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Information Processing Letters 92 (2004) 31–37
Information Processing Letters
www.elsevier.com/locate/ipl

Generalized honeycomb torus is Hamiltonian ☆

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Received 18 September 2003; received in revised form 10 May 2004

Communicated by F.Y.L. Chin

Abstract

Generalized honeycomb torus is a candidate for interconnection network architectures, which includes honeycomb torus, honeycomb rectangular torus, and honeycomb parallelogramic torus as special cases. Existence of Hamiltonian cycle is a basic requirement for interconnection networks since it helps map a "token ring" parallel algorithm onto the associated network in an efficient way. Cho and Hsu [Inform. Process. Lett. 86 (4) (2003) 185–190] speculated that every generalized honeycomb torus is Hamiltonian. In this paper, we have proved this conjecture.
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Keywords: Interconnection networks; Generalized honeycomb torus; Hamiltonian cycle

1. Introduction

The effectiveness of the interconnection network in a parallel computing system is a crucial factor of performance of the system [7]. Stojmenovic [9] proposed three classes of *honeycomb torus* architectures: *honeycomb hexagonal torus*, *honeycomb rectangular torus*, and *honeycomb parallelogramic torus*. Due to lower node degree and lower implementation cost than those of a standard torus of the same size, these architectures have incurred great research interest [1,2,5,6,8–11]. Cho and Hsu [3] found that all these honeycomb torus networks can be characterized in a unified way, and thereby proposed a class of interconnection networks known as the *generalized honeycomb torus*.

Existence of a Hamiltonian cycle is one of the basic requirements for interconnection networks since it helps map a "token ring" parallel algorithm onto the associated network in an efficient way [7]. The Hamiltonicity of honeycomb torus networks has been extensively studied. Megson et al. [5,6] proved that hon-

☆ This research was partly supported by the Visiting Scholar's Funds of National Education Ministry's Key Laboratory of Electro-Optical Technique and System, Chongqing University.

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doi:10.1016/j.ipl.2004.05.017

eycomb hexagonal torus is Hamiltonian, even in the presence of node failures. Cho and Hsu [2] discovered a Hamiltonian cycle for faulty honeycomb rectangular torus. As for generalized honeycomb torus, Cho and Hsu [3], and Yang and Megson [12] proved the existence of Hamiltonian cycles for some special cases. Furthermore, Cho and Hsu [3] speculated that every generalized honeycomb torus is Hamiltonian.

This paper aims at proving this conjecture by constructing a Hamiltonian cycle for each generalized honeycomb torus.

2. Preliminaries

Henceforth we say *graph* instead of the interconnection network modeled by the graph. For fundamental graph-theoretical terminology the reader is referred to [4].

Definition 1 [3]. Let n be a positive even integer, m be a positive integer, and d be a nonnegative integer that is less than n and is of the same parity as m. An (m,n,d) *generalized honeycomb torus*, denoted by $\mathrm{GHT}(m,n,d)$, is a graph with vertex set

$$V\big(\mathrm{GHT}(m,n,d)\big)=\big\{\langle i,j\rangle\colon\ i\in\{0,1,\ldots,m-1\},\ j\in\{0,1,\ldots,n-1\}\big\}.$$

We call m,n, and d as the *width*, *height*, and *slope* of $\mathrm{GHT}(m,n,d)$, respectively. For a vertex $\langle i,j\rangle$ of $\mathrm{GHT}(m,n,d)$, i and j are called as its *first* and *second components*, respectively. Here and in what follows, all arithmetic operations carried out on the first and second components are modulo m and n, respectively. Two vertices $\langle i,j\rangle$ and $\langle k,l\rangle$ with $i\leqslant k$ are adjacent if and only if one of the following three conditions is satisfied:

(a) $\langle k,l\rangle=\langle i,j+1\rangle$ or $\langle k,l\rangle=\langle i,j-1\rangle$;
(b) $0\leqslant i\leqslant m-2$, $i+j$ is odd, and $\langle k,l\rangle=\langle i+1,j\rangle$;
(c) $i=0$, j is even, and $\langle k,l\rangle=\langle m-1,j+d\rangle$.

Clearly, every generalized honeycomb torus is a 3-regular bipartite graph. See Fig. 1 for two examples of generalized honeycomb torus. It is known [3] that generalized honeycomb torus includes honeycomb torus, honeycomb rectangular torus, and honeycomb parallelogramic torus as special cases. For convenience, let us introduce the following notations.

Definition 2. A *path decomposition of graph G* is a set of disjoint paths $P_1,P_2,\ldots,P_k$ in G satisfying $\bigcup_{i=1}^{k}V(P_i)=V(G)$, where $V(P_i)$ denotes the set of vertices on P_i.

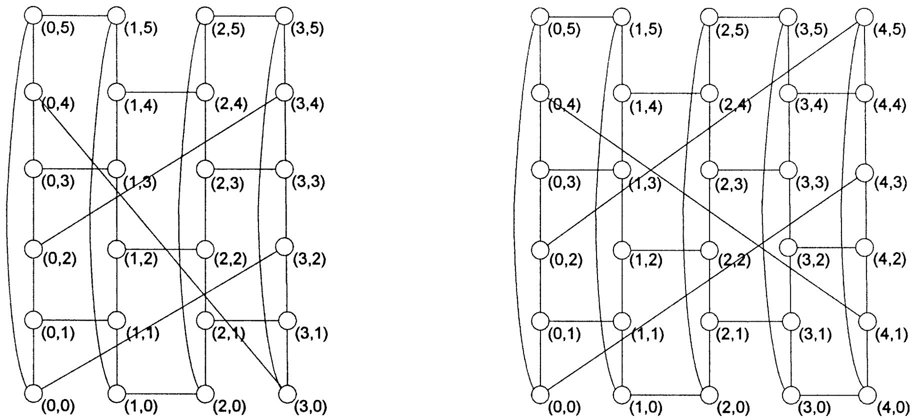
Given two vertices (i,j) and (i,k) of $\mathrm{GHT}(m,n,d)$, the path

$$\langle i,j\rangle\to\langle i,j+1\rangle\to\langle i,j+2\rangle\to\cdots\to\langle i,k\rangle$$

is denoted as $\langle i,j\rangle\uparrow\langle i,k\rangle$ and the path

$$\langle i,j\rangle\to\langle i,j-1\rangle\to\langle i,j-2\rangle\to\cdots\to\langle i,k\rangle$$

is denoted as $\langle i,j\rangle\downarrow\langle i,k\rangle$.



(a) (b)

Fig. 1. Two examples of generalized honeycomb torus. (a) GHT(4, 6, 2). (b) GHT(5, 6, 3).

Given two positive integers p and q, let $\gcd(p,q)$ denote the greatest common divisor of p and q, and let $p \mid q$ denotes that q is divisible by p.

Lemma 1. *Let p and q be two positive integers. Let $g(p,q)$ denote the smallest positive integer s satisfying $p \times s = 0 \pmod q$. Then $g(p,q) = \frac{q}{\gcd(p,q)}$.*

Proof. Since

$$p \times \frac{q}{\gcd(p,q)} = \frac{p}{\gcd(p,q)} \times q \equiv 0 \pmod q,$$

we derive $g(p,q) \leqslant \frac{q}{\gcd(p,q)}$. On the other hand, let s be an integer with $1 \leqslant s \leqslant \frac{q}{\gcd(p,q)} - 1$. If $p \times s = 0 \pmod q$, there would be a positive integer r such that $p \times s = r \times q$. Dividing both sides of this equality by $\gcd(p,q)$, we could obtain $\frac{p}{\gcd(p,q)} \times s = r \times \frac{q}{\gcd(p,q)}$. Since $\gcd(\frac{p}{\gcd(p,q)}, \frac{q}{\gcd(p,q)}) = 1$, we would derive $\frac{q}{\gcd(p,q)} \mid s$. This would contradict the assumption that $1 \leqslant s \leqslant \frac{q}{\gcd(p,q)} - 1$. Hence, $g(p,q) \geqslant \frac{q}{\gcd(p,q)}$. □

Given two positive integers a and b, we need to consider a graph $G(a,b)$ that has $\{0,1,\ldots,a-1\}$ as the vertex set and $\{\langle i, i+b\rangle\colon 0 \leqslant i \leqslant a-1\}$ as the edge set, where the arithmetic is modulo a.

Lemma 2. *If $\gcd(a,b) = 1$, then $G(a,b)$ is a cycle (loop and multiple edges inclusive).*

Proof. Consider the infinite sequence $(0, b, 2b, 3b, \ldots)$ of neighboring vertices. It follows from Lemma 1 that $(0, b, 2b, 3b, \ldots, (a-1)b, 0)$ forms a cycle. □

3. Hamiltonicity of generalized honeycomb torus

When $d = 0$, $\mathrm{GHT}(m,n,d)$ is a honeycomb rectangular torus, which is clearly a Hamiltonian graph. Henceforth, we investigate the Hamiltonicity of generalized honeycomb torus for the case $d > 0$ by distinguishing two possibilities.

3.1. The width is even

We first investigate the Hamiltonicity of $\mathrm{GHT}(m, n, d)$ in case m is even. Let k be an even integer satisfying $0 \leqslant k \leqslant n-1$. Let h be an even integer satisfying $1 \leqslant h \leqslant n$. Then $\mathrm{GHT}(m,n,d)$ contains a path starting from vertex $\langle 0,k\rangle$ and terminating at vertex $\langle m-1,k\rangle$, which is shown below:

$$\begin{aligned}
&\langle 0,k\rangle \uparrow \langle 0,k+h-1\rangle \to \langle 1,k+h-1\rangle \downarrow \langle 1,k\rangle \\
&\quad\to \langle 2,k\rangle \uparrow \langle 2,k+h-1\rangle \to \langle 3,k+h-1\rangle \downarrow \langle 3,k\rangle \\
&\quad\to \langle 4,k\rangle \uparrow \langle 4,k+h-1\rangle \to \langle 5,k+h-1\rangle \downarrow \langle 5,k\rangle \\
&\quad\to \cdots \\
&\quad\to \langle m-2,k\rangle \uparrow \langle m-2,k+h-1\rangle \\
&\quad\to \langle m-1,k+h-1\rangle \downarrow \langle m-1,k\rangle.
\end{aligned}$$

We denote this path by $P(k,h)$. Obviously, we have

Lemma 3. *If m is even, then set $\{P(k \times \gcd(n,d), \gcd(n,d))\colon 0 \leqslant k \leqslant \frac{n}{\gcd(n,d)} - 1\}$ constitutes a path decomposition of $\mathrm{GHT}(m,n,d)$. (We call this path decomposition as* standard path decomposition.)

Figs. 2(a) and 3(a) gives two instances of standard path decomposition of generalized honeycomb torus.

Theorem 4. *If m is even. Then $\mathrm{GHT}(m,n,d)$ is Hamiltonian.*

Proof. We construct a graph $G(\frac{n}{\gcd(n,d)}, \frac{d}{\gcd(n,d)})$ in the way given in Lemma 2. By Lemma 2, the sequence of neighboring vertices

$$\left(0, \frac{d}{\gcd(n,d)}, 2 \times \frac{d}{\gcd(n,d)}, \ldots, \left(\frac{n}{\gcd(n,d)} - 1\right) \times \frac{d}{\gcd(n,d)}, 0\right)$$

forms a cycle. This cycle can be extended to a Hamiltonian cycle of $\mathrm{GHT}(m,n,d)$ according to the following steps:

Step 1: For each i with $0 \leqslant i \leqslant \frac{n}{\gcd(n,d)} - 1$, let

$$V(i) = \bigl\{\langle p,q\rangle\colon 0 \leqslant p \leqslant m-1, i \times \gcd(n,d) \leqslant q \leqslant (i+1) \times \gcd(n,d) - 1\bigr\}.$$

Step 2: Replace each vertex i of $G(\frac{n}{\gcd(n,d)}, \frac{d}{\gcd(n,d)})$ with $V(i)$.

Step 3: Replace each edge $(i, i+d)$ of $G(\frac{n}{\gcd(n,d)}, \frac{d}{\gcd(n,d)})$ with a path of $\mathrm{GHT}(m,n,d)$ obtained from

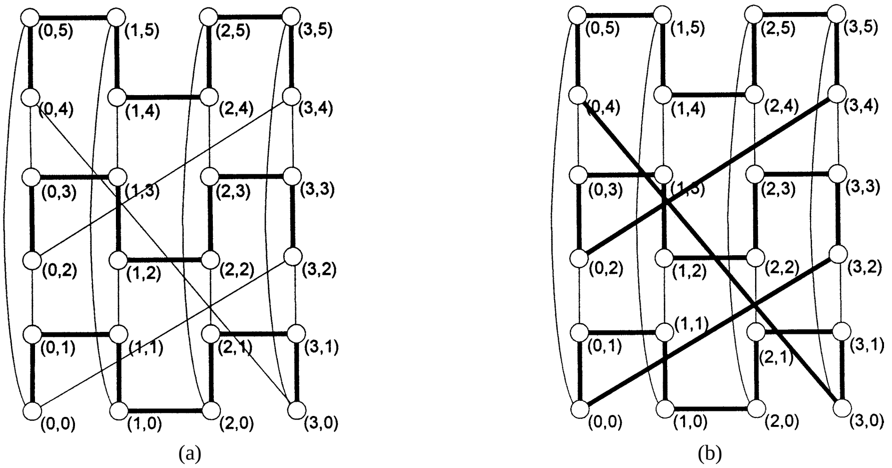


Fig. 2. Standard path decomposition and standard Hamiltonian cycle of $\mathrm{GHT}(4, 6, 2)$. (Notice that $\gcd(n, d) = \gcd(6, 2) = 2$.) (a) $\{P(0, 2), P(2, 2), P(4, 2)\}$. (b) Standard Hamiltonian cycle.

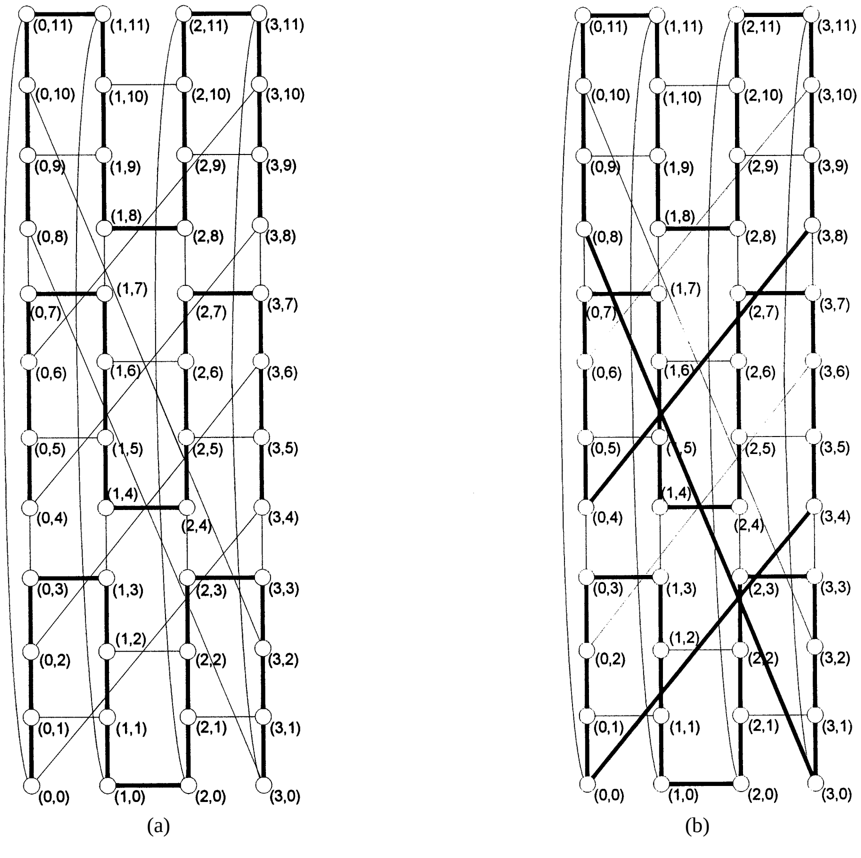


Fig. 3. Standard path decomposition and standard Hamiltonian cycle of $\mathrm{GHT}(4, 12, 4)$ (Notice that $\gcd(n, d) = \gcd(12, 4) = 4$.) (a) $\{P(0, 4), P(4, 4), P(8, 4)\}$. (b) Standard Hamiltonian cycle.

path $P(i \times \gcd(n,d), \gcd(n,d))$ by adding the following edge:

$$\bigl(\langle 0, i \times \gcd(n,d)\rangle, \langle m-1, i \times \gcd(n,d)+d\rangle\bigr). \qquad \square$$

We call a Hamiltonian cycle thus constructed as a *standard Hamiltonian cycle*. Figs. 2(b) and 3(b) give two instances of standard Hamiltonian cycle.

3.2. The width is odd

Next, we investigate the Hamiltonicity of GHT(m, n, d) in case m is odd. Let k be an even integer satisfying $0 \leqslant k \leqslant n-1$. Let h be an even integer satisfying $1 \leqslant h \leqslant n-1$. Then GHT$(m,n,d)$ contains a path starting from vertex $\langle 0,k\rangle$ and terminating at vertex $\langle m-1, k+h-1\rangle$:

$$\begin{aligned}&\langle 0,k\rangle \uparrow \langle 0,k+h-1\rangle \to \langle 1,k+h-1\rangle \downarrow \langle 1,k\rangle\\ &\quad\to \langle 2,k\rangle \uparrow \langle 2,k+h-1\rangle \to \langle 3,k+h-1\rangle \downarrow \langle 3,k\rangle\\ &\quad\to \langle 4,k\rangle \uparrow \langle 4,k+h-1\rangle \to \langle 5,k+h-1\rangle \downarrow \langle 5,k\rangle\\ &\quad\to \cdots \to \langle m-1,k\rangle \uparrow \langle m-1,k+h-1\rangle.\end{aligned}$$

We denote this path by $P(k,h)$. Obviously, we have

Lemma 5. *If m is odd, then the set*

$$\Bigl\{P\bigl(k \times \gcd(n,d+1), 2\bigr):\ 0 \leqslant k \leqslant \frac{n}{\gcd(n,d+1)} - 1\Bigr\}$$
$$\cup \Bigl\{P\bigl(k \times \gcd(n,d+1)+2, \gcd(n,d+1)-2\bigr):\ 0 \leqslant k \leqslant \frac{n}{\gcd(n,d+1)} - 1\Bigr\}$$

of paths constitutes a path decomposition of GHT(m,n,d). (*We call this path decomposition as* standard path decomposition.)

Figs. 4(a) and 5(a) gives two instances of standard path decomposition of generalized honeycomb torus.

Theorem 6. *If m is odd, then* GHT(m,n,d) *is Hamiltonian.*

Proof. We construct a graph $G(\frac{n}{\gcd(n,d+1)}, \frac{d+1}{\gcd(n,d+1)})$ in the way given in Lemma 2. By Lemma 2, the sequence

$$\Bigl(0, \frac{d+1}{\gcd(n,d+1)}, 2 \times \frac{d+1}{\gcd(n,d+1)}, \ldots, \Bigl(\frac{n}{\gcd(n,d+1)} - 1\Bigr) \times \frac{d+1}{\gcd(n,d+1)}, 0\Bigr)$$

forms a cycle. This cycle can be extended to a Hamiltonian cycle of GHT(m,n,d) according to the following steps:

Step 1: For each i with $0 \leqslant i \leqslant \frac{n}{\gcd(n,d+1)} - 1$, let

$$\begin{aligned}V(i) = \bigl\{\langle p,q\rangle:\ &0 \leqslant p \leqslant m-1,\\ &i \times \gcd(n,d+1) \leqslant q \leqslant (i+1)\\ &\times \gcd(n,d+1) - 1\bigr\}.\end{aligned}$$

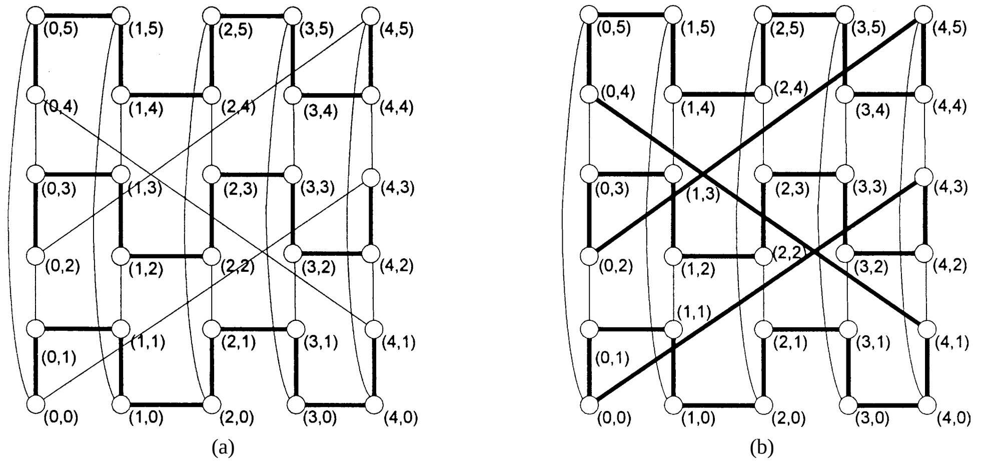


Fig. 4. Standard path decomposition and standard Hamiltonian cycle in GHT(5, 6, 3). (Notice that $\gcd(n, d+1) = \gcd(6,4) = 2$.) (a) $\{P(0,2), P(2,2), P(4,2)\}$. (b) Standard Hamiltonian cycle.

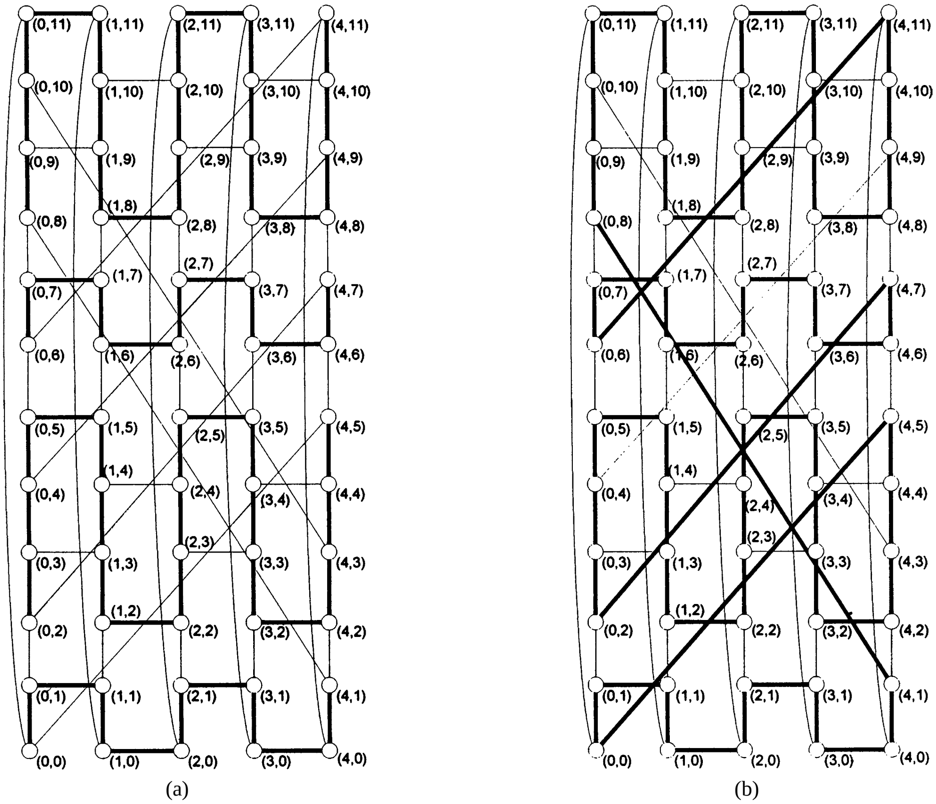


Fig. 5. Standard path decomposition and standard Hamiltonian cycle in GHT$(5, 12, 5)$. (Notice that $\gcd(n, d+1) = \gcd(12, 6) = 6$.) (a) $\{P(0, 2), P(2, 4), P(6, 2), P(8, 4)\}$. (b) Standard Hamiltonian cycle.

Step 2: Replace each vertex i of $G(\frac{n}{\gcd(n,d+1)}, \frac{d}{\gcd(n,d+1)})$ with $V(i)$.

Step 3: Replace each edge $(i, i+d)$ of $G(\frac{n}{\gcd(n,d+1)}, \frac{d}{\gcd(n,d+1)})$ with a path of GHT(m, n, d) obtained from the following two paths:

$$P\big(i \times \gcd(n, d+1), 2\big) \quad \text{and}$$
$$P\big(i \times \gcd(n, d+1) + 2, \gcd(n, d+1) - 2\big),$$

by adding the following two edges

$$\big(\langle 0, i \times \gcd(n, d+1)\rangle,$$
$$\langle m-1, i \times \gcd(n, d+1) + d\rangle\big) \quad \text{and}$$
$$\big(\langle 0, i \times \gcd(n, d+1) + 2\rangle,$$
$$\langle m-1, i \times \gcd(n, d+1) + d + 2\rangle\big). \qquad \square$$

We call a Hamiltonian cycle thus constructed as a *standard Hamiltonian cycle*. Figs. 4(b) and 5(b) gives two instances of standard Hamiltonian cycle.

Combining Theorems 4 and 6, we obtain

Corollary 7. *Every generalized honeycomb torus is Hamiltonian.*

Acknowledgements

The authors wish to express their gratitude to the two anonymous referees for their valuable suggestions that greatly improve the quality of the paper.

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