Eulerian Subgraphs in 3-Edge-Connected Graphs and Hamiltonian Line Graphs

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Received August 9, 2000; Revised September 19, 2002

DOI 10.1002/jgt.10094

Abstract: In this paper, we show that if G is a 3-edge-connected graph with $S \subseteq V(G)$ and $|S| \leq 12$, then either G has an Eulerian subgraph H such that $S \subseteq V(H)$, or G can be contracted to the Petersen graph in such a way that the preimage of each vertex of the Petersen graph contains at least one vertex in S. If G is a 3-edge-connected planar graph, then for any $|S| \leq 23$, G has an Eulerian subgraph H such that $S \subseteq V(H)$. As an application, we obtain a new result on Hamiltonian line graphs.
 J Graph Theory 42: 308–319, 2003

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Keywords: *Eulerian subgraphs; Hamiltonian line graphs; the Petersen graph*

1. INTRODUCTION

We follow Bondy and Murty [4] for terms and notations, unless otherwise stated. Graphs considered in this paper are finite and loopless, but multiple edges are allowed. As in Body and Murty [4], $\kappa(G), \kappa'(G)$, and $d_G(v)$ denote the connectivity of G, the edge-connectivity of G and the degree of a vertex v in G, respectively. Let G be a graph and let $O(G)$ denote the set of odd degree vertices of G. If G is connected and $O(G) = \emptyset$, then G is an *Eulerian* graph. Note that K_1 is Eulerian. A Eulerian subgraph H is called a *dominating* Eulerian subgraph of G if $G - V(H)$ is edgeless.

Let G be a graph and let $X \subseteq E(G)$. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. A graph is *trivial* if it is edgeless. If H is a subgraph of G, then we write G/H for $G/E(H)$. If H is a connected subgraph of G, and if v_H denotes the vertex in G/H to which H is contracted, then H is called the *preimage* of v_H. A vertex v in a contraction of G is *nontrivial* if v has a nontrivial preimage. If $G_0 = G/X$ and if every vertex of G_0 is a nontrivial vertex, then G_0 is a *nontrivial contraction* of G.

For a graph G, the line graph $L(G)$ has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. The following relates dominating Eulerian subgraphs and Hamiltonian line graphs.

Theorem A (Harary and Nash-Williams [11]). *Let G be a graph with at least three edges. Then $L(G)$ is Hamiltonian if and only if G has a dominating Eulerian subgraph.*

In this paper, we first study the existence of an Eulerian subgraph H in a 3-edge-connected graph G such that H contains a given set of vertices of G.

Theorem 1.1. *Let G be a 3-edge-connected graph and let $S \subseteq V(G)$ be a vertex subset such that $|S| \leq 12$. Then either G has an Eulerian subgraph H such that $S \subseteq V(H)$, or G can be contracted to the Petersen graph in such a way that the preimage of each vertex of the Petersen graph contains at least one vertex in S.*

Theorem 1.2. *Let G be a 3-edge-connected planar graph, and let $S \subseteq V(G)$ be a vertex subset such that $|S| \leq 23$. Then there is an Eulerian subgraph in G containing S.*

When G is a cubic graph, $\kappa(G) = \kappa'(G)$ and every Eulerian subgraph H of G is a cycle of G. Therefore, Theorems 1.1 and 1.2 extend the following results in [2] and [1], respectively.

Theorem B (Bau and Holton [2]). *Let G be a 3-connected cubic graph and let $S \subseteq V(G)$ be a vertex subset such that $|S| \leq 12$. Then either G has a cycle H such*

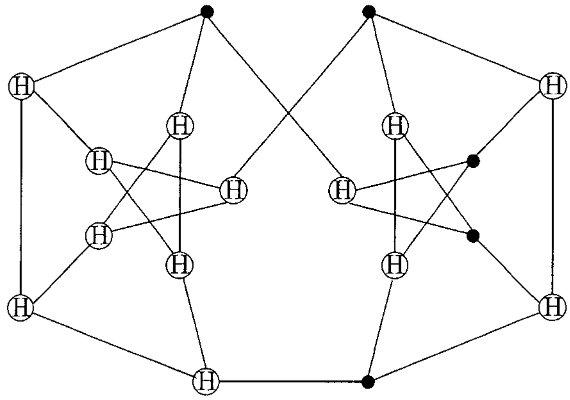

FIGURE 1.

that $S \subseteq V(H)$, or G is contractible to the Petersen graph in such a way that the preimage of each vertex of the Petersen graph contains at least one vertex in S.

Theorem C (Aldred et al. [1]). *If S is a set of at most 23 vertices in a 3-connected cubic planar graph G, then there is a cycle in G containing S.*

Let G be the graph shown in Figure 1 above, where each H is a single vertex (see Holton and Sheehan [13], page 243). If S is the set of those 13 vertices marked by H, then there is no cycle through the 13 vertices. This shows that the requirement on 12 vertices in Theorem 1.1 and Theorem B can not be replaced by 13 vertices. The sharpness of Theorem C and Theorem 1.2 are demonstrated in [12] that there are 3-edge-connected cubic planar graphs in which there is a set of 24 vertices that does not lie on a common cycle.

Next, we apply Theorem 1.1 to Hamiltonian line graphs. For an integer $i \geq 0$,

$$D_i(G) = \{v \in V(G) : d_G(v) = i\}.$$

An edge $e \in E(G)$ is a *pendant edge* if e is incident with a vertex in $D_1(G)$. The following was conjectured by Benhocine et al in 1986 [3], and proved by Veldman in 1994 [15].

Theorem D (Veldman [15]). *Let G be a simple graph on n vertices such that $\kappa'(G - D_1(G)) \geq 2$. If for every edge $uv \in E(G)$,*

$$d_G(u) + d_G(v) > \frac{2n}{5} - 2, \tag{1}$$

then for n sufficiently large, $L(G)$ is Hamiltonian.

When the edge-connectivity is higher, the lower bound in (1) becomes lower as shown in [9]. The following was proved.

Theorem E (Chen and Lai [9]). *Let G be a simple graph on n vertices such that* $\kappa'(G) \geq 3$. *If for every edge* $uv \in E(G)$,

$$d_G(u) + d_G(v) \geq \frac{n}{6} - 2, \tag{2}$$

then for n sufficiently large, either $L(G)$ is Hamiltonian, or the Petersen graph is a nontrivial contraction of G.

Theorem E improves a previous result in [7] and [15]. The authors in [9] conjectured that the lower bound in Theorem E can be reduced to $n/9 - 1$, with the conclusion that either $L(G)$ is Hamiltonian or G is contractible to the Petersen graph. This conjecture, if proved, will be best possible, due to a construction using the Blanuša snarks [9]. For more on the literature of related problems, readers are referred to Caltin's survey [6] and its update [8].

Noting that (1) implies that

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{5} - 1, \text{ for every edge } uv \in E(G) \tag{3}$$

Lai considered (3) as a relaxation of (1), and made an improvement of Theorem D.

Theorem F (Lai [14]). *Let G be a simple graph on n vertices such that* $\kappa'(G - D_1(G)) \geq 2$. *If (3) holds, then for n sufficiently large, either* $L(G)$ *is Hamiltonian, or (1) is violated and G can be contracted to one of seven specified graphs.*

The following theorem extends Theorem E.

Theorem 1.3. *Let G be a simple graph on n vertices such that* $\kappa'(G - (D_1(G) \cup D_2(G))) \geq 3$, *and let* $\varepsilon \geq 1$ *be a constant. If*

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{12} - \varepsilon, \text{ for every edge } u(v) \in E(G), \tag{4}$$

then for n sufficiently large, $L(G)$ is Hamiltonian if and only if G does not have the Petersen graph as a nontrivial contraction.

Remark. $n/12$ *can not be replaced by* $n/13$ *in Theorem 1.3. Let r be an integer, and* $n = 13r + 5$. *Let G be the graph in Figure 1, where each H is a complete graph* K_r. *Then G is a graph of order* $n = 13r + 5$. *Obviously, one can see that for every edge* $uv \in E(G)$, $\max\{d_G(u), d_G(v)\} \geq r - 1 = \frac{n-5}{13} - 1 > \frac{n}{13} - 2$. *Although G can be contracted to the Petersen graph, at least one vertex of the Petersen graph is a trivial contraction. Also* $L(G)$ *is not Hamiltonian. The statement of Theorem 1.3 is not true for this case.*

In Section 2, we present Catlin's reduction method, which is the needed mechanism in the proof of Theorem 1.3. In Section 3, we assume the truth of Theorem 1.1 to prove Theorem 1.3. The last section will be devoted to the proofs of Theorems 1.1 and 1.2.

2. CATLIN'S REDUCTION METHOD

Let G be a graph and let $F \subseteq V(G)$ be a vertex subset. An Eulerian subgraph H of G is called an *F-Eulerian subgraph* if $F \subseteq V(H)$. A graph G is *supereulerian* if it has a spanning Eulerian subgraph, i.e., a $V(G)$-Eulerian subgraph (See Catlin [6] for supereulerian graphs). Catlin [5] invented a reduction method to find spanning Eulerian subgraphs for given G.

A graph G is *collapsible* if for every subset $R \subseteq V(G)$ with $|R|$ even, G has a spanning connected subgraph H_R such that $O(H_R) = R$. In [5], Catlin showed that every graph G has a unique collection of maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The *reduction* of G is $G' = G/(\cup_{i=1}^{c} H_i)$, the graph obtained from G by contracting all nontrivial maximal collapsible subgraphs of G. A graph G is reduced if the reduction of G is G. Note that any subgraph of a *reduced* graph is reduced.

Theorem G (Theorem 8 of [5]). *If G is a reduced graph, then G is simple and K_3-free, with $\delta(G) \leq 3$, and for any subgraph H of G, either $H \in \{K_1, K_2\}$ or $|E(H)| \leq 2|V(H)| - 4$.*

Lemma 2.1 (Theorem 3 of [5]). *Let G be a graph. Let L be a collapsible subgraph of G, and let v_L be the vertex in G/L to which L is contracted, and $M \subseteq V(G) - V(L)$. Then G has an Eulerian subgraph H such that $M \cup V(L) \subseteq V(H)$ if and only if G/L has an Eulerian subgraph H' such that $M \cup \{v_L\} \subseteq V(H')$.*

Let G be a graph. Let $v \in D_2(G)$ be a vertex which is incident with edges e_1, e_2. We say that the contraction G/e_1 is the graph obtained by *eliminating a degree 2 vertex v*. Define $\tilde{G}$ to be the graph obtained from $G - D_1(G)$ by eliminating all vertices in $D_2(G)$. Since deleting vertices in $D_1(G)$ is equivalent to contracting all the pendant edge in G, one can always view $\tilde{G}$ as a contraction of G.

Lemma 2.2 (Proposition 3.2 of [14]). *Let G be a graph and* let $F \subseteq V(G) - (D_1(G) \cup D_2(G))$. *The following are equivalent.*

(i) G has an F-Eulerian subgraph.
(ii) $\tilde{G}$ has an F-Eulerian subgraph.

Let H be a collapsible subgraph of G, and let $G' = G/H$. Let v_H denote the vertex in G' onto which the subgraph H is contracted. Let $F \subseteq V(G)$ be a vertex subset. Define $F' \subseteq V(G')$ such that

$$F' = \begin{cases} F & \text{if } F \cap V(H) = \emptyset \\ (F - V(H)) \cup \{v_H\} & \text{if } F \cap V(H) \neq \emptyset. \end{cases}$$

Lemma 2.3. *Let G be a graph and let $F \subseteq V(G) - (D_1(G) \cup D_2(G))$. Let H be a collapsible subgraph of G, and let H' denote the graph obtained from H by eliminating all vertices in $D_2(G) \cap D_2(H)$. (Thus H' is a subgraph of $\tilde{G}$.) Let G'' denote $\tilde{G}/H'$. The following are equivalent.*

(i) G has an F-Eulerian subgraph which is either disjoint from H, or contains every vertex of H.
(ii) G'' has an F'-Eulerian subgraph.

Proof. Since $F \subseteq V(G) - (D_1(G) \cup D_2(G))$, one can view $F \subseteq V(\tilde{G})$, and so $F' \subseteq V(G'')$.

If L is an F-Eulerian subgraph satisfying (i), and if X is the edge subset such that $G'' = G/X$, then $L/(X \cap E(L))$ is an F'-Eulerian subgraph in G''.

Suppose that G'' has an F'-Eulerian subgraph L'. By Lemma 2.2 and by Lemma 2.1, if $v_H \in V(L')$, then G has an F-Eulerian subgraph L that contains every vertex of H; if $v_H \notin V(L')$, then G has an F-Eulerian subgraph that does not intersect H, and so (i) must hold. ■

3. PROOF OF THEOREM 1.3

Throughout this section, we assume that Theorem 1.1 holds. Let G be a simple graph with n vertices. Following closely the method of Lai [14], we consider the condition

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{p} - \varepsilon, \text{for every edge } uv \in E(G), \tag{5}$$

where ε is a constant. Let $p \geq 2$ be an integer, and define

$$J_p(G) = \left\{ v \in V(G) \ : \ d(v) \geq \frac{n}{p} - \varepsilon \right\}.$$

Lemma 3.1. *Suppose that (5) holds for a simple graph G with n vertices. Let $\tilde{G}'$ denote the reduction of $\tilde{G}$, and let W' denote the set of all nontrivial vertices of $\tilde{G}'$. Then there exists a number $n(p)$ such that when $n \geq n(p)$, each of the following holds.*

(i) $|W'| \leq p$.
(ii) Every vertex in $J_p(G)$ lies in the preimage of a vertex in W'.
(iii) If $\tilde{G}'$ has a W'-Eulerian subgraph, then G has a dominating Eulerian subgraph.

Proof. Note that $\tilde{G}'$ is reduced by definition. Let $n' = |V(\tilde{G}')|$. Let

$$c = 3p + 7 \tag{6}$$

and let

$$W = \{v \in V(\tilde{G}') : d_{\tilde{G}'}(v) \leq c\} \text{ and } W' = \{v \in W : \{v \text{ is nontrivial}\}. \quad ■$$

Claim 1. *For any $v \in W'$, if H_v denotes the preimage of v in G, then*

$$|V(H_v)| \geq \frac{n}{p} - \varepsilon + 1 - d_{\tilde{G}'}(v).$$

Proof. Let $Out(H_v) = \{x \in V(H_v) : N_G(x) \neq N_{H_v}(x)\}$ and $In(H_v) = V(H_v) - Out(H_v)$. If there is an edge $xy \in E(H_v)$ such that $d(x) \geq d(y)$ and $x \in In(H_v)$, then by (5), $|V(H_v)| \geq d(x) + 1 \geq n/p - \varepsilon + 1$, and so Claim 1 holds. Therefore, we assume that for any edge $xy \in E(H_v)$ with $d(x) \geq d(y)$, we always have $x \in Out(H_v)$. Thus $|Out(H_v)| \geq 1$, and so by (5)

$$|V(H_v)| = |In(H_v)| + |Out(H_v)| \geq 1 + d_G(x) - d_{\tilde{G}'}(v) \geq \frac{n}{p} - \varepsilon + 1 - d_{\tilde{G}'}(v).$$

Claim 1 is proved. ∎

Claim 2. $|W'| \leq p$.

Proof. Since $d_{\tilde{G}'}(v) \leq c$ for $v \in W$, and $W' \subset W$, by Claim 1, we have $n \geq |W'|(n/p - \varepsilon + 1 - c)$. Thus, $|W'| \leq np/(n - p(\varepsilon + c - 1))$. Since ε and c are constants, when n is sufficiently large (say $n > (p+1)p(\varepsilon + c - 1)$), $|W'| \leq p$. This proves Claim 2. ∎

Claim 3. $V(\tilde{G}') = W$.

Proof. By contradiction, we assume that $V(\tilde{G}') - W \neq \emptyset$. Note that every vertex in $V(\tilde{G}') - W$ has degree at least $c + 1$ in $\tilde{G}'$. Since $\tilde{G}'$ is simple and K_3-free by Theorem G, this means that

$$n' = |V(\tilde{G}')| \geq c + 2. \tag{7}$$

Count the incidences to get $c|V(\tilde{G}') - W| \leq 2|E(\tilde{G}')| \leq 4n' - 8$, which means $|V(\tilde{G}') - W| \leq (4n' - 8)/c$. It follows that

$$|W| = n' - |V(\tilde{G}') - W| \geq (1 - 4/c)n' + \frac{8}{c}. \tag{8}$$

Note that every vertex in $W - W'$ has degree at least 3 in $\tilde{G}'$, and when n is sufficiently large, $W - W'$ is independent in $\tilde{G}'$. By Theorem G, by Claim 2 and by (8)

$$2n' - 4 \geq |E(\tilde{G}')| \geq 3|W - W'| \geq (3 - 12/c)n' + \frac{24}{c} - 3p.$$

It follows that

$$3p - 4 \geq \left(1 - \frac{12}{c}\right)n' + \frac{24}{c}. \tag{9}$$

By (7), $n' \geq c + 2$. Thus, (9) implies that

$$3p - 4 \geq \left(1 - \frac{12}{c}\right)(c + 2) + \frac{24}{c} = c - 10,$$

and so by (6) $3p + 6 \geq c = 3p + 7$, a contradiction. This shows that $V(\tilde{G}') = W$. ■

Claim 3 implies that W' is the set of all nontrivial vertices of $\tilde{G}'$. This shows that Lemma 3.1 (i) holds.

Claim 4. *Every vertex in $J_p(G)$ is contained in the preimage of some vertex in W'.*

Proof. Since ε and p are constants, when n is large enough, the degree of vertices in $J_p(G)$ will exceed c, and so Claim 4 follows from Claim 3. This shows that Lemma 3.1 (ii) holds.

It remains to prove Lemma 3.1(iii). By the hypothesis of Lemma 3.1(iii), by Lemma 2.3 and by Lemma 3.1(ii), G has an Eulerian subgraph L such that $J_p(G) \subseteq V(L)$. Since (5) holds for G, $G - V(L)$ must be edgeless, and so L is dominating. The proof is complete. ■

Lemma 3.2. *Suppose that $D_2(G)$ is an independent set of G. If $\kappa'(G - (D_1(G) \cup D_2(G))) \geq 3$, then $\kappa'(\tilde{G}) \geq 3$.*

Proof. Let $G_{1,2} = G - (D_1(G) \cup D_2(G))$ and let $D_2 = \{z_1, z_2, \ldots, z_s\}$. For each z_i, let u_i and v_i be the two neighbors of z_i, where $1 \leq i \leq s$. Note that $\tilde{G}$ is obtained from $G_{1,2}$ by adding edges $\{u_i v_i : 1 \leq i \leq s\}$ and deleting all possibly resulting loops (which may occur when $u_i = v_i$, for some i). Since $\kappa'(G_{1,2}) \geq 3$, it follows that $\kappa'(\tilde{G}) \geq 3$. ■

Lemma 3.3. *If $\kappa'(G) \geq 3$ and if G' is the reduction of G, then $\kappa'(\tilde{G}') \geq 3$.*

Proof Let $X \subseteq E(\tilde{G}')$ be an edge cut of $\tilde{G}'$. Note that by the definition of $\tilde{G}$, X can be viewed as a subset of $E(G)$. Therefore, X is also an edge cut of G. Since $X \subseteq E(\tilde{G}')$, X does not contain any pendant edges of G, and so X cannot consist of the two edges incident with a vertex in $D_2(G)$. Therefore, X is an edge cut of $\tilde{G}$. Since it is assumed that $\kappa'(G) \geq 3$, one must have $|X| \geq 3$ and so $\kappa'(\tilde{G}') \geq 3$. ■

We shall prove a slightly stronger version of Theorem 1.3. By Lemma 3.2, Theorem 3.4 below implies Theorem 1.3.

Theorem 3.1. *Let G be a simple graph with n vertices such that $\kappa'(\tilde{G}) \geq 3$. If (5) holds, then for n sufficiently large, $L(G)$ is Hamiltonian if and only if G does not have the Petersen graph as a nontrivial contraction.*

Proof. Let G' denote the reduction of G, and let $\tilde{G}'$ denote the reduction of $\tilde{G}$. Let W' denote the set of nontrivial vertices of G. By Lemma 3.1(i) with $p = 12$, $|W'| \leq 12$. Since $\kappa'(\tilde{G}) \geq 3$, one has $W' \cap (D_1(G') \cup D_2(G')) = \emptyset$. Therefore, $W' \subseteq V(\tilde{G}')$.

Since $|W'| \leq 12$ and since $\kappa'(\tilde{G}') \geq 3$, it follows by Theorem 1.1 that either $\tilde{G}'$ has the Petersen graph as a nontrivial contraction, or $\tilde{G}'$ has a W'-Eulerian subgraph H'.

Since $\tilde{G}'$ is a contraction of G', and G' is a contraction of G, if the Petersen graph is a nontrivial contraction of $\tilde{G}'$, then the Petersen graph is also a nontrivial contraction of G. Note that if G has the Petersen graph as a nontrivial contraction, then $L(G)$ cannot be Hamiltonian, and so we are done. Therefore, we assume that $\tilde{G}'$ has a W'-Eulerian subgraph H'. By Lemma 3.1, we conclude that G has an Eulerian subgraph H that contains all vertices of $J_{12}(G)$, and G has a dominating Eulerian subgraph. Therefore by Theorem A, $L(G)$ is Hamiltonian. ■

4. PROOFS OF THEOREM 1.1 AND THEOREM 1.2

We shall argue by contradiction. Some more notations and lemmas are needed.

Let G be a graph and let $v \in V(G)$. Then $N_G(v)$ denote the set of vertices in $V(G)$ that are adjacent to v in G; $E_G(v)$ denote the set of edges incident with v in G and for each $i \geq 1$,

$$D_i^*(G) = \bigcup_{j \geq i} D_j(G).$$

For $v \in D_4^*(G)$, let $N_G(v) = \{v_1, v_2, \ldots, v_d\}$, where $d = d_G(v) \geq 4$. For a 4-cycle C_4, let $V(C_4) = \{x, y, z, w\}$ and let $E(C_4) = \{xy, yz, zw, wx\}$. Let G_v be a graph obtained from $G - v$ and C_4 by joining x to v_1, y to v_2, z to v_3, and w to v_i for all $i \geq 4$ as shown below.

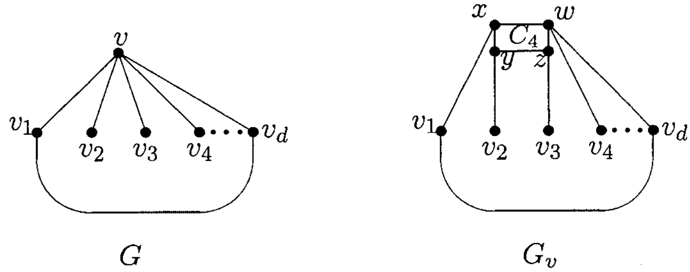

FIGURE 2.

Lemma 4.1. *Let G be a 3-edge-connected graph and let $v \in D_4^*(G)$. If v is not a cut-vertex, then G_v is 3-edge-connected.*

Proof. By way of contradiction, suppose that G_v has an edge cut $E_0 \subseteq E(G)$ with $|E_0| \leq 2$. Let G_1 and G_2 be the two components of $G_v - E_0$. Note that if all the vertices of $N_G(v) = \{v_1, v_2, \ldots, v_d\}$ are in the same component (say G_1), then E_0 is an edge cut of G with $|E_0| \leq 2$, contrary to that G is 3-edge-connected. Therefore, $E_0 \subseteq E(C_4) = \{xy, yz, zw, wx\}$ and $|E_0| = 2$. However, this implies that v is a cut-vertex, a contradiction. Hence, G_v is 3-edge-connected. ∎

Let $S \subseteq V(G)$ be a vertex set, and let $v \in D_4^*(G)$ and $v' \in V(C_4)$. Define

$$S' = \begin{cases} S & \text{if } v \notin S, \\ (S - v) \cup v' & \text{otherwise.} \end{cases}$$

Then $|S'| = |S|$ and $S' \subseteq V(G_v)$.

Lemma 4.2. *Let G be a graph, and let $v \in D_4^*(G)$. Let S be a vertex subset of $V(G)$, and S' be the set defined above. Then each of the following holds*:

- (i) *If G_v has an Eulerian subgraph H_1 such that $S' \subseteq V(H_1)$, then G has an Eulerian subgraph H such that $S \subseteq V(H)$.*
- (ii) *If G_v can be contracted to the Petersen graph such that the contraction preimage of each vertex in the Petersen graph contains at least one vertex in S' then G can be contracted to the Petersen graph such that the contraction preimage of each vertex in the Petersen graph contains at least one vertex in S.*

Proof. Obvious. ∎

Proof of Theorem 1.1. We argue by induction on

$$f(G) = \sum_{v \in D_4^*(G)} d_G(v).$$

If $f(G) = 0$, then G is a cubic 3-connected graph, and so the theorem follows from Theorem B. Assume that $f(G) > 0$. Then $|D_4^*(G)| \geq 1$. Pick $v \in D_4^*(G)$. If v is not a cut-vertex, then we define G_v as shown in Figure 2. By Lemma 4.1, G_v is also 3-edge-connected. By the definition of $f(G)$ and G_v, we have $f(G_v) = f(G) - 1$. Pick $v' \in V(C_4) = \{x, y, z, w\}$. Define

$$S' = \begin{cases} S & \text{if } v \notin S, \\ (S - v) \cup v' & \text{otherwise.} \end{cases}$$

Then $|S'| = |S| \leq 12$ and $S' \subseteq V(G_v)$. By induction, either G_v has an S'-Eulerian subgraph H', or G_v can be contracted to the Petersen graph such that the contraction preimage of each vertex in the Petersen graph contains at least one vertex in S'. Therefore, Theorem 1.1 follows from Lemma 4.2.

Next we only need to consider the case that v is a cut-vertex.

Let H_1 and H_2 be the two components of $G - v$. Let $G_1 = G[V(H_1) \cup v]$ and $G_2 = G[V(H_2) \cup v]$. Note that since G is 3-edge-connected, G_i $(i = 1, 2)$ is also 3-edge-connected. Obviously,

$$f(G_i) < f(G) \quad (i = 1, 2). \tag{10}$$

Let $S_i = S \cap V(G_i)$ $(i = 1, 2)$. We may assume that $|S_2| \le |S_1|$. If $S_2 = \emptyset$, then $S \subseteq V(G_1)$. By (10) and then by induction, the theorem statement holds for G_1, and so the theorem holds in this case. Without loss of generality, we assume that $1 \le |S_2| \le |S_1| \le 11$. For $i = 1, 2$, define

$$S'_i = \begin{cases} S_i & \text{if } v \in S, \\ S_i \cup v & \text{otherwise.} \end{cases}$$

Then $2 \le |S'_2| \le |S'_1| \le 12$. By (10) and then by induction, we know that either G_i has an Eulerian subgraph H_i such that $S'_i \subseteq V(H_i)$ or G_i can be contracted to the Petersen graph such that the contraction preimage of each vertex in the Petersen graph contains at least one vertex of S'_i. There are two cases to be considered here.

Case 1. G_1 is contractible to the Petersen graph such that the contraction preimage of each vertex in the Petersen graph contains at least one vertex of S'_1.

Since $v \in S'_1$, v is in one of the preimages of a vertex in the Petersen graph. Note that $G_1 = G/G_2$. Then G can be contracted to the Petersen graph in such a way that by contracting G_2 to v, and then successively contracting the preimage of each vertex of the Petersen graph in G_1. Obviously, The contraction preimage of each vertex in the Petersen graph contains at least one vertex of S. The theorem is proved in this case.

Case 2. G_1 has an Eulerian subgraph H_1 such that $S'_1 \subseteq V(H_1)$.

Since $|S'_2| \le |S'_1| \le 12$, $|S'_2| \le 6$. By (10) and by induction, G_2 has an Eulerian subgraph H_2 such that $S'_2 \subseteq V(H_2)$. Since $v \in S'_i \subseteq V(H_i)$, $H = H_1 \cup H_2$ is an Eulerian subgraph in G with $S \subseteq V(H) = V(H_1) \cup V(H_2)$. The proof is completed. ■

Remark. *Theorem* 1.2 *can be proved by using Theorem C and the same techniques used in the proof of Theorem* 1.1.

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