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Global asymptotic stability in a rational recursive sequence ☆

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Abstract

In this paper, we study the global stability of the difference equation

$$x_n = \frac{a + bx_{n-1} + cx_{n-1}^2}{d - x_{n-2}}, \quad n = 1, 2, \ldots,$$

where a, $b \geqslant 0$ and c, $d > 0$. We show that one nonnegative equilibrium point of the equation is a global attractor with a basin that is determined by the parameters, and every positive solution of the equation in the basin exponentially converges to the attractor.

Keywords: Difference equation; Recursive sequence; Equilibrium; Global attractor; Basin; Exponential convergence

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1. Introduction

Kocic et al. [1] examined the periodicity and oscillating properties of the positive solutions as well as the global attractivity of the nonnegative equilibrium of the difference equation

$$x_n = \frac{a + bx_{n-1}}{d + x_{n-k}}, \quad n = 1, 2, \ldots, \tag{1.1}$$

where $a, b, d \geqslant 0$, $a + b > 0$, and $k \in \{2, 3, \ldots\}$. There are yet three other types of recursive sequences that are formally similar to sequence (1.1), which are listed below:

$$x_n = \frac{a - bx_{n-1}}{d + x_{n-k}}, \quad n = 1, 2, \ldots, \tag{1.2}$$

$$x_n = \frac{a - bx_{n-1}}{d - x_{n-k}}, \quad n = 1, 2, \ldots, \tag{1.3}$$

$$x_n = \frac{a + bx_{n-1}}{d - x_{n-k}}, \quad n = 1, 2, \ldots, \tag{1.4}$$

where $a, b, d \geqslant 0$, $a + b > 0$, and $k \in \{2, 3, \ldots\}$. Aboutaleb et al. [2] studied the global asymptotic stability of Eq. (1.2) with $k = 2$, and Li and Sun [3] extended the results to Eq. (1.2) with $k \geqslant 2$. Yan and Li [4] investigated the global attractivity of Eq. (1.3) with $k = 2$, and Yan et al. [5] extended the results to Eq. (1.3) with $k \geqslant 2$. Yan and Li [6] examined the global asymptotic behavior of Eq. (1.4) with $k = 2$. Sequences (1.1)–(1.4) have the common feature that the numerator and the denominator in the fraction are both linear in x_n. For more recursive sequences with this feature, the reader is referred to [7–16].

Some rational recursive sequences were also investigated in which the numerator or/and the denominator is quadratic in x_n. For instances, Li [17] found some sufficient conditions for the global attractivity of the positive equilibrium point of the difference equation

$$x_n = \frac{a + cx_{n-1}^2}{d + x_{n-k}^2}, \quad n = 1, 2, \ldots. \tag{1.5}$$

Zhang et al. [18] investigated the global stability of the sequence (1.5) with $k = 2$ and $d = 1$.

In this paper, we study the global asymptotic behavior of the following recursive sequence:

$$x_n = \frac{a + bx_{n-1} + cx_{n-1}^2}{d - x_{n-2}}, \quad n = 1, 2, \ldots, \tag{1.6}$$

where $a, b \geqslant 0$ and $c, d > 0$. Eq. (1.6) is similar to Eq. (1.4) with $k = 2$, with the only difference that the former contains a quadratic term in the numerator of the fraction while the numerator of the latter is linear. Also Eq. (1.6) is similar to (1.5) in that they both contain x_n^2 term in the numerator.

2. Preliminaries

Let I be a real interval and let $f\colon I \times I \to I$ be a continuous function. For every initial condition $\langle x_{-1}, x_0 \rangle \in I \times I$, the difference equation

$$x_n = f(x_{n-1}, x_{n-2}), \quad n = 1, 2, \ldots, \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$, which is called a *recursive sequence*. An *equilibrium point* of Eq. (2.1) is a point $\alpha \in I$ with $f(\alpha, \alpha) = \alpha$.

Definition 2.1. Let α be an equilibrium point of Eq. (2.1).

(i) α is *locally stable* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $\langle x_{-1}, x_0 \rangle \in I \times I$ with $|x_{-1} - \alpha| + |x_0 - \alpha| < \delta$, $|x_n - \alpha| < \varepsilon$ holds for $n = 1, 2, \ldots$.
(ii) α is a *local attractor* if there exists $\gamma > 0$ such that for each $\langle x_{-1}, x_0 \rangle \in I \times I$ with $|x_{-1} - \alpha| + |x_0 - \alpha| < \gamma$, $x_n \to \alpha$ holds.
(iii) α is *locally asymptotically stable* if it is locally stable and is a local attractor.
(iv) α is a *global attractor* if for each $\langle x_{-1}, x_0 \rangle \in I \times I$, $x_n \to \alpha$ holds.
(v) α is *globally asymptotically stable* if it is locally stable and is a global attractor.
(vi) α is a *repeller* if there exists $\gamma > 0$ such that for each $\langle x_{-1}, x_0 \rangle \in I \times I$ with $|x_{-1} - \alpha| + |x_0 - \alpha| < \gamma$, there exists N such that $|x_N - \alpha| \geqslant \gamma$.
(vii) α is a *saddle point* if it is neither a local attractor nor a repeller.

Assume α is an equilibrium point of Eq. (2.1). Let $p = -\frac{\partial f(\alpha,\alpha)}{\partial x_{n-1}}$ and $q = -\frac{\partial f(\alpha,\alpha)}{\partial x_{n-2}}$. Then the linearized equation associated with Eq. (2.1) about the equilibrium α is

$$y_n + p y_{n-1} + q y_{n-2} = 0. \tag{2.2}$$

The characteristic equation is

$$\lambda^2 + p\lambda + q = 0. \tag{2.3}$$

Theorem 2.1 (Linearized stability theorem [19])
(i) *If* $|p| < 1 + q$ *and* $q < 1$, *and, then* α *is locally asymptotically stable.*
(ii) *If* $|q| > 1$ *and* $|p| < |1 + q|$, *then* α *is a repeller.*
(iii) *If* $p^2 > 4q$ *and* $|p| > |1 + q|$, *then* α *is a saddle point.*

Definition 2.2. Let $\{x_n\}$ be a sequence of real numbers, and α be a real number.

(i) $\{x_n\}$ *oscillates about* α if for every positive integer N, there exist $n, m > N$ such that $x_n \geqslant \alpha$ and $x_m \leqslant \alpha$.
(ii) $\{x_n\}$ *strictly oscillates about* α if for every positive integer N, there exist $n, m > N$ such that $x_n > \alpha$ and $x_m < \alpha$.
(iii) A *negative semicycle* of $\{x_n\}$ about α is a string of consecutively negative terms of the sequence $\{x_n - \alpha\}$ preceded by a nonnegative term and followed by a nonnegative term. A *positive semicycle* is a string of consecutively nonnegative terms of the sequence $\{x_n - \alpha\}$ preceded by a negative term and followed by a negative term.

For other basic terminologies and results of difference equations the reader is referred to [19].

3. Equilibria and local asymptotic stability

Consider the difference equation

$$x_n = f(x_{n-1}, x_{n-2}) = \frac{a + bx_{n-1} + cx_{n-1}^2}{d - x_{n-2}}, \quad n = 1, 2, \ldots, \tag{3.1}$$

where

$$a, b \geqslant 0, \quad c, d > 0. \tag{3.2}$$

The equilibrium points of this equation are the solutions of the quadratic equation

$$(1 + c)x^2 - (d - b)x + a = 0. \tag{3.3}$$

Suppose

$$d > b, (d - b)^2 > 4a(1 + c). \tag{3.4}$$

Then Eq. (3.1) has one nonnegative equilibrium α and one positive equilibrium $\beta > \alpha$, where

$$\alpha = \frac{(d - b) - \sqrt{(d - b)^2 - 4a(1 + c)}}{2(1 + c)}, \tag{3.5}$$

$$\beta = \frac{(d - b) + \sqrt{(d - b)^2 - 4a(1 + c)}}{2(1 + c)}. \tag{3.6}$$

Since

$$d-\alpha > d-\beta = \frac{d+b+2cd-\sqrt{(d-b)^2-4a(1+c)}}{2(1+c)}$$
$$\geqslant \frac{d+b+2cd-(d-b)}{2(1+c)} = \frac{b+cd}{1+c} > 0,$$

we have

$$0 \leqslant \alpha < \beta < d. \tag{3.7}$$

Theorem 3.1. *Assume* (3.2) *and* (3.4) *hold. Then*

(i) α *is locally asymptotically stable*; *and*
(ii) *if* $c \geqslant 1$ *or if* $c > \frac{d-3b}{3d-b}$, *then* β *is a saddle point of Eq.* (3.1).

Proof. The linearized equation associated with Eq. (3.1) about the equilibrium α is

$$y_n - \frac{b+2c\alpha}{d-\alpha} y_{n-1} - \frac{\alpha}{d-\alpha} y_{n-2} = 0.$$

Let $p = -\frac{b+2c\alpha}{d-\alpha}$ and $q = -\frac{\alpha}{d-\alpha}$. Then

$$p+q+1 = -\frac{b+2c\alpha}{d-\alpha} - \frac{\alpha}{d-\alpha} + 1 = \frac{\sqrt{(d-b)^2-4a(1+c)}}{d-\alpha} > 0,$$
$$-p+q+1 \geqslant p+q+1 > 0, \text{and } q \leqslant 0 < 1.$$

It follows from Theorem 2.1(i) that α is locally asymptotically stable.

Similarly, the linearized equation associated with Eq. (3.1) about the equilibrium β is

$$y_n - \frac{b+2c\beta}{d-\beta} y_{n-1} - \frac{\beta}{d-\beta} y_{n-2} = 0.$$

Let $p = -\frac{b+2c\beta}{d-\beta}$ and $q = -\frac{\beta}{d-\beta}$. Then

$$p^2 - 4q = \left(\frac{b+2c\beta}{d-\beta}\right)^2 + \frac{\beta}{d-\beta} > 0,$$

$$|p| - q - 1 = \frac{b+2c\beta}{d-\beta} + \frac{\beta}{d-\beta} - 1 = \frac{\sqrt{(d-b)^2-4a(1+c)}}{d-\beta} > 0,$$

$$|p| + q + 1 = \frac{b + 2c\beta}{d - \beta} - \frac{\beta}{d - \beta} + 1$$
$$= \frac{2(b + cd) + (c - 1)\sqrt{(d - b)^2 - 4a(1 + c)}}{(d - \beta)(1 + c)}.$$

If $c \geqslant 1$, then $|p| + q + 1 > 0$. If $c < 1$ and $c > \frac{d-3b}{3d-b}$, then

$$|p| + q + 1 \geqslant \frac{2(b + cd) + (c - 1)(d - b)}{(d - \beta)(1 + c)} = \frac{(3d - b)c + (3b - d)}{(d - \beta)(1 + c)} > 0.$$

It follows from Theorem 2.1(iii) that β is a saddle point. □

4. Global asymptotic stability

In this section, we deal with the global attractivity of α. To this end, we need to make an estimation on the gap between x_n and α.

$$\begin{aligned}
x_n - \alpha &= f(x_{n-1}, x_{n-2}) - f(\alpha, \alpha) \\
&= [f(x_{n-1}, x_{n-2}) - f(\alpha, x_{n-2})] + [f(\alpha, x_{n-2}) - f(\alpha, \alpha)] \\
&= \left[\frac{a + bx_{n-1} + cx_{n-1}^2}{d - x_{n-2}} - \frac{a + b\alpha + c\alpha^2}{d - x_{n-2}}\right] \\
&\quad + \left[\frac{a + b\alpha + c\alpha^2}{d - x_{n-2}} - \frac{a + b\alpha + c\alpha^2}{d - \alpha}\right] \\
&= \frac{b + c(x_{n-1} + \alpha)}{d - x_{n-2}}(x_{n-1} - \alpha) + \frac{a + b\alpha + c\alpha^2}{(d - x_{n-2})(d - \alpha)}(x_{n-2} - \alpha).
\end{aligned}$$

In view of $a + b\alpha + c\alpha^2 = \alpha(d - \alpha)$, we derive

$$x_n - \alpha = \frac{b + c\alpha + cx_{n-1}}{d - x_{n-2}}(x_{n-1} - \alpha) + \frac{\alpha}{d - x_{n-2}}(x_{n-2} - \alpha). \tag{4.1}$$

The following lemma follows from Eq. (4.1).

Lemma 4.1. *Assume* (3.2) *and* (3.4) *hold and* $\langle x_{n-2}, x_{n-1}\rangle \in [0, \beta) \times [0, \beta)$.

(i) *If* $x_{n-1} \geqslant \alpha$ *and* $x_{n-2} \geqslant \alpha$, *then*

$$0 \leqslant x_n - \alpha \leqslant \frac{b + (1 + c)\alpha + cx_{n-1}}{d - x_{n-2}} \times \max\{x_{n-1} - \alpha, x_{n-2} - \alpha\}. \tag{4.2}$$

(ii) *If* $x_{n-1} \leqslant \alpha$ *and* $x_{n-2} \leqslant \alpha$, *then*

$$0 \leqslant \alpha - x_n \leqslant \frac{b + (1 + c)\alpha + cx_{n-1}}{d - x_{n-2}} \times \max\{\alpha - x_{n-1}, \alpha - x_{n-2}\}. \tag{4.3}$$

(iii) *If* $(x_{n-1}-\alpha)(x_{n-2}-\alpha)<0$, *then*

$$|x_n-\alpha|\leqslant \max\left\{\frac{b+c\alpha+cx_{n-1}}{d-x_{n-2}}\times|x_{n-1}-\alpha|,\frac{\alpha}{d-x_{n-2}}\times|x_{n-2}-\alpha|\right\} \quad (4.4)$$

and hence

$$|x_n-\alpha|\leqslant \max\left\{\frac{b+c\alpha+cx_{n-1}}{d-x_{n-2}},\frac{\alpha}{d-x_{n-2}}\right\}\times\max\{|x_{n-1}-\alpha|,|x_{n-2}-\alpha|\}. \quad (4.5)$$

Lemma 4.2. *Assume* (3.2) *and* (3.4) *hold and* $\langle x_{-1},x_0\rangle\in[0,\beta)\times[0,\beta)$, *then*

$$x_n\in[0,\beta)\quad \textit{for } n=1,0,1,\ldots.$$

Proof. By induction on n. The assertion is true for $n=1$, 0. Suppose for some integer $n\geqslant 0$, x_{n-1} and $x_{n-2}\in[0,\beta)$. Note that $a+bx+cx^2\geqslant 0$ for $x\geqslant 0$, we have

$$0\leqslant x_n=\frac{a+bx_{n-1}+cx_{n-1}^2}{d-x_{n-2}}<\frac{a+bx_{n-1}+cx_{n-1}^2}{d-\beta}<\frac{a+b\beta+c\beta^2}{d-\beta}=\beta.$$

This completes our inductive proof. □

Lemma 4.3. *Assume* (3.2) *and* (3.4) *hold*

(i) *If* $\langle x_{n-2},x_{n-1}\rangle\in[\alpha,\beta)\times[\alpha,\beta)$ *and* $\langle x_{n-2},x_{n-1}\rangle\neq\langle\alpha,\alpha\rangle$, *then*

$$\alpha\leqslant x_n<\max\{x_{n-1},x_{n-2}\},\quad \textit{for } n=1,2,\ldots.$$

(ii) *If* $\langle x_{n-2},x_{n-1}\rangle\in[0,\alpha]\times[0,\alpha]$ *and* $\langle x_{n-2},x_{n-1}\rangle\neq\langle\alpha,\alpha\rangle$, *then*

$$\max\{x_{n-1},x_{n-2}\}<x_n\leqslant\alpha,\quad \textit{for } n=1,2,\ldots.$$

Proof

(i) By Lemma 4.1(i), we have

$$0\leqslant x_n-\alpha\leqslant\frac{b+(1+c)\alpha+cx_{n-1}}{d-x_{n-2}}\times\max\{x_{n-1},x_{n-2}\}.$$

Clearly, $(1+c)(\alpha+\beta)=d-b$, or, equivalently, $\frac{b+(1+c)\alpha+c\beta}{d-\beta}=1$. By Lemma 4.2(i), $x_n\in[0,\beta)$ for $n=-1,0,1,\ldots$. So

$$0\leqslant\frac{b+(1+c)\alpha+cx_{n-1}}{d-x_{n-2}}<\frac{b(1+c)\alpha+c\beta}{d-\beta}=1.$$

Thus, $0 \leqslant x_n - \alpha < \max\{(x_{n-1}, x_{n-2}\}$. This implies the desired result.
(ii) The proof is similar to that of (i). □

Lemma 4.4. *Assume* (3.2) *and* (3.4) *hold and* $\langle x_{-1}, x_0\rangle \in [0,\beta) \times [0,\beta)$.

(i) *If* $\langle x_{-1}, x_0\rangle \in [\alpha,\beta) \times [\alpha,\beta)$ *and* $\langle x_{-1}, x_0\rangle \neq \langle \alpha,\alpha\rangle$, *then*

$$\alpha \leqslant x_n < \max\{x_0, x_{-1}\} \quad \textit{for } n = 1, 2, \ldots.$$

(ii) *If* $\langle x_{-1}, x_0\rangle \in [0,\alpha] \times [0,\alpha]$ *and* $\langle x_{-1}, x_0\rangle \neq \langle \alpha,\alpha\rangle$, *then*

$$\max\{x_0, x_{-1}\} < x_n \leqslant \alpha \quad \textit{for } n = 1, 2, \ldots.$$

Proof. By induction on n and using Lemma 4.3. □

Lemma 4.5. *Assume* (3.2) *and* (3.4) *hold. Let*

$$L = \frac{b + (1+c)\alpha + c\max\{x_{-1}, x_0\}}{d - \max\{x_{-1}, x_0\}}. \tag{4.6}$$

(i) *If* $\langle x_{-1}, x_0\rangle \in [\alpha,\beta) \times [\alpha,\beta)$, *then* $0 \leqslant L < 1$ *and*

$$0 \leqslant x_n - \alpha \leqslant L^{\lceil n/2\rceil} \times \max\{x_0 - \alpha, x_{-1} - \alpha\} \quad \textit{for } n = 1, 2, \ldots. \tag{4.7}$$

(ii) *If* $\langle x_{-1}, x_0\rangle \in [0,\alpha] \times [0,\alpha]$, *then* $0 \leqslant L < 1$ *and*

$$0 \leqslant \alpha - x_n \leqslant L^{\lceil n/2\rceil} \times \max\{x_0 - \alpha, x_{-1} - \alpha\} \quad \textit{for } n = 1, 2, \ldots. \tag{4.8}$$

Proof

$$0 \leqslant L = \frac{b + (1+c)\alpha + c\max\{x_{-1}, x_0\}}{d - \max\{x_{-1}, x_0\}} < \frac{b + (1+c)\alpha + c\beta}{d - \beta} = 1.$$

We prove assertion (i) by induction on n. By Lemma 4.1(i), we have

$$\begin{aligned} 0 \leqslant x_1 - \alpha &\leqslant \frac{b + (1+c)\alpha + cx_0}{d - x_{-1}} \times \max\{x_{-1} - \alpha, x_0 - \alpha\} \\ &\leqslant L \times \max\{x_{-1} - \alpha, x_0 - \alpha\} \end{aligned}$$

and

$$0 \leqslant x_2 - \alpha \leqslant \frac{b + (1+c)\alpha + cx_1}{d - x_0} \times \max\{x_1 - \alpha, x_0 - \alpha\}.$$

By Lemma 4.4(i), $\alpha \leqslant x_1 < \max\{x_0, x_{-1}\}$. So

$$\frac{b + (1 + c)\alpha + cx_1}{d - x_0} < \frac{b + (1 + c)\alpha + c\max\{x_0, x_{-1}\}}{d - \max\{x_0, x_{-1}\}} = L$$

and thus

$$\begin{aligned} 0 &\leqslant x_2 - \alpha \leqslant L \times \max\{x_1 - \alpha, x_0 - \alpha\} \\ &\leqslant L \times \max\{L \times \max\{x_0 - \alpha, x_{-1} - \alpha\}, x_0 - \alpha\} \\ &\leqslant L \times \max\{\max\{x_0 - \alpha, x_{-1} - \alpha\}, x_0 - \alpha\} = L \times \max\{x_0 - \alpha, x_{-1} - \alpha\}. \end{aligned}$$

Hence the assertion is true for $n = 1, 2$. Suppose the assertion is true for $n - 1$ and $n - 2$ ($n \geqslant 3$). By Lemma 4.1(i),

$$0 \leqslant x_n - \alpha \leqslant \frac{b + (1 + c)\alpha + cx_{n-1}}{d - x_{n-2}} \times \max\{x_{n-1} - \alpha, x_{n-2} - \alpha\}.$$

By Lemma 4.4(i), $\alpha \leqslant x_{n-1} \leqslant \max\{x_0, x_{-1}\}$ and $\alpha \leqslant x_{n-2} \leqslant \max\{x_0, x_{-1}\}$. So

$$\frac{b + (1 + c)\alpha + cx_{n-1}}{d - x_{n-2}} \leqslant \frac{b + (1 + c)\alpha + c\max\{x_0, x_{-1}\}}{d - \max\{x_0, x_{-1}\}} = L.$$

Thus,

$$0 \leqslant x_n - \alpha \leqslant L \times \max\{x_{n-1} - \alpha, x_{n-2} - \alpha\}.$$

By the inductive hypothesis, we have

$$0 \leqslant x_{n-1} - \alpha \leqslant L^{\lceil (n-1)/2 \rceil} \times \max\{x_0 - \alpha, x_{-1} - \alpha\}$$

and

$$0 \leqslant x_{n-2} - \alpha \leqslant L^{\lceil (n-2)/2 \rceil} \times \max\{x_0 - \alpha, x_{-1} - \alpha\}.$$

It follows that

$$\begin{aligned} 0 &\leqslant x_n - \alpha \leqslant L \times \max\{L^{\lceil (n-1)/2 \rceil}, L^{\lceil (n-2)/2 \rceil}\} \times \max\{x_0 - \alpha, x_{-1} - \alpha\} \\ &= L^{\lceil n/2 \rceil} \times \max\{x_0 - \alpha, x_{-1} - \alpha\}. \end{aligned}$$

This completes our proof. The proof of assertion (ii) is similar and hence is omitted. □

Similarly, we can deduce the following result.

Lemma 4.6. *Assume* (3.2) *and* (3.4) *hold. Let*

$$M = \max\left\{\frac{b + c\alpha + c\max\{x_0, x_{-1}\}}{d - \max\{x_0, x_{-1}\}}, \frac{\alpha}{d - \max\{x_0, x_{-1}\}}\right\}. \tag{4.9}$$

If $\langle x_{-1}, x_0\rangle \in [0, \beta) \times [0, \beta) - [0, \alpha] \times [0, \alpha] - [\alpha, \beta) \times [\alpha, \beta)$, *then* $0 \leqslant M < 1$ *and*

$$|x_n - \alpha| \leqslant M^{\lceil n/2 \rceil} \times \max\{|x_{-1} - \alpha|, |x_0 - \alpha|\} \quad \text{for } n = 1, 2, \ldots. \tag{4.10}$$

By combining Lemmas 4.5 and 4.6, we establish the following result related to the global attractivity of α.

Theorem 4.7. *Assume* (3.2) *and* (3.4) *hold. Then* α *is a global attractor with a basin* $[0, \beta) \times [0, \beta)$. *Furthermore, for any* $\langle x_{-1}, x_0 \rangle \in [0, \beta) \times [0, \beta)$, *the sequence* $\{x_n\}$ *exponentially converges to* α. *Moreover, the convergence is subject to*

$$|x_n - \alpha| < \max\{|x_{n-1} - \alpha|, |x_{n-2} - \alpha|\} \quad \text{for } n = 1, 2, \ldots.$$

To make a more exact estimation on the basin of α, we need the following preliminary result.

Lemma 4.8. *Assume* (3.2) *and* (3.4) *hold.*

(i) *If* $b^2 \leqslant 4ac$ *and* $\langle x_{-1}, x_0 \rangle \in (-\infty, \beta) \times (-\beta - b/c, \beta)$, *then*

$$x_n \in [0, \beta) \quad \textit{for } n = 1, 2, \ldots.$$

(ii) *If* $b^2 > 4ac$ *and* $\langle x_{-1}, x_0 \rangle \in (-\infty, \beta) \times ((-\beta - b/c, \beta) - (\omega_1, \omega_2))$, *then*

$$x_n \in [0, \beta] \quad \textit{for } n = 1, 2, \ldots,$$

where

$$\omega_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2c}, \quad \omega_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2c}. \tag{4.11}$$

Proof

(i) Since $b^2 \leqslant 4ac$, the quadratic function $a + bx + cx^2 \geqslant 0$ for any real number x. Notice that

$$a + bx + cx^2 < a + b\beta + c\beta^2 \quad \text{for } x \in (-\beta - b/c, \beta),$$

we have

$$0 \leqslant x_1 = \frac{a + bx_0 + cx_0^2}{d - x_{-1}} < \frac{a + bx_0 + cx_0^2}{d - \beta} < \frac{a + b\beta + c\beta^2}{d - \beta} = \beta$$

and

$$0 \leqslant x_2 = \frac{a + bx_1 + cx_1^2}{d - x_0} < \frac{a + bx_1 + cx_1^2}{d - \beta} < \frac{a + b\beta + c\beta^2}{d - \beta} = \beta.$$

By induction on n and in view of Lemma 4.2, we conclude the desired result.

(ii) Since $b^2 > 4ac$, $a + bx + cx^2 \geqslant 0$ for any $x \notin (\omega_1, \omega_2)$. The remaining part of the proof is similar to that of assertion (i). □

From Theorem 4.7 and Lemma 4.8, we obtain the following result.

Theorem 4.9. *Assume* (3.2) *and* (3.4) *hold.*

(i) *If* $b^2 \leqslant 4ac$, *then* α *is a global attractor with a basin* $(-\infty, \beta) \times (-\beta - b/c, \beta)$.
(ii) *If* $b^2 > 4ac$, *then* α *is a global attractor with a basin*

$$(-\infty, \beta) \times ((-\beta - b/c, \beta) - (\omega_1, \omega_2)).$$

5. Asymptotic behavior of positive solutions

In this section, we investigate the asymptotic behavior of positive solutions of Eq. (3.1). To this end, we need to make the following estimation:

$$\begin{aligned} x_{n+1} - x_n &= f(x_n, x_{n-1}) - f(x_{n-1}, x_{n-2}) \\ &= [f(x_n, x_{n-1}) - f(x_{n-1}, x_{n-1})] + [f(x_{n-1}, x_{n-1}) - f(x_{n-1}, x_{n-2})] \\ &= \frac{b + c(x_n + x_{n+1})}{d - x_{n-1}}(x_n - x_{n-1}) + \frac{a + bx_{n-1} + cx_{n-1}^2}{(d - x_{n-1})(d - x_{n-2})}(x_{n-1} - x_{n-2}). \end{aligned} \tag{5.1}$$

From Eq. (5.1) and by induction on n, we obtain

Lemma 5.1. *Assume* (3.2) *and* (3.4) *hold.*

(i) *If there exists N such that* $x_N \leqslant x_{N+1} \leqslant x_{N+2}$, *then* $x_n \leqslant x_{n+1}$ *for* $n \geqslant N$.
(ii) *If there exists N such that* $x_N \geqslant x_{N+1} \geqslant x_{N+2}$, *then* $x_n \geqslant x_{n+1}$ *for* $n \geqslant N$.

From Eqs. (4.2) and (4.3), we derive

Lemma 5.2. *Assume* (3.2) *and* (3.4) *hold.*

(i) *If* $x_{n-1} \leqslant x_{n-2} \leqslant \alpha$, *then* $x_n \geqslant x_{n-1}$.
(ii) *If* $x_{n-1} \geqslant x_{n-2} \geqslant \alpha$, *then* $x_n \leqslant x_{n-1}$.

From Lemmas 5.1 and 5.2, we establish the following result related to the asymptotic behavior of the positive solutions of Eq. (3.1) in the basin of α.

Theorem 5.3. *Assume* (3.2) *and* (3.4) *hold, and assume* $\langle x_{-1}, x_0 \rangle \in [0, \beta) \times [0, \beta)$.

(i) *If* $\langle x_{-1}, x_0 \rangle \in [0, \alpha] \times [0, \alpha]$, *then the sequence* $\{x_n\}$ *converges to* α *in one of the following three ways*:
- *There exists an integer N such that the sequence* $\{x_n\}_{n=N}^{\infty}$ *is monotonically increasing*.
- $x_n \in [0, \alpha]$ *and* $x_{2n-1} \leqslant x_{2n} \geqslant x_{2n+1}$ *for* $n = 1, 2, \ldots$.
- $x_n \in [0, \alpha]$ *and* $x_{2n} \leqslant x_{2n+1} \geqslant x_{2n}$ *for* $n = 0, 1, 2, \ldots$.

(ii) *If* $\langle x_{-1}, x_0 \rangle \in [\alpha, \beta) \times [\alpha, \beta)$, *then the sequence* $\{x_n\}$ *approaches* α *in one of the following three ways*:
- *There is an integer N such that the sequence* $\{x_n\}_{n=N}^{\infty}$ *is monotonically decreasing*.
- $x_n \in [\alpha, \beta)$ *and* $x_{2n-1} \leqslant x_{2n} \geqslant x_{2n+1}$ *for* $n = 1, 2, \ldots$.
- $x_n \in [\alpha, \beta)$ *and* $x_{2n} \leqslant x_{2n+1} \geqslant x_{2n}$ *for* $n = 0, 1, 2, \ldots$.

(iii) *If* $\langle x_{-1}, x_0 \rangle \in [0, \beta) \times [0, \beta) - [0, \alpha] \times [0, \alpha] - [\alpha, \beta) \times [\alpha, \beta)$, *then the sequence* $\{x_n\}$ *tends to* α *in one of the following two ways*:
- *There is an integer N such that the sequence* $\{x_n\}_{n=N}^{\infty}$ *behaves in one of the six ways described in* (*i*) *or* (*ii*).
- $\{x_n\}$ *strictly oscillates about* α, *with each positive semicycle having a length of one*, *and each negative semicycle having a length of one*.

At the end of this paper, we give a sufficient condition for some positive solutions of Eq. (3.1) to converge to α bi-monotonically.

Lemma 5.4. *Suppose* (3.2) *and* (3.4) *hold, and assume* $c \geqslant 1$. *Then* $3\alpha < d$, *and*

$$\alpha(x - \alpha) \geqslant (b + c\alpha + cx)(\alpha - x) \quad \text{for } 0 \leqslant x \leqslant y \leqslant \alpha.$$

Proof

$$3\alpha = 3 \times \frac{d - b - \sqrt{(d - b)^2 - 4a(1 + c)}}{2(1 + c)} \leqslant \frac{3d}{4} < d.$$

Consider the two functions

$$g(x) = (b + c\alpha + cx)(x - \alpha) = cx^2 + bx - \alpha(b + c\alpha)$$

and $h(x) = \alpha(x - \alpha)$. The two corresponding curves intersect with each other at point $\langle \alpha, 0 \rangle$ and point $\langle \frac{(1-c)\alpha - b}{c}, \frac{(1-2c)\alpha^2 - b\alpha}{c} \rangle$. Since $c \geqslant 1$, then $\frac{(1-c)\alpha - b}{c} \leqslant 0$. So $h(x) \geqslant g(x)$ for $0 \leqslant x \leqslant \alpha$. This together with the fact that $g(x)$ is monotonically increasing in the interval $[0, \alpha]$ implies that $h(y) \geqslant g(y) \geqslant g(x)$ for $0 \leqslant x \leqslant y \leqslant \alpha$. $\square$

Theorem 5.5. *Assume* (3.2) *and* (3.4) *hold and* $c \geqslant 1$. *Then for any* $\langle x_0, x_{-1}\rangle \in [0,\alpha) \times [0,\alpha)$, *the two sequences* $\{x_{2n}\}$ *and* $\{x_{2n+1}\}$ *are strictly monotonically increasing*, *respectively*.

Proof. Consider two consecutive terms, say x_{2n-2} and x_{2n}, of sequence $\{x_{2n}\}$. We examine two cases.

Case 1. $x_{2n-2} \leqslant x_{2n-1}$. By Lemma 4.3(ii), we have $x_{2n} > x_{2n-2}$.
Case 2. $x_{2n-2} \geqslant x_{2n-1}$. By Lemma 4.1(ii) and Lemma 5.4, we have

$$\begin{aligned}\alpha - x_{2n} &= \frac{b + c\alpha + cx_{2n-1}}{d - x_{2n-2}}(\alpha - x_{2n-1}) + \frac{\alpha}{d - x_{2n-2}}(\alpha - x_{2n-2}) \\ &\leqslant \frac{2\alpha}{d-\alpha} \times (\alpha - x_{2n-2}) < \alpha - x_{2n-2}.\end{aligned}$$

So we also have $x_{2n} > x_{2n-2}$.

From the above discussions, we conclude that the sequence $\{x_{2n}\}$ is strictly monotonically increasing. Similarly, we can prove that the sequence $\{x_{2n+1}\}$ is strictly monotonically increasing. □

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