Minimally (k, k)-Edge-Connected Graphs

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Abstract: For an integer $l > 1$, the l-edge-connectivity of a connected graph with at least l vertices is the smallest number of edges whose removal results in a graph with l components. A connected graph G is (k, l)-edge-connected if the l-edge-connectivity of G is at least k. In this paper, we present a structural characterization of minimally (k, k)-edge-connected graphs. As a result, former characterizations of minimally $(2, 2)$-edge-connected graphs in [J of Graph Theory 3 (1979), 15–22] are extended.
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1. INTRODUCTION

We use Bondy and Murty [2] for basic notations except that we use $\lambda(G)$ to denote the edge-connectivity of a graph G, and our notion of contraction. For an edge subset $X \subseteq E(G)$, then the contraction G/X is the graph obtained from G by identifying the two ends of each edge in X, and then deleting the edges in X. Note that new loops or multiple edges may be resulted in the contraction. Graphs in this paper are finite and may have loops and multiple edges. As in [2], a minimal edge cut of G is a bond. If an edge cut of G consisting of a single edge e, then e is a bridge (or a cut edge) of G. Throughout this paper, k denotes an integer at least 2.

For an integer $l \geq 2$, Boesch and Chen [1] defined the l-edge-connectivity $\lambda_l(G)$ of a connected graph G to be the smallest number of edges whose removal leaves a graph with at least l components, if $|V(G)| \geq l$, and $\lambda_l(G) = |E(G)|$ if $|V(G)| < l$. Note that $\lambda_2(G) = \lambda(G)$. (Goldsmith in [7] and [8] defined the same concept and called it higher order of edge connectivity). This generalized edge connectivity has been studied by many. See [3,7–12], among others. For an overview on this parameter, see Oellermann's recent survey [10].

Following [10], call a graph G (k,l)-edge-connected if $\lambda_l(G) \geq k$. Therefore a (2,2)-edge-connected graph is just a 2-edge-connected graph. A graph G is minimally (k,l)-edge-connected if $\lambda_l(G) \geq k$ but for any edge $e \in E(G)$, $\lambda_l(G-e) < k$.

In [4], Chaty and Chein presented a structural characterization of minimally (2,2)-edge-connected graphs (Theorem 4.2 and Theorem 4.4 in Section 4). In this paper, we consider the same problem for characterizing minimally (k,k)-edge-connected graphs, for all integral values $k \geq 2$. As we shall see, when $k > 2$, a minimally (k,k)-edge-connected graph may have bridges, and so some of these graphs will fall out of the pattern given in [4].

In Section 2, we present some lemmas and observations that will be used in the proofs of the main results. Section 3 and Section 4 will be devoted to the graphs that have bridges and that do not have bridges, respectively.

2. PRELIMINARIES

For a graph G, define a relation on $E(G)$ as follows: $\forall e, e' \in E(G)$, $e \sim e'$, if and only if, either $e = e'$, or $\{e, e'\}$ is a bond of G. The relation $\sim$ is clearly reflexive and symmetric. For distinct edges $e, e', e'' \in E(G)$, if $\{e, e'\}$ and $\{e, e''\}$ are bonds of G, then G has a component H containing e, e', and e''; and $H - \{e, e', e''\}$ has three components. It follows that $\{e', e''\}$ is also an bond of G, and so $\sim$ is an equivalence relation on $E(G)$.

Let $[e]$ denote the equivalence class that contains $e \in E(G)$. For an $e \in E(G)$, define $G_{[e]} = G/(E(G) - [e])$. Note that $E(G_{[e]}) = [e]$ and that $G_{[e]}$ is obtained from G by contracting each component of $G - [e]$ into a single vertex.

Define, for a connected graph G,

$$B(G) = \{e : e \text{ is a bridge of } G\}.$$

Let $G' = G/(E(G) - B(G))$, called the B-reduction of G. Note that $E(G') = B(G)$ and that G' is the graph obtained from G by contracting the edges in $E(G) - B(G)$. We put some observations into a proposition below.

Proposition 2.1. *Each of the following holds for a connected graph G:*

(i) *If G has a bridge, then G' is a tree with edge set $B(G)$.*
(ii) *If G has no bridges, then $G_{[e]}$ is a cycle with edge set $[e]$.*
(iii) *If G has no bridges, then for any $e' \in [e]$, $(G - e')'$ is a path with edge set $[e] - \{e'\}$.*
(iv) *If $|V(G)| \leq k$, then G is minimally (k, k)-edge-connected, if and only if, $|E(G)| = k$.*
(v) *Let $l \geq 2$ and $k > 0$ be integers. If $|V(G)| \geq l$, and if for some integer $k > 0$ such that $\lambda_l(G) = k$, then for any an edge subset $T \subseteq E(G)$ with $|T| = k$ and with $\omega(G - T) \geq l$, the graph $G - T$ has exactly l components.*

Proof. Suppose that G has a bridge. Since $B(G)$ is the set of all bridges of G, each component of $G - B(G)$ is either a K_1 or a maximal 2-edge-connected subgraph of G. By the definition of contraction, $E(G') = B(G)$. For each $e \in B(G)$, let G_1 and G_2 be the two components of $G - e$. Then $B(G) = \{e\} \cup B(G_1) \cup B(G_2)$, by the definition of $B(G)$. It follows by induction on $|B(G)|$ that the number of components of $G - B(G)$ is $|B(G)| + 1$. Since G is connected, G' is also connected, and $|V(G')|$ is the number of components of $G - B(G)$. If follows that G' is a connected graph with $|V(G')| = |E(G')| + 1$, and so G' must be a tree. This proves (i).

Now assume that G has no bridges. Let $e \in E(G)$ be an edge. If $[e] = \{e\}$, then $G_{[e]}$ is a loop, and so is a cycle with edge set $\{e\}$. Assume that $|[e]| \geq 2$. Since for any edge $e' \in [e] - \{e\}$, $\{e, e'\}$ is a bond of G, we have $B(G - e) = [e] - \{e\}$, and so by Proposition 2.1(i), $(G - e)'$ must be a tree. Since G has no bridge and since e' can joint only two components of $(G - e') - ([e] - \{e'\}) = G - [e]$, $(G - e')'$ must be a path whose two ends being the two components in $G - [e]$ that are joined by e' in G. Therefore, $G_{[e]}$ is a cycle with edge set $[e]$. This proves (ii) and (iii).

Proposition 2.1(iv) follows from the definition of minimally (k, k)-edge-connectedness.

Now, we assume that $|V(G)| \geq l$, and $\lambda_l(G) = k$. Let $T \subseteq E(G)$ be an edge subset with $|T| = k$ and with $\omega(G - T) = c \geq l$. If $c \geq l + 1$, then since G is connected, there must be an edge $e \in T$ such that e joins two components of $G - T$ in G. It follows that $\omega(G - (T - \{e\})) \geq l$, and so $\lambda_l(G) \leq |T| - 1 = k - 1$, contrary to the assumption that $\lambda_l(G) = k$. Thus $G - T$ must have exactly l components. ■

Lemma 2.1. *Let G be a connected graph with $|V(G)| \geq k \geq 2$. Then $\lambda_k(G) \geq k - 1$. Moreover, the following are equivalent.*

(i) $\lambda_k(G) = k - 1$.

(ii) *If $T \subseteq E(G)$ is an edge subset with $|T| = k - 1$ such that $G - T$ has k components, then $T \subseteq B(G)$; and there exists at least one of such edge subset T.*

(iii) *G' is a tree of at least $k - 1$ edges.*

Proof. It is well known that any connected graph of order k has at least $k - 1$ edges, and so if G is a connected graph with at least k vertices, then $\lambda_k(G) \geq k - 1$.

Assume first Lemma 2.1(i) and let $T \subseteq E(G)$ be an edge subset such that $|T| = k - 1$ and such that $G - T$ has exactly k components (Proposition 2.1(v)). Contract each of these components of $G - T$ into a single vertex to get the graph $G/(E(G) - T)$. Since G is connected, $G/(E(G) - T)$ is a connected graph with k vertices and with $|T| = k - 1$ edges. Therefore, $G/(E(G) - T)$ must be a tree, and so every edge of T must be a bridge of G. This proves Lemma 2.1(ii).

Assume Lemma 2.1(ii). Since there exists at least one of such subset T satisfying Lemma 2.1(ii), $T \subseteq B(G)$, and so G' is a tree with $|B(G)| \geq |T| = k - 1$ edges. This proves Lemma 2.1(iii).

Assume Lemma 2.1(iii). Let $T \subseteq B(G)$ be a subset with $|T| = k - 1$. Then $G - T$ has exactly k components, since G' is a tree. It follows that $\lambda_k(G) \leq k - 1$. This implies Lemma 2.1(i). ■

Lemma 2.2. *Let G be a minimally (k, k)-edge-connected graph and let $e \in E(G)$. Then $\lambda_k(G - e) = k - 1$. Moreover, if T' is a minimal edge subset of $G - e$ such that $(G - e) - T'$ has k components, then $T' \cup \{e\}$ is a minimum edge subset of G such that $G - (T' \cup \{e\})$ has k components.*

Proof. Since G is minimally (k, k)-edge-connected, $\lambda_k(G - e) \leq k - 1$. Let $T' \subseteq E(G - e)$ such that $|T'| = \lambda_k(G - e) \leq k - 1$ and such that $(G - e) - T'$ has k components. If $|T'| < k - 1$, then $G - (T' \cup \{e\})$ has k components, contrary to the assumption that $\lambda_k(G) = k$. Thus $\lambda_k(G - e) = k - 1$.

If both ends of e are in one of the k components of $(G - e) - T'$, then T' is an edge subset of G such that $G - T'$ has k components, contrary to $\lambda_k(G) = k$. Hence, the two ends of e must be in two distinct components of $(G - e) - T'$, and so $T' \cup \{e\}$ is a minimum edge subset such that $G - (T' \cup \{e\})$ has k components. ■

Theorem 2.1. *Let $k \geq 2$ be an integer, and let G be a connected graph. The following are equivalent.*

(i) *For any edge $e \in E(G)$, $\lambda_k(G - e) = k - 1$.*

(ii) *G is minimally (k, k)-edge-connected.*

Proof. By Lemma 2.2, Theorem 2.1(ii) implies Theorem 2.1(i). To show that Theorem 2.1(i) implies Theorem 2.1(ii), we assume Theorem 2.1(i) to show that $\lambda_k(G) = k$.

If $|V(G)| < k$, then by Proposition 2.1(iv), Theorem 2.1(i) implies that $|E(G)| = k$, and so by Proposition 2.1(iv) again, Theorem 2.1(ii) holds. Therefore, we assume that $|V(G)| \geq k$.

By Theorem 2.1(i), $\lambda_k(G) \leq k$. By Lemma 2.1 and by the assumption that $|V(G)| \geq k$, $\lambda_k(G) \geq k-1$. If $\lambda_k(G) = k-1$, then by Lemma 2.1, G has $k-1 > 0$ bridges. It follows that for each $e \in B(G)$, $\lambda_k(G-e) \leq k-2$, contrary to Theorem 2.1(i). This completes the proof. ■

For a connected graph G with a bridge, by Proposition 2.1, G' is a tree with $E(G') = B(G)$. Let $e \in B(G)$ be a bridge of G. Let G_1 and G_2 be the two components of $G - e$. Then we have

$$|B(G_1)| + |B(G_2)| = |B(G)| - 1. \tag{1}$$

If $E(G_i) = B(G_i)$, then G_i is called a tree component of $G - e$. In Lemma 2.3 below, we let s' be the number of components of $G - B(G)$, and s the number of components of $G - B(G)$ that are not isomorphic to K_1. When $s \geq 1$, we let $H_1, H_2, \ldots, H_s$ be the components of $G - B(G)$ that are not isomorphic to K_1. Note that if $B(G) \neq \emptyset$, $s' = |B(G)| + 1$.

Lemma 2.3. *Let G be a connected graph with a bridge, and let $k \geq 2$ be an integer. Each of the following holds.*

(i) *If G is (k,k)-edge-connected, then $k-2 \geq |B(G)|$ and $s \geq 1$.*
(ii) *If G is minimally (k,k)-edge-connected, then for any $e \in E(G)$, G has an edge subset $T \subseteq E(G)$ with $|T| = k$ and with $e \in T$, such that $B(G) \subset T$ and such that $G - T$ has k components.*
(iii) *If G is (k,k)-edge-connected, then for any $T \subset E(G)$ with $B(G) \subset T$ and with $|T| = k$ such that $G - T$ has k components, there exists exactly one i with $1 \leq i \leq s$ such that $T - B(G) \subseteq E(H_i)$.*
(iv) *Suppose that $v \in V(G)$ is a vertex of degree 1 in G. Then G is minimally (k,k)-edge-connected, if and only if, $G - v$ is minimally $(k-1,k-1)$-edge-connected.*

Proof. If $|B(G)| \geq k-1$, then by Proposition 2.1(i), G', the B-reduction of G, is a tree with at least $k-1$ edges. It follows that $|V(G)| \geq k$ and so by Lemma 2.1, $\lambda_k(G) = k-1$, contrary to the assumption that G is (k,k)-edge-connected. Note that $s = 0$, if and only if, G is a tree, and so by Lemma 2.1, if G is (k,k)-edge-connected, we must have $s \geq 1$. This proves (i).

Since G is minimally (k,k)-edge-connected, by Theorem 2.1, G has an edge subset T such that $e \in T$ and $|T| = k$, and such that $G - T$ has k components. Choose such T so that $|T \cap B(G)|$ is maximized. Suppose by contradiction that there exists an edge $e' \in B(G) - T$. By Lemma 2.3(i), there exist two of these k components which are joined to each other by edges $T_1 \subset T - B(G)$ with $e \notin T_1$; and so $G - (T - T_1)$ has $k-1$ components. Since $e' \notin T$, e' must be in one of these $k-1$ components, and so $G-((T-T_1)\cup\{e'\})$ has k components. Let $T'=(T-T_1)\cup\{e'\}$. Note that $|T'|=|T|-|T_1|+1$. If $|T_1|>1$, then $|T'|\leq k-1$, contrary to the assumption that G is minimally (k,k)-edge-connected. Hence, $|T_1|=1$ and $|T'|=k$. But then, $T'\cap B(G)-T\cap B(G)=\{e'\}$, contrary to the choice of T. Hence, we must have $B(G)\subset T$. This proves (ii).

Suppose that G is (k,k)-edge-connected and that $T\subset E(G)$ is an edge subset with $B(G)\subset T$ and with $|T|=k$ such that $G-T$ has k components. Since $B(G)\subset T$, we may assume, without loss of generality, that for some integer $m\geq 1$, $E(H_i)\cap T=\emptyset$, for all $i\geq m+1$. Therefore, $G-T$ has components $G_1^1,\ldots,G_{i_1}^1,G_1^2,\ldots,G_{i_2}^2,\ldots,G_1^m,\ldots,G_{i_m}^m,\ldots,H_{m+1},\ldots,H_{s'}$, where $G_1^j,\ldots,G_{i_j}^j$ are components of $H_j-(T\cap E(H_j))$, $1\leq j\leq m\leq s$. Then counting the number of components of $G-T$, $k=i_1+i_2+\cdots+i_m+(s'-m)$. Since each H_j, $1\leq j\leq m\leq s$, is a component of $G-B(G)$, H_j is 2-edge-connected. Moreover, as $|V(H_j)|\geq\omega(H_j-(T\cap E(H_j)))=i_j$, it follows by Lemma 2.1 that $\lambda_{i_j}(H_j)\geq i_j$, $1\leq j\leq m\leq s$. Thus by counting the number of edges in T, we have $|T|\geq i_1+i_2+\cdots+i_m+|B(G)|$. However, by $s'=|B(G)|+1$, and by $|T|=k$, we have $s'-1=|B(G)|\leq s'-m$, and so we must have $m=1$, as desired. This proves (iii).

Suppose first that $|V(G)|\leq k$. Then by Proposition 2.1(iv), G is minimally (k,k)-edge-connected, if and only if, $|E(G)|=k$. But as $|V(G-v)|=|V(G)|-1\leq k-1$ and $|E(G-v)|=|E(G)|-1=k-1$, $|E(G)|=k$, if and only if, $G-v$ is minimally $(k-1,k-1)$-edge-connected.

Hence we assume that $|V(G)|\geq k+1$. Let $e\in E(G)$ denote the edge incident with v. Suppose first that $G-v$ is minimally $(k-1,k-1)$-edge-connected. Then by Theorem 2.1, G is minimally (k,k)-edge-connected.

Conversely, assume that G is minimally (k,k)-edge-connected. We want to apply Theorem 2.1 to show that $G-v$ is minimally $(k-1,k-1)$-edge-connected. Note that by definition, every (2,2)-edge-connected graph has no bridges, we assume $k\geq 3$ here.

First, we show that $\lambda_{k-1}(G-v)\geq k-1$. By the definition of minimally (k,k)-edge-connectedness, $\lambda_k(G)=k$. If $\lambda_{k-1}(G-v)<k-1$, then by $|V(G)|\geq k+1$ and by Proposition 2.1(v), $G-v$ has an edge subset $T'\subseteq E(G-v)$ such that $|T'|=\lambda_{k-1}(G-v)<k-1$, and $\omega((G-v)-T')=k-1$. It follows that G has a edge subset $T'\cup\{e\}$ such that $G-(T'\cup\{e\})$ has k components, and so $\lambda_k(G)\leq|T'\cup\{e\}|\leq k-1$, contrary to the assumption that $\lambda_k(G)=k$. Hence, we must have $\lambda_{k-1}(G-v)\geq k-1$.

Let $e'\in E(G-v)$ such that $e'\notin B(G)$. Since G is minimally (k,k)-edge-connected and by Theorem 2.1, $\lambda_k(G-e')=k-1$. By the assumption that $|V(G-v)|=|V(G)|\geq k+1$ and by Lemma 2.1, $(G-e')'$ is a tree with at least $k-1$ edges. As $B(G)\subseteq B(G-e')$, $e\in E(G-e')$. Let $T\subseteq E((G-e')')$ be an edge subset such that $e\in T$ and $|T|=k-1$. Then $\omega((G-e')-T)=k$. Since $e\in T$ and since e is the only edge incident with v in G, we have $\omega((G-v-e')-\ (T-e))=k-1$, and $|T-e|=k-2$. It follows that $\lambda_{k-1}(G-v-e')=k-2$.

Now assume that $e' \in E(G-v) \cap B(G)$. Since G is minimally (k,k)-edge-connected and by Theorem 2.1, G is not a tree, and so G has an edge $e'' \in E(G) - B(G)$. Similar to the arguments above, we conclude that $(G-e'')'$ is a tree with at least $k-1$ edges and $e, e' \in B(G) \subseteq E((G-e'')')$. Since $k \geq 3$, we can find an edge subset $T \subseteq E((G-e'')')$ such that $e, e' \in T$ and $|T| = k-1$. Then $\omega((G-e'')-T) = k$. Since $e, e' \in T \cap B(G)$ and since e is the only edge incident with v in G, we have $\omega((G-v-e') - (T - \{e,e'\}) \cup \{e''\}) = k-1$, and $|(T - \{e,e'\}) \cup \{e''\}| = k-2$. It follows that $\lambda_{k-1}(G-v-e') = k-2$.

Thus, by Theorem 2.1, we conclude that $G-v$ is minimally $(k-1,k-1)$-edge-connected. This proves (iv). ■

Lemma 2.4. *Let G be a minimally (k,k)-edge-connected graph and $e \in B(G)$ be a bridge of G. Let G_1 and G_2 be the two components of $G-e$. Then one of the following holds.*

(i) *For some integer i with $1 \leq i \leq 2$, $|V(G_i)| = 1$ and G_{3-i} is a minimally $(k-1,k-1)$-edge-connected graph.*

(ii) *Neither G_1 nor G_2 is a single vertex graph, and for some integer i with $1 \leq i \leq 2$, G_i is a tree components of $G-e$, and G_{3-i} is minimally (t,t)-edge-connected, where $t = k - |E(G_i)| - 1$.*

(iii) *$|V(G)| \geq k$ and neither G_1 nor G_2 is a single vertex graph nor a tree component of $G-e$, and for each i with $1 \leq i \leq 2$, G_i is minimally (t_i,t_i)-edge-connected, where $t_i = k - |B(G_{3-i})| - 1$.*

(iv) *$|V(G)| < k$ and neither G_1 nor G_2 is a single vertex graph nor a tree component of $G-e$, and for each i with $1 \leq i \leq 2$, G_i is minimally (t_i,t_i)-edge-connected, where $t_i = k - |E(G_{3-i})| - 1$.*

Proof. Since a $(2,2)$-edge-connected graph does not have a bridge, Lemma 2.4 holds vacuously for $k=2$, and so we assume that $k \geq 3$.

Note that Lemma 2.4(i) and (ii) follow from Lemma 2.3(iv) or repeated application of Lemma 2.3(iv). It suffices to show Lemma 2.4(iii) and (iv).

Since G is a minimally (k,k)-edge-connected graph, when $|V(G)| \geq k$, choose an edge subset $T \subseteq E(G)$ with $|T| = k$ such that $G-T$ has k components $L_1, L_2, \ldots, L_k$; and when $|V(G)| \leq k$, let $T = E(G)$.

To prove Lemma 2.4(iii), we assume that $|V(G)| \geq k$ and neither G_1 nor G_2 is a single vertex graph nor a tree component of $G-e$.

By symmetry, it suffices to show that G_1 is minimally (t_1,t_1)-edge-connected. If $\lambda_{t_1}(G_1) \leq t_1 - 1$, then G_1 has an edge subset T_1 with $|T_1| \leq t_1 - 1$ such that $G_1 - T_1$ has t_1 components. Thus, $T_1 \cup B(G_2)$ is an edge subset with $|T_1| + |B(G_2)| < k$ edges such that $G - (T_1 \cup B(G_2))$ has k components, contrary to the assumption that $\lambda_k(G) = k$. Therefore, we must have $\lambda_{t_1}(G_1) \geq t_1$. It remains to show, by Theorem 2.1, that for any $e' \in E(G_1)$, G_1 has an edge subset T_1 with $e' \in T_1$ and with $|T_1| = t_1$ such that $G_1 - T_1$ has t_1 components. Let $e' \in E(G_1)$.

Suppose first that $e' \in E(G_1) - B(G_1)$. Then $G - e'$ is connected. By Theorem 2.1, $\lambda_k(G - e') = k - 1$. By $|V(G)| \geq k$ and by Lemma 2.1, $|B(G - e')| \geq k - 1$. Note that $e \in B(G) \subseteq B(G - e')$. Thus $(G_1 - e')'$ is a subtree of the tree $(G - e')'$, and so $|E(G_1 - e')'| \geq k - 1 - |B(G_2) \cup \{e\}| = t_1 - 1$. It follows that $|V(G_1)| \geq |B(G_1 - e')| + 1 = |E(G_1 - e')'| + 1 \geq t_1$. By Lemma 2.1, $\lambda_{t_1}(G_1 - e') = t_1 - 1$, and there exists a $T_1' \subseteq B(G_1 - e')$ with $|T_1'| = t_1 - 1$ such that $G_1 - (T_1' \cup \{e'\})$ has t_1 components.

Now we assume that $e' \in B(G_1)$. Since G_1 is not a tree component nor a single vertex graph, and since $e \in B(G)$, we may assume that G_1 contains a non-trivial component H_1 of $G - B(G)$. Pick another edge $e'' \in E(H_1)$. Since G is minimally (k, k)-edge-connected and by Lemma 2.3(ii), G has an edge subset $T \subset E(G)$ with $B(G) \subset T$, $e'' \in T$ and $|T| = k$ such that $G - T$ has k components. By Lemma 2.3(iii), (we may assume that $i = 1$ in Lemma 2.3(iii), and so) $T - B(G) \subseteq E(H_1)$. Let $T_1 = T \cap E(G_1)$ and note that $e' \in B(G_1) \subset T_1$. Then as $T - T_1 = B(G_2) \cup \{e\}$, $|T_1| = |T| - |B(G_2) \cup \{e\}| = t_1$. Moreover, $G - T$ has $|T - T_1| = |B(G_2)| + 1$ components not in G_1, and so $G_1 - T_1$ must have $k - |B(G_2)| - 1 = t_1$ components. This completes the proof for Lemma 2.4(iii).

Finally, we assume that $|V(G)| < k$ and neither G_1 nor G_2 is a single vertex graph nor a tree component of $G - e$.

Since $|V(G)| < k$, by Proposition 2.1(iv), $|E(G)| = k$. For each i with $1 \leq i \leq 2$, since G_{3-i} is connected and not a tree, $|V(G_i)| = |V(G)| - |V(G_{3-i})| \leq k - 1 - |E(G_{3-i})| = t_i$ and $|E(G_i)| = |E(G)| - |E(G_{3-i})| = t_i$. It follows by Proposition 2.1(iv) that G_i is minimally (t_i, t_i)-edge-connected. This proves Lemma 2.4(iv).

Lemma 2.5. *Let G be a 2-edge-connected graph with $|V(G)| \geq k$. The following are equivalent.*

(i) *G is minimally (k, k)-edge-connected.*
(ii) *for each edge e of $E(G)$, $G - e$ has at least $k - 1$ bridges $e_1, e_2, \ldots, e_{k-1}$ that separates the two ends of e.*

Proof. Let G be a 2-edge-connected graph.

Assume Lemma 2.5(i) and let $e = xy \in E(G)$. Since $\lambda_k(G) = k$, and by Theorem 2.1, $\lambda_k(G - e) = k - 1$ for any edge $e \in E(G)$. By Lemma 2.1, $(G - e)'$ is a tree with at least $k - 1$ edges. Let $e_1, e_2, \ldots, e_{k-1} \in B(G - e)$ be any $k - 1$ edges. Since $B(G) = \emptyset$, none of these e_i's is a bridge of G, and so $(G - e)'$ must be a path, and the two ends of this path, as subgraphs of G, must contain x and y, respectively.

Conversely, assume Lemma 2.5(ii). Then for any $e \in E(G)$, $|B(G - e)| \geq k - 1$, and so $(G - e)'$ is a tree with at least $k - 1$ edges. It follows that $\lambda_k(G - e) \leq k - 1$. Since G is 2-edge-connected, $G - e$ is connected, and so by Lemma 2.1, $\lambda_k(G - e) \geq k - 1$. Therefore, for any edge $e \in E(G)$, we must have $\lambda_k(G - e) = k - 1$. By Theorem 2.1, G is minimally (k, k)-edge-connected. ∎

Lemma 2.6. *Let G be a 2-edge-connected graph with $|V(G)| \geq k$. The following are equivalent.*

(i) *G is minimally (k, k)-edge-connected.*
(ii) *$\forall e \in E(G)$, $G_{[e]}$ is a cycle of length at least k.*
(iii) *$\forall e \in E(G)$, $(G - e)'$ is a path of length at least $k - 1$.*

Proof. By Lemma 2.5, Lemma 2.6(i) implies Lemma 2.6(ii); and it is straight forward to see that Lemma 2.6(ii) implies Lemma 2.6(iii). It remains to show that Lemma 2.6(iii) implies Lemma 2.6(i).

Assume Lemma 2.6(iii). By Lemma 2.1, $\forall e \in E(G)$, $\lambda_k(G - e) = k - 1$; and so by Theorem 2.1, Lemma 2.6(i) holds. ■

Let G_1 and G_2 be two vertex disjoint graphs with distinguished vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Let $G_1 \oplus G_2$ denote the graph obtained from the union of G_1 and G_2 by identifying v_1 with v_2. Note that the identified vertex is a new cut vertex of $G_1 \oplus G_2$.

Corollary 2.1. *Let G_1 and G_2 be 2-edge-connected graphs, where $i \in \{1, 2\}$, and let $G = G_1 \oplus G_2$. The following are equivalent.*

(i) *G is minimally (k, k)-edge-connected with $|V(G)| \geq k$.*
(ii) *Both G_1 and G_2 are minimally (k, k)-edge-connected, and both $|V(G_1)| \geq k$ and $|V(G_2)| \geq k$.*

Proof. Assume Corollary 2.1(i). Pick $e \in E(G_1)$. By Lemma 2.6(iii), $(G - e)'$ is a path of length at least $k - 1$. Since $\lambda_2(G_2) \geq 2$, we must have $B(G - e) \subseteq E(G_1)$, which implies that $(G_1 - e)'$ is a path of length at least $k - 1$, and so $|V(G_1)| \geq k$. By Lemma 2.6, G_1 is minimally (k, k)-edge-connected. Similarly, G_2 is minimally (k, k)-edge-connected with $|V(G_2)| \geq k$.

The proof for the other direction is similar, again applying Lemma 2.6. This completes the proof. ■

3. MINIMALLY (K, K)-EDGE-CONNECTED GRAPH WITH BRIDGES

The main result of this section is Theorem 3.1 below, which gives a structural characterization of a minimally (k, k)-edge-connected graph with bridges, and which indicates that every minimally (k, k)-edge-connected graph with bridges contains some minimally (k', k')-edge-connected graph without bridges, for some values $k' > 1$. We start with one more lemma.

Lemma 3.1. *Let G be a connected graph with a bridge and with $|V(G)| \geq k$. If for some integer $k > 1$, G is minimally (k, k)-edge-connected, then every non-trivial component of $G - B(G)$ is minimally (t, t)-edge-connected, where $t = k - |B(G)|$.*

Proof. Let G be a minimally (k,k)-connected graph with a bridge.

We shall prove Lemma 3.1 by induction on k. Let $H_1, H_2, \ldots, H_s$ be the components of $G - B(G)$ that are not isomorphic to K_1, where $1 \leq s \leq |B(G)| + 1$ (Lemma 2.3(i)). It suffices to prove that H_1 is a minimally (t,t)-edge-connected graph with $t = k - |B(G)|$.

Since G is connected and has a bridge, G' is a tree by Lemma 2.1. Let e be an edge of $B(G)$ such that e is incident with a vertex of degree one in the tree G'. Let G_1 and G_2 be the two subgraphs of $G - e$. Assume, without loss of generality, that H_1 is a subgraph of G_2.

By Lemma 2.4, one of the following must hold.

(A) Lemma 2.4(i) holds with $|V(G_1)| = 1$, whence G_2 is a minimally $(k-1, k-1)$-edge-connected graph. By induction, for $t = (k-1) - |B(G_2)| = k - |B(G)|$, H_1 is minimally (t,t)-edge-connected graph.

(B) Lemma 2.4(ii) holds with G_1 being a tree component, whence G_2 is a minimally $(k - |E(G_1)| - 1, k - |E(G_1)| - 1)$-edge-connected graph. By (1) and by $E(G_1) = B(G_1)$, $t = (k - |E(G_1)| - 1) - |B(G_2)| = k - |B(G)|$. By induction, H_1 is a minimally (t,t)-edge-connected graph.

(C) Lemma 2.4(iii) holds, whence G_i is a minimally $(k - |B(G_{3-i})| - 1, k - |B(G_{3-i})| - 1)$-edge-connected graph. By (1), $t = (k - |B(G_1)| - 1) - |B(G_2)| = k - |B(G)|$. By induction, H_1 is a minimally (t,t)-edge-connected graph. This proves Lemma 3.1. ∎

We are to give a characterization for minimally (k,k)-edge-connected graphs with $B(G) \neq \emptyset$. By Lemma 2.3(i), we should only consider the cases when $k \geq 3$. Let G be a graph with at least one bridge. Let $H_1, H_2, \ldots, H_s$ be the components of $G - B(G)$ that are not isomorphic to K_1, where $0 \leq s \leq |B(G)| + 1$. If $s \geq 1$ and if there is an integer $k \geq 3$ such that $k - |B(G)| = t \geq 2$ (Lemma 2.3(i)) and such that for $1 \leq i \leq s$, each H_i is minimally (t,t)-edge-connected, then G is called a k-arbor.

Theorem 3.1. *Let $k \geq 3$ be an integer and let G be a connected graph with a bridge and with $|V(G)| \geq k$. The following are equivalent.*

(i) *G is minimally (k,k)-edge-connected.*
(ii) *G is a k-arbor.*

Proof. Let $H_1, H_2, \ldots, H_s$ be the components of $G - B(G)$ that are not isomorphic to K_1, and let $t = k - |B(G)|$.

Assume first that G is minimally (k,k)-edge-connected with a bridge. By Lemma 2.3, $s \geq 1$, and $t = k - |B(G)| \geq 2$. By Lemma 3.1, each of these H_i's is minimally (t,t)-edge-connected. Thus G is a k-arbor. This shows that Theorem 3.1(i) implies Theorem 3.1(ii).

Now assume Theorem 3.1(ii). Then G is a k-arbor and each H_i, where $1 \leq i \leq s$, is a minimally (t,t)-edge-connected graph without bridges. We shall apply Theorem 2.1 to show Theorem 3.1(i).

Let $e \in E(G)$. Assume first that $e \in B(G)$. Note that H_1 is a minimally (t,t)-edge-connected graph. There is an edge subset $T_1 \subseteq E(H_1)$ with $|T_1| = t$ such that $H_1 - T_1$ has t components. Thus $(G-e) - (T_1 \cup B(G-e))$ has $t + |B(G)| = k$ components and $|T_1 \cup B(G-e)| = k-1$. Therefore, $\lambda_k(G-e) \leq k-1$.

Assume then $e \in E(G) - B(G)$. Then we may assume that $e \in E(H_1)$, without loss of generality. Note that H_1 is a 2-edge-connected graph. By Lemma 2.6(iii), $|B(H_1 - e)| \geq t-1$. Let $T' \subseteq B(H_1 - e)$ be such that $|T'| = t-1$, and $T = T' \cup B(G)$. Then $|T| = |T'| + |B(G)| = k-1$ and $(G-e) - T$ has k components. Once again we have $\lambda_k(G-e) \leq k-1$.

Thus by Theorem 2.1, G is minimally (k,k)-edge-connected, and so Theorem 3.1(i) follows. ■

4. MINIMALLY (K, K)-EDGE-CONNECTED GRAPH WITHOUT BRIDGES

In this section, we only consider graph that are $(2,2)$-edge-connected. What Corollary 2.1 indicates is that to study the structure of minimally (k,k)-edge-connected graph without bridges, it suffices to study the structure of 2-connected minimally (k,k)-edge-connected graph. Motivated from this view point and from the similar concepts in [4], we present the following definitions.

A k-necklace is a 2-connected minimally (k,k)-edge connected simple graph with at least k vertices. A graph G is k-extensible between x and y if, for two distinct vertices x and y in G, the graph $G_{xy}^{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}$ obtained from G by adding $k-1$ distinct new vertices $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ and the k new edges $x\alpha_1, \alpha_1\alpha_2, \ldots, \alpha_{k-1}y$, is minimally (k,k)-edge-connected. Since a cycle of length at least k is minimally (k,k)-edge-connected, an edge xy is k-extensible between x and y. A graph G is called an E_k-chain if it can be represented by $G_1a_1G_2a_2 \cdots a_{l-1}G_l$ where all of the following properties are satisfied:

(E1) $l \geq k-1$.
(E2) For each i with $1 \leq i \leq l$, G_i is either an edge or a k-necklace.
(E3) There exist at least $k-1$ of the G_j's that are edges, where $1 \leq j \leq l$.
(E4) For each i with $1 \leq i \leq l-1$, $V(G_i) \cap V(G_{i+1}) = \{a_i\}$.
(E5) For each i with $1 \leq i \leq l-2$ and for each j with $i+2 \leq j \leq l$, $V(G_i) \cap V(G_j) = \emptyset$.

Each G_i is called a pearl of the E_k-chain. If a pearl G_i consists of only one edge, then it is called an edge pearl.

Lemma 4.1. *Let G be a 2-connected graph with $|V(G)| \geq k$. The following are equivalent.*

(i) *G is a k-necklace.*
(ii) *$\forall e \in E(G)$, $G_{[e]}$ is a cycle of length at least k.*
(iii) *$\forall e \in E(G)$, $(G - e)'$ is a path of length at least $k - 1$.*
(iv) *$\forall e = ab \in E(G)$, $G - e$ is an E_k-chain $G_1 a_1 G_2 a_2 \cdots a_{l-1} G_l$ such that $l \geq k$, $a \in V(G_1)$, $b \in V(G_l)$, $a \neq a_1$, $b \neq a_{l-1}$.*

Proof. By Lemma 2.6 and by the assumption that G is 2-connected, Lemma 4.1(i) implies Lemma 4.1(ii) and Lemma 4.1(ii) implies Lemma 4.1(iii). It is straight forward to check that Lemma 4.1(iii) implies Lemma 4.1(iv). It remains to show that Lemma 4.1(iv) implies Lemma 4.1(i).

Assume Lemma 4.1(iv) holds. For any $e \in E(G)$, by Lemma 4.1(iv) and by (E3), $G - e$ has at least $k - 1$ edge pearls. By (E4) and by (E5), these edges are all in $B(G - e)$, and so $|B(G - e)| \geq k - 1$. Thus by Lemma 2.1, $\lambda_k(G - e) = k - 1$, $\forall e \in E(G)$. It follows by Theorem 2.1 that G is minimally (k, k)-edge-connected. ■

Lemma 4.2. *Let G be a k-necklace and let $x, y \in V(G)$ be two distinct vertices. The following are equivalent.*

(i) *G is k-extensible between x and y.*
(ii) *$\forall e \in E(G)$, the E_k-chain $G - e$ can be described by $G_1 a_1 G_2 a_2 \cdots a_{l-1} G_l$, such that for some i and j, $1 \leq i \leq j \leq l$, $x \in V(G_i)$, $y \in V(G_j)$, and such that*

$$\text{there are at least } k - 1 \text{ edge pearls } G_h, \text{ where } h \notin \{i, i+1, \ldots, j\}. \tag{2}$$

Proof. Assume Lemma 4.2(i). By Lemma 4.1 and since G is a k-necklace, $G - e$ is an E_k-chain $G_1 a_1 G_2 a_2 \cdots a_{l-1} G_l$ satisfying Lemma 4.1(iv). As $x, y \in V(G) = V(G - e)$, we may assume that for some i and j, $1 \leq i \leq j \leq l$, $x \in V(G_i)$, $y \in V(G_j)$. By contradiction, assume further that (2) fails. Then there are at most $k - 2$ edges in $B(G - e)$ that are not between G_i and G_j in the E_k-chain. It follows that $|B((G_{xy}^{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}})_{[e]})| \leq k - 1$, and so by Lemma 4.1, $G_{xy}^{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}}$ is not minimally (k, k)-edge-connected, contrary to the assumption that G is k-extensible between x and y.

Conversely, assume Lemma 4.2(ii). We shall show that $\forall e \in E(G_{xy}^{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}})$, the graph $(G_{xy}^{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}})_{[e]}$ is a cycle of length at least k. By Lemma 4.1, it suffices to show that $\forall e \in E(G_{xy}^{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}})$,

$$|E((G_{xy}^{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}})_{[e]})| \geq k. \tag{3}$$

Pick $e \in E(G_{xy}^{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}})$. If $e \in E(G)$, then (3) follows from (2). If $e \notin E(G)$, then by the definition of $G_{xy}^{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}}$, e must be one of the new edges in $\{x\alpha_1, \alpha_1\alpha_2, \ldots, \alpha_{k-1}y\}$, and so (3) holds as well. Therefore, Lemma 4.2(i) follows from Lemma 4.1. ■

Lemma 4.3. *Let $G = G_1a_1G_2a_2 \cdots a_{l-1}G_l$ denote an E_k-chain. The following are equivalent.*

(i) *G is k-extensible between a_0 and a_l, where $a_0 \in V(G_1 - a_1)$ and $a_l \in V(G_l - a_{l-1})$.*
(ii) *for each $i = 1, 2, \ldots, l$, G_i is k-extensible between a_{i-1} and a_i.*

Proof. Assume Lemma 4.3(i) and by contradiction, assume also that G_i is not k-extensible between a_{i-1} and a_i, for some i with $1 \leq i \leq l$. Thus, G_i is not an edge pearl and hence it is a k-necklace. By Lemma 4.2, for some edge $e \in E(G_i)$, there are at most $k-2$ edges in $B(G_i - e)$ which not between a_{i-1} and a_i. It follows that $(G_{a_0a_l}^{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}})_{[e]}$ is a cycle of length at most $k-1$, and so by Lemma 4.1, $G_{a_0a_l}^{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}$ is not minimally (k,k)-edge-connected, contrary to Lemma 4.3(i).

Conversely assume Lemma 4.3(ii). Note that since each pearl in the E_k-chain G is either a k-necklace or an edge, $G_{a_0a_l}^{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}$ is 2-connected. If G is not k-extensible between a_0 and a_l, then $G_{a_0a_l}^{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}$ is not minimally (k,k)-edge-connected. By Lemma 4.1, there is an edge $e \in E(G_{a_0a_l}^{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}})$ such that $(G_{a_0a_l}^{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}})_{[e]}$ is a cycle of length at most $k-1$. Since $a_0\alpha_1\alpha_2\cdots\alpha_{k-1}a_l$ is a path of length k, this edge e must be an edge of G and must not be an edge of any edge pearl G_i in this E_k-chain G. Therefore, there is some non edge pearl G_i such that $e \in E(G_i)$.

By Lemma 4.3(ii), and by Lemma 4.2, there are at least $k-1$ edges in $B(G_i - e)$ that are not lying between a_{i-1} and a_i in the E_k-chain $G_i - e$. It follows that these $k-1$ edges in $B(G_i - e)$ together with e will be the k edges in $(G_{a_0a_l}^{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}})_{[e]}$, a contradiction. This proves Lemma 4.3(i). ■

Lemma 4.4. *Let $k \geq 2$ and let G be a k-necklace and $e = xy \in E(G)$. Then $G - e$ is k-extensible between x and y.*

Proof. By Lemma 4.1, $G - e$ is an E_k-chain $G_1a_1G_2a_2 \cdots a_{l-1}G_l$ with $x \in V(G_1)$ and $y \in V(G_l)$. By (E3)–(E5), $|B(G-e)| \geq k-1$.

Let $H = (G-e)_{xy}^{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}$. We shall apply Lemma 4.1 to show that H is minimally (k,k)-edge-connected.

Let $e' \in E(H)$. If $e' \in B(G-e) \cup \{x\alpha_1, \alpha_1\alpha_2, \ldots, \alpha_{k-1}y\}$, then $E((G-e)_{[e']}) \subseteq (B(G-e) \cup \{x\alpha_1, \alpha_1\alpha_2, \ldots, \alpha_{k-1}y\}) - \{e'\}$ and so by $k \geq 2$, $|E(H_{[e']})| \geq k$.

Now assume that $e' \in E(G_i)$ for some non edge pearl G_i. Since G_i is a k-necklace, by Lemma 4.1, $G_{i[e']}$ is a cycle of length at least k. Since G is also a k-necklace, $G_{[e']}$ is also a cycle of length at least k. If there is an edge $e'' \in E(G_{[e']}) - E(G_{i[e']})$, then $\{e', e''\}$ is an edge cut of G and G_i, a subgraph of G, has a cycle contains e' and avoids e'', which is impossible. Therefore, $E(G_{[e']}) \subseteq E(G_{i[e']})$, and so $|E(G_{i[e']})| \geq |E(G_{[e']})| \geq k$.

Moreover, any $e''' \in E(H_{[e']}) - E(G_{i[e']}) \subseteq E(H_{[e']}) - E(G_{[e']})$ must be an edge in $\{x\alpha_1, \alpha_1\alpha_2, \ldots, \alpha_{k-1}y\}$. If such an e''' exists, then e' and $x\alpha_1$ would form an edge cut of H, which would force $e' \in B(G-e)$, contrary to the assumption that $e' \in E(G_i)$ for some non edge pearl G_i.

It follows that $E(H_{[e']}) = E(G_{i[e']})$ and so $|E(H_{[e']})| = |E(G_{i[e']})| \geq k$. By Lemma 4.1, H is minimally (k,k)-edge-connected and so $G-e$ is k-extensible between x and y. ■

Let $\mathcal{F}_k^o$ denote the family of graphs obtained by the following.

(FO1) Every cycle of length $\geq k$ belongs to $\mathcal{F}_k^o$.

(FO2) A graph G that is not a cycle and that is in $\mathcal{F}_k^o$, if and only if, for some integer $l \geq k$, there are l vertex-disjoint graphs $G_1, G_2, \ldots, G_l$ such that

(a) $\forall i$ with $1 \leq i \leq l$, $G_i \in \mathcal{F}_k^o$ or $G_i \cong K_2$, and at least k of the G_i's are isomorphic to a K_2;

(b) there exist two distinct vertices $\{x_i, y_i\} \subset V(G_i)$ such that G_i is k-extensible between x_i and y_i, for each i with $1 \leq i \leq l$; and such that G can be obtained from these G_i's by identifying y_l and x_1, and for $1 \leq i \leq l-1$, identifying y_i and x_{i+1}.

Theorem 4.1. *The family of the k-necklaces is* $\mathcal{F}_k^o$.

Proof. First, we show that every k-necklace belongs to $\mathcal{F}_k^o$. Let G be a k-necklace. If G is a cycle, then by Lemma 4.1, G must be a cycle of length at least k, and so by (FO1), $G \in \mathcal{F}_k^o$. Assume that G is not a cycle. Pick $e = xy \in E(G)$. By Lemma 4.1(iv) and Lemma 4.4, $G - e = G_1a_1G_2a_2 \cdots a_{l-1}G_l$ is an E_k-chain such that $l \geq k$, $x \in V(G_1 - a_1)$ and $y \in V(G_l - a_{l-1})$, and such that $G-e$ is k-extensible between x and y. By Lemma 4.3, this E_k-chain satisfies (FO2) and so $G \in \mathcal{F}_k^o$.

Conversely, let $G \in \mathcal{F}_k^o$. We shall argue by induction on $|V(G)|$ to show that G is a k-necklace. If G is a cycle, then by (FO1), G has length at least k. and so by Lemma 4.1(ii), G is a k-necklace. Hence, we assume that G is obtained via (FO2).

Note that since $x_i \neq y_i$, for each i in (FO2), G is 2-connected. Also, by induction, each G_i in (FO2) is either isomorphic to a K_2 or is a k-necklace.

For any $e \in E(G)$, if e is one of the G_i's which is isomorphic to a K_2, then by (a) of (FO2), there are at least k such G_i's and so $G_{[e]}$ is a cycle of length at least k; if $e \in E(G_i)$ for some $G_i \not\cong K_2$, then since G_i is a k-necklace and is k-extensible between x_i and y_i, by Lemma 4.2, there are at least $k-1$ edges in $B(G_i - e)$ that are not lying between x_i and y_i in the E_k-chain $G_i - e$. It follows that these $k-1$ edges are also in $[e]$, and so $G_{[e]}$ is a cycle of length at least k. By Lemma 4.1, G is a k-necklace. ■

Let $\mathcal{F}_k$ denote the family of graphs obtained by the following.

(F1) $\mathcal{F}_k^o \subset \mathcal{F}_k$.

(F2) A graph $G \in \mathcal{F}_k$, if and only if, either $G \in \mathcal{F}_k^o$, or there are two edge disjoint proper subgraphs G_1 and G_2 of G with $|V(G_1) \cap V(G_2)| = 1$ and $G = G_1 \cup G_2$ such that $G_1, G_2 \in \mathcal{F}_k$.

Theorem 4.2. *The family of 2-edge-connected, minimally (k,k)-edge-connected graphs with at least k vertices is $\mathcal{F}_k$.*

Proof. Let G be a 2-edge-connected, minimally (k,k)-edge-connected graph. If G is also 2-connected, then $G \in \mathcal{F}_k^o \subset \mathcal{F}_k$ by Theorem 4.1. If G has a cut vertex, then by Corollary 1, $G \in \mathcal{F}_k$ by (F2). Hence every 2-edge-connected, minimally (k,k)-edge-connected graph is in $\mathcal{F}_k$.

Conversely, let $G \in \mathcal{F}_k$. If $G \in \mathcal{F}_k^o$, then by Theorem 4.1, G is a 2-edge-connected, minimally (k,k)-edge-connected graph. If G is obtained via (F2), then by Corollary 1, G is also a 2-edge-connected, minimally (k,k)-edge-connected graph. This proves Theorem 4.2. ∎

When $k=2$, one obtains the following former characterizations by Chaty and Chein in [4].

Theorem 4.3. (Chaty and Chein, Theorem 3(a) of [4]). *The family of the 2-necklaces is $\mathcal{F}_2^o$.*

Theorem 4.4. (Chaty and Chein, Theorem 3(b) of [4]). *The family of 2-edge-connected, minimally $(2,2)$-edge-connected graphs is $\mathcal{F}_2$.*

REFERENCES

[1] F. T. Boesch and S. Chen, A generalization of line connectivity and optimally invulnerable graphs, SIAM J Appl Math 34 (1978), 657–665.

[2] J. A. Bondy and U. S. R. Murty, Graph theory with applications, American Elsevier, New York, 1976.

[3] G. Chartrand, S. F. Kapoor, L. Lesniak, and D. R. Lick, Generalized connectivity in graphs. Bull Bombay Math Colloq 2 (1984), 1–6.

[4] G. Chaty and M. Chein, Minimally 2-edge-connected graphs. J Graph Theory 3 (1979), 15–22.

[5] C. C. Chen, K. M. Koh, and Y. H. Peng, On the higher-order of edge-toughness of a graph, Discrete Math 111 (1993), 113–123.

[6] Z. Chen and H.-J. Lai, The higher order edge toughness of a graph and truncated uniformly dense matroids, J Comb Math Comb Computing 22 (1996), 157–160.

[7] D. L. Goldsmith, On the second order edge-connectivity of a graph, Congressus Numerantium 29 (1980), 479–484.

[8] D. L. Goldsmith, On the *n*th order edge-connectivity of a graph, Congressus Numerantium 32 (1981), 375–382.

[9] K. Hennayake, H.-J. Lai, and L. Xu, The strength and the *l*-edge-connectivity of a graph, Bull Inst Comb Appl 26 (1999), 55–70.

[10] O. R. Oellermann, Explorations into graph connectivity, The notices of the South African Mathematical Society 20 (1988), 117–151.

[11] O. R. Oellermann, Generalized connectivity in graphs, Ph.D. Dissertation, Western Michigan University, 1986.

[12] O. R. Oellermann, On the l-connectivity of a graph, Graphs and combinatorics 3 (1987), 285-291.