - 2,$$

we have

$$|F - (N[u] \cup N[v])| = |F| - |F \cap (N[u] \cup N[v])| \leq (3n-6) - (2n-2) = n - 4.$$

By Lemma 3.2, $Q_n - (N[u] \cup N[v])$ is $(n-2)$-connected. So the graph

$$Q_n - (N[u] \cup N[v]) - (F - (N[u] \cup N[v])) = Q_n - F - \{u, v\}$$

is connected. ■

LEMMA 3.6 *Let $F$ be a set of three distinct vertices of $Q_n (n \geq 5)$ that induces a connected subgraph. Then $|N(F)| = 3n - 5$ and $Q_n - N[F]$ is connected.*

*Proof* Let $F = \{u, v, w\}$. Without loss of generality, we may assume that $(u, v, w)$ forms a path of $Q_n$. Clearly, $u$ and $v$ share no common neighbours (otherwise $Q_n$ would contain cycles of odd length), $v$ and $w$ have no common neighbours, and $w$ and $u$ share exactly two common neighbours. Let $x$ be the common neighbor of $w$ and $u$ other than $v$, then

$$\begin{aligned}|N(F)| &= |(N(u) - \{v\}) \cup (N(v) - \{u, w\}) \cup (N(w) - \{u, x\})| \\ &= |N(u) - \{v\}| + |N(v) - \{u, w\}| + |N(w) - \{u, x\}| \\ &= (n-1) + (n-2) + (n-2) = 3n - 5.\end{aligned}$$

Now, we prove the second assertion by induction on $n$. The assertion is clearly true for $n = 3$. Suppose the assertion is true for $n = k - 1 (k \geq 4)$. Now assume $F = \{u, v, w\}$ is a set of three vertices in $Q_k$ such that $(u, v, w)$ forms a path. In view of the symmetry of hypercube, we may assume that $F \subseteq V(0Q_{k-1})$. Let $F' = \{u', v', w'\}$ denotes the set of the neighbours of $u$, $v$, and $w$ in $1Q_{k-1}$. Let

$$N^*(F) = N(F) - F', \quad N^*[F] = N[F] - F'.$$

By applying the inductive hypothesis to $0Q_{k-1}$ and $N^*(F)$, we derive that the graph $0Q_{k-1} - N^*[F]$ is connected. Next, we prove that $1Q_{k-1} - F'$ is connected. This can be easily verified for the case $k = 4$. When $k \geq 5$, $\kappa(1Q_{k-1}) = k - 1 \geq 4$. So $1Q_{k-1} - F'$ is connected. Now consider a vertex $x$ in $0Q_{k-1} - N^*[F]$. Let $x'$ denote the neighbour of $x$ in $1Q_{k-1}$. Clearly, $x'$ belongs to $1Q_{k-1} - F'$. So the two connected graphs $0Q_{k-1} - N^*[F]$ and $1Q_{k-1} - F'$ are connected to each other via the edge $(x, x')$ (Fig. 4). Hence, the graph $Q_k - N[F]$ is connected. The inductive proof is completed. ■

Now, we are ready to establish the second main result of this article.

THEOREM 3.7 *Let $F$ be a set of at most $3n - 6$ vertices in $Q_n (n \geq 4)$. Then*

$$m(Q_n - F) \geq V(Q_n) - |F| - 2.$$

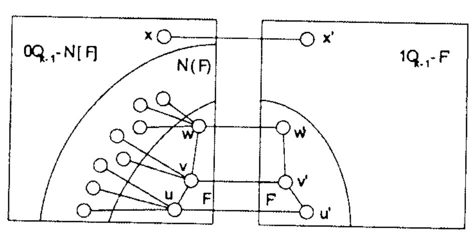



FIGURE 4 The graph $Q_k - N[F]$ is connected.

*Moreover, this inequality is optimal in the sense that there is a set $F$ of $3n-5$ vertices in $Q_n$ such that*

$$m(Q_n - F) = V(Q_n) - |F| - 3.$$

*Proof* If $N(u) \not\subseteq F$, for any vertex $u$ in $Q_n$ and $N(\{u, v\}) \not\subseteq F$, for any two adjacent vertices $u$ and $v$ in $Q_n$, then Theorem 2.3 implies that $Q_n - F$ is connected and, hence,

$$m(Q_n - F) = V(Q_n) - |F|.$$

Let $F_0 = F \cap V(0Q_{n-1})$, $F_1 = F \cap V(1Q_{n-1})$. Two possibilities have yet to be investigated.

*Case 1* There is a node $u$ in $Q_n$ such that $N(u) \subseteq F$. By Lemma 3.4, we have two possible subcases.

*Subcase 1.1* $Q_n - F - \{u\}$ is connected (Fig. 5(a)). Then $Q_n - F - \{u\}$ is the maximal component of $Q_n - F$. Thus,

$$m(Q_n - F) = V(Q_n) - |F| - 1.$$

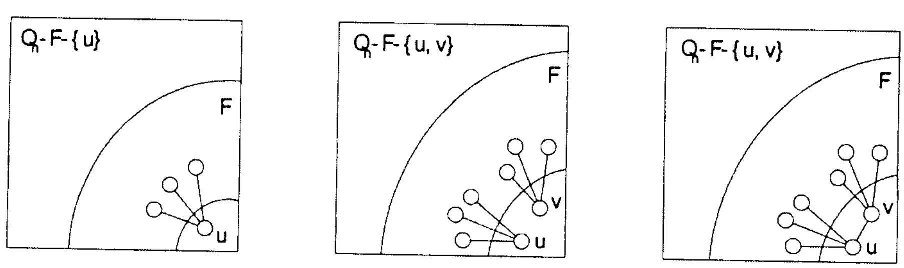
*Subcase 1.2* $Q_n - F - \{u\}$ contains a node $v$ such that $N(v) \subseteq F$ and the graph $Q_n - F - \{u, v\}$ is connected (Fig. 5(b)). Then $Q_n - F - \{u, v\}$ is the maximal component of $Q_n - F$. Thus,

$$m(Q_n - F) = V(Q_n) - |F| - 2.$$

*Case 2* There is a pair $\{u, v\}$ of adjacent nodes in $Q_n$ such that $N(\{u, v\}) \subseteq F$ (Fig. 5(c)). By Lemma 3.5, the graph $Q_n - F - \{u, v\}$ is connected, which is the maximal component of $Q_n - F$. Thus,

$$m(Q_n - F) = V(Q_n) - |F| - 2.$$

Combining the above discussions, we conclude the first desired result. Now, consider a set $F = \{u, v, w\}$ of three distinct vertices in $Q_n$ that induce a connected component. By Lemma 3.6, the graph $Q_n - F$ consists of two connected components, one being induced by $F$ and the other being induced by $V(Q_n) - N[F]$. Thus, $m(Q_n - F) = V(Q_n) - |F| - 3$. ■




(a) Case 1 (b) Case 2 (c) Case 3

FIGURE 5 Schematic illustrations of the proof of Theorem 3.7.

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