Nowhere-Zero 3-Flows in Locally Connected Graphs

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Abstract: Let G be a graph. For each vertex $v \in V(G)$, N_v denotes the subgraph induces by the vertices adjacent to v in G. The graph G is locally k-edge-connected if for each vertex $v \in V(G)$, N_v is k-edge-connected. In this paper we study the existence of nowhere-zero 3-flows in locally k-edge-connected graphs. In particular, we show that every 2-edge-connected, locally 3-edge-connected graph admits a nowhere-zero 3-flow. This result is best possible in the sense that there exists an infinite family of 2-edge-connected, locally 2-edge-connected graphs each of which does not have a 3-NZF. © 2003 Wiley Periodicals, Inc. J Graph Theory 42: 211–219, 2003

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1. INTRODUCTION

Graphs in this note are finite and may have loops and multiple edges. Undefined terms and notation may be found in [1]. A connected graph is *nontrivial* if it has at least 2 vertices.

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Let $D = D(G)$ be an orientation of an undirected graph G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let *tail*$(e) = u$ and *head*$(e) = v$. For a vertex $v \in V(G)$, let

$$E_D^-(v) = \{e \in E(D) : v = \text{tail}(e)\}, \quad \text{and} \quad E_D^+(v) = \{e \in E(D) : v = \text{head}(e)\}.$$

The subscript D may be omitted when $D(G)$ is understood from the context. Let $E_G(v)$ denote the subset of edges incident with v in G.

Let A denote an (additive) Abelian group with identity 0, and let $F(G,A)$ denote the set of all functions from $E(G)$ to A. Given a function $f \in F(G,A)$, let $\partial f : V(G) \mapsto A$ be given by

$$\partial f(v) = \sum_{e \in D^+(v)} f(e) - \sum_{e \in D^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. Unless otherwise stated, we shall adopt the following convention: if $X \subseteq E(G)$ and if $f : X \mapsto A$ is a function, then we regard f as a function $f : E(G) \mapsto A$ such that $f(e) = 0$ for all $e \in E(G) - X$. We also use notation (D,f) for a function $f \in F(G,A)$ to emphasize the orientation D.

Let $\mathbf{Z}$ denote the set of integers. For an integer $k \geq 2$, a *nowhere-zero k-flow* (abbreviated as a k-NZF) of G is an orientation D of G together with a map $f \in F(G,\mathbf{Z})$ such that for each $e \in E(G)$, $0 < |f(e)| < k$, and for each vertex $v \in V(G)$, $\partial f(v) = 0$. As noted in [3], the existence of a nowhere-zero k-flow of a graph G is independent of the choice of the orientation D.

Tutte has several well known conjectures on the existence of k-NZFs ([3]).

Tutte's 3-flow conjecture. *Every 4-edge-connected graph admits a 3-NZF.*

Tutte's 4-flow conjecture. *Every* 2*-edge-connected graph either admits a* 4*-NZF, or has a subgraph contractible to the Petersen graph.*

Tutte's 5-flow conjecture. *Every* 2*-edge-connected graph admits a 5-NZF.*

Jeager [4] showed that every 2-edge-connected graph has an 8-NZF, and Seymour [8] improved Jaeger's result by showing that every 2-edge-connected graph has a 6-NZF.

Let G be a graph. For each vertex $v \in V(G)$, $N_{G,v}$ denotes the subgraph induced by the vertices adjacent to v in G. When no confusion arises, we use N_v to denote $N_{G,v}$. The graph G is *locally k-edge-connected* if for each vertex $v \in V(G)$, N_v is k-edge-connected. Note that K_1, the edgeless graph with just one vertex, is locally k-edge-connected for any integer $k \geq 1$. Note that any connected, locally connected graph G with $|V(G)| \geq 3$ must be 2-edge-connected.

It has been observed (Corollary 2.1 below) that every locally connected graph has a k-NZF, for any $k \geq 4$. In this paper, we shall prove the following:

Theorem 1.1. *Every 2-edge-connected, locally 3-edge-connected graph has a 3-NZF.*

In the last section, we construct an infinite family of graphs that are 2-edge-connected and locally 2-edge-connected, but no graph in this family has a 3-NZF. Thus, the locally 3-edge-connected condition in Theorem 1.1 may not be relaxed. A stronger version of Theorem 1.1 will be proved in the last section. In the next section, we display and develop some tools needed for the proof of the main result.

2. GROUP CONNECTIVITY OF A GRAPH

Throughout this section, A denotes an Abelian group with $|A| \geq 3$ and G denotes an undirected graph. The propositions and lemmas developed in this section will be needed in the proofs in Section 3.

For a subset $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. Note that even when G is simple, G/X may have multiple edges. For convenience, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G, then G/H denotes $G/E(H)$.

Fix an orientation D of G. Let A be a nontrivial Abelian group with identity 0, and let A^* denote the set of nonzero elements in A. Recall $F(G,A) = \{f : E(G) \mapsto A\}$ and define $F^*(G,A) = \{f : E(G) \mapsto A^*\}$.

Let $Z(G,A)$ denote the collection of all functions $b : V(G) \mapsto A$ satisfying $\sum_{v\in V(G)} b(v) = 0$. For a given $b \in Z(G,A)$, a function $f \in F^*(G,A)$ is an (A,b)-*NZF* if $\partial f = b$. A graph G is A-*connected* if G has an orientation D such that for every function $b \in Z(G,A)$, there is an (A,b)-NZF $f \in F^*(G,A)$. For an Abelian group A, let $\langle A\rangle$ denote the family of graphs that are A-connected. As noted in [5], $G \in \langle A\rangle$ is independent of the orientation D of G. An A-*nowhere-zero-flow* (abbreviated as an A-NZF) in G is a function $f \in F^*(G,A)$ such that $\partial f = 0$. (See Jaeger's survey [3] for more literature on nowhere-zero-flow problems).

Theorem 2.1 (Tutte [12]). *Let A be an Abelian group with $|A| = k$. Then a graph G has an A-NZF, if and only if, G has a k-NZF.*

Following Jaeger [3], for an integer $k \geq 2$, F_k denotes the collection of all graphs admitting a k-NZF. Let Z_k denote the cyclic group of order k. It follows by Theorem 2.1 that $\langle Z_k\rangle \subseteq F_k$.

The concept of A-connectivity was introduced by Jaeger et al in [5], where A-NZF's were successfully generalized to A-connectivities. A concept similar to the group connectivity was independently introduced in [6], with a different motivation from [5]. Proposition 2.1 below summarizes some former observations on group connectivities. The proofs for parts (i)–(iv) of Proposition 2.1 are in [7], while Parts (v) and (vi) are in [5].

Proposition 2.1. *Let H be a subgraph of G. Then each of the following holds.*

(i) *If $H \in \langle A \rangle$ and if $e \in E(H)$, then $H/e \in \langle A \rangle$.*
(ii) *If $H \in \langle A \rangle$*, then *$G/H \in \langle A \rangle \Leftrightarrow G \in \langle A \rangle$.*
(iii) *If $H \in \langle Z_k \rangle$, then $G/H \in F_k \Leftrightarrow G \in F_k$.*
(iv) *The wheel W_4 with 5 vertices is in $\langle A \rangle$, for each Abelian group A with $|A| \geq 3$.*
(v) *Let C_n denote the cycle of length n (also called an n-cycle), where $n \geq 2$, and let A be an Abelian group. Then, $C_n \in \langle A \rangle$ if and only if $|A| \geq n + 1$.*
(vi) *If $G \in \langle A \rangle$, then G is connected.*

Lemma 2.1. *Let T be a connected spanning subgraph of G. If for each edge $e \in E(T)$, G has a subgraph $H_e \in \langle A \rangle$ with $e \in E(H_e)$, then $G \in \langle A \rangle$.*

Proof. We argue by induction on $|V(G)|$. Lemma 2.1 holds if $|V(G)| = 1$. Assume that $|V(G)| > 1$ and pick an edge $e' \in E(T)$. Then, G has a subgraph $H' \in \langle A \rangle$ such that $e' \in E(H')$. Let $G' = G/H'$ and let $T' = T/(E(H') \cap E(T))$. Since T is a connected spanning subgraph of G, T' is a connected spanning subgraph of G'. For each $e \in E(T')$, $e \in E(T)$, and so by assumption, G has a subgraph $H_e \in \langle A \rangle$ with $e \in E(H_e)$. By Proposition 2.1(i), $H'_e = H_e/(E(H_e) \cap E(H')) \in \langle A \rangle$ and $e \in H'_e$. Therefore, by induction $G' \in \langle A \rangle$. Then, by Proposition 2.1(ii), and by the assumption that $H' \in \langle A \rangle$, $G \in \langle A \rangle$. ■

Note that in a locally connected graph, every edge lies in a 3-cycle. An immediate consequence of Proposition 2.1 and Lemma 2.1 is the following.

Corollary 2.1. *Let A be an Abelian group with $|A| \geq 4$. Every 2-edge-connected, locally connected graph G is in $\langle A \rangle$.*

Let G be a graph and let H_1 and H_2 be two subgraphs of G. Then, $H_1 \cup H_2$ denotes the subgraph of G whose vertex set is $V(H_1) \cup V(H_2)$, and whose edge set is $E(H_1) \cup E(H_2)$.

Lemma 2.2. *Let A be an Abelian group with $|A| \geq 3$ and let H_1, H_2 be subgraphs of G such that $H_1, H_2 \in \langle A \rangle$. If $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2 \in \langle A \rangle$.*

Proof. By Proposition 2.1(vi), both H_2 are connected. As $V(H_1) \cap V(H_2) \neq \emptyset$, G is connected and so G has a spanning tree T. Each edge $e \in E(T)$ is either in $E(H_1)$ or in $E(H_2)$, and so by Lemma 2.1, $G \in \langle A \rangle$. ■

Lemma 2.3. *Let G be a graph. For every vertex $v \in V(G)$, there is a unique maximal subgraph H_v of G that is A-connected and contains v.*

Proof. Since $K_1 \in \langle A \rangle$, every vertex $v \in V(G)$ is in a maximal subgraph $H(v) \in \langle A \rangle$. Let H denote the union of these subgraphs $\{H(v) : v \in V(G)\}$. We need to show that H is unique. Suppose H' is another subgraph of G such that each component of H' is a maximal subgraph of G in $\langle A \rangle$. For each vertex $v \in V(G)$, let $H'(v)$ denote the component of H' that contains v. If $H \neq H'$, then there must be a vertex $v \in V(G)$, such that $H(v) \neq H'(v)$. Since both $H(v), H'(v) \in \langle A \rangle$ and since $v \in V(H(v)) \cap V(H'(v)) \neq \emptyset$, by Lemma 2.2, $H(v) \cup H'(v) \in \langle A \rangle$. As both $H(v)$ and $H'(v)$ are maximal subgraphs in $\langle A \rangle$, it must be $H(v) = H(v) \cup H'(v) = H'(v)$, contrary to the assumption that $H(v) \neq H'(v)$. ■

Corollary 2.2. *Let H be a subgraph of G such that $H \in \langle A \rangle$, and let v_H denote the vertex in G/H into which H is contracted. If $L' \in \langle A \rangle$ is a subgraph of G/H such that $v_H \in V(L')$, then $L = G[E(L') \cup E(H)]$, an edge induced subgraph, is also in $\langle A \rangle$.*

Proof. Note that H is a subgraph of L and $L/H = L'$. Thus, Corollary 2.2 follows from Proposition 2.1(ii). ■

Therefore, for each graph G and for a fixed abelian group A, G has a unique subgraph $M_A(G)$ such that each component of $M_A(G)$ is a maximal subgraph of G that is in $\langle A \rangle$. Call the contraction $R_A(G) = G/M_A(G)$ the *A-reduction* of G. If $G = R_A(G)$, then G is *A-reduced*.

Corollary 2.3. *Let G be a graph. Each of the following holds.*

(i) $R_A(R_A(G)) = R_A(G)$.
(ii) $G \in \langle A \rangle$ *if and only if* $R_A(G) \cong K_1$.

Proof. The proof for part(ii) is trivial and so we only prove part(i). Note that the vertices of $R_A(G)$ are in 1–1 correspondence with the components of $M_A(G)$. If $R_A(G)$ is not reduced, then there is a nontrivial component C of $M_A(R_A(G))$. Expanding C back to G, we get a union U of at least two components of $M_A(G)$ that contract to the vertices of C. The subgraph of G induced by the vertices of U is connected and is in $\langle A \rangle$ by Corollary 2.2, contrary to the fact that each component of $M_A(G)$ is a maximal subgraph that is in $\langle A \rangle$. Thus, $M_A(G)$ has no nontrivial subgraphs that are in $\langle A \rangle$, and so $M_A(R_A(G))$ is an edgeless spanning subgraph of $R_A(G)$. It follows that $R_A(R_A(G)) = R_A(G)/M_A(R_A(G)) = R_A(G)$. ■

Let $v \in V(G)$ be a vertex. Partition the edges of G incident with v into two nonempty sets E' and E''. Split v into two vertices v' and v'', each incident with edges in E' and E'', respectively. This yields the graph G_v, which is an *elementary detachment* of G. A *detachment* of G is a graph Γ obtained from G by a finite number of elementary detachments.

Lemma 2.4. *Let G_v be an elementary detachment of G and let $b \in Z(G, A)$. Define $b' : V(G_v) \mapsto A$ by*

$$b'(z) = \begin{cases} b(v) & \text{if } z = v' \\ 0 & \text{if } z = v'' \\ b(z) & \text{otherwise.} \end{cases} \quad (1)$$

If G_v has an (A, b')-NZF, then G has an (A, b)-NZF.

Proof. Note that $b' \in Z(G_v, A)$. By assumption, there is an (A, b')-NZF $f \in F^*(G_v, A)$ such that $\partial f = b'$. However, as $E(G_v) = E(G), f \in F^*(G, A)$ also. By the definition of b', we also have $\partial f = b$, and so f is an (A, b)-NZF, as desired. ■

By Lemma 2.4. and by induction, it is easy to see that if a detachment of G is in $\langle A \rangle$, then $G \in \langle A \rangle$. Lemma 2.4 also has the following corollary.

Corollary 2.4. *Let G be a graph, and $v \in V(G)$ be a vertex of degree at least 4 in G. Let e_1, e_2 be two edges in G incident with v. Let G_v be the graph obtained from G by splitting v into v' and v'' such that v'' is incident with e_1 and e_2, while v' is incident with all the other edges formerly incident with v in G. If $G_v/e_1 \in \langle A \rangle$, then $G \in \langle A \rangle$.*

Proof. We use the same notations in Lemma 2.4. It suffices to show that for each $b \in Z(G, A)$, G has an (A, b)-NZF.

Fix a function $b \in Z(G, A)$. Define b' as in (1), and let b'' be the restriction of b' in $V(G_v) - \{v''\}$. Since $b'(v'') = 0$, $b'' \in Z(G_v/e_1, A)$. *By assumption*, G_v/e_1 has an (A, b'')-NZF f''. Define $f : E(G) \mapsto A$ by

$$f(e) = \begin{cases} f''(e) & \text{if } e \neq e_1 \\ f''(e_2) & \text{if } e = e_1. \end{cases}$$

As $b'(v'') = 0$ and $f'' \in F^*(G_v/e_1, A)$, it is easy to check that $\partial f = b'$ and $f \in F^*(G_v, A)$. By Lemma 2.4, G has an (A, b)-NZF. ■

3. GROUP CONNECTIVITY OF LOCALLY CONNECTED GRAPHS

The main purpose of this section is to prove a stronger version of Theorem 1.1, stated as Theorem 3.1 below.

Theorem 3.1. *Let A be an Abelian group with $|A| \geq 3$. Then, every 2-edge-connected, locally 3-edge-connected graph is in $\langle A \rangle$.*

We start with some lemmas. A θ-graph $L(v, w)$ consists of three (v, w)-paths P_1, P_2, and P_3 such that $V(P_i) \cap V(P_j) = \{v, w\}$, whenever $i \neq j$, where v, w are two distinct vertices of the graph. Lemma 3.1 follows from Menger's Theorem immediately.

Lemma 3.1. *If H is a nontrivial 3-edge-connected graph, then H contains a θ-graph.*

Lemma 3.2. *Let H be a graph with a distinguished vertex u such that u is adjacent to every vertex in $V(H - u)$ in H. If $H - u$ is a θ-graph $L(v, w)$, then $H \in \langle A \rangle$.*

Proof. Since $H - u = L(v, w)$ is a θ-graph, we assume that $H - u$ consists of three internally disjoint paths $P_1 = x_1x_2 \cdots x_l$, $P_2 = y_1y_2 \cdots y_m$, and $P_3 = z_1z_2 \cdots z_n$, where $x_1 = y_1 = z_1 = v$ and $x_l = y_m = z_n = w$, and where $l \geq m \geq n$.

By Proposition 2.1(v), we may assume that $l > 1$. Let $e_1 = vu$, $e_2 = vx_2 = x_1x_2$, H_v be the graph obtained from H by splitting v into v' and v'' such that v'' is incident with e_1 and e_2, while v' is incident with all the other edges formerly incident with v in H. Let $L = H_v/e_1$, and we can view $V(L)$ as $V(H) \cup \{v'\} - \{v\}$. Since u is adjacent to every vertex in $H - u$, L has a 2-cycle $C_1 = \{e_2, ux_2\}$. Note that L/C_1 has a 2-cycle $C_2 = \{x_2x_3, ux_3\}$. By Proposition 2.1, and by the fact that u is adjacent to every vertex in $V(P_1)$, we can repeat this process to conclude that $L[V(P_1 - v) \cup \{u\}] \in \langle A \rangle$. Now, we consider $L/L[V(P_1 - v) \cup \{u\}]$, which has two 2-cycles $\{uy_{m-1}, y_{m-1}y_m\}$ and $\{uz_{n-1}, z_{n-1}z_n\}$. Contracting these 2-cycles, and repeatedly contracting the 2-cycles arised from contraction, it follows that the resulting graph after all the 2-cycles have been repeatedly contracted is a K_1. By Proposition 2.1, $L \in \langle A \rangle$. By Corollary 2.4, $H \in \langle A \rangle$. ■

Lemma 3.3. *Let H be a graph with a distinguished vertex u such that u is adjacent to every vertex in $V(H - u)$. If $H - u$ is connected and contains a nontrivial subgraph $L \in \langle A \rangle$, then $H \in \langle A \rangle$.*

Proof. Let L_1 be the subgraph of H that induced by the edges in L and the edges in H with one end as u and the other end in $V(L)$. Since every edge in L_1/L is in a 2-cycle, by Proposition 2.1(ii) and (vi), $L_1 \in \langle A \rangle$. Now consider H/L_1. Since every vertex in $H - u$ is adjacent to u and $u \in V(L_1)$, H/L_1 must have a 2-cycle. Since $H - u$ is connected, by repeatedly contracting the 2-cycles arised from contractions, we can see that the resulting graph must be a K_1. By Proposition 2.1(ii) and (v) again, $H \in \langle A \rangle$. ■

Proof of Theorem 3.1. Since G is connected, G has a spanning tree T. For each edge $e = uv \in E(T)$, by assumption, N_u is 3-edge-connected. Since G is 2-edge-connected, N_u is nontrivial. Therefore by Lemma 3.1, N_u has a θ-subgraph. Let $H_e = G[V(N_u) \cup \{u\}]$ be the subgraph of G induced by $V(N_u) \cup \{u\}$. By Lemma 3.2 and since $H_e - u = N_u$ has a θ-subgraph, H_e contains a nontrivial subgraph in $\langle A \rangle$. Therefore by Lemma 3.3, $H_e \in \langle A \rangle$. It follows that each edge $e \in E(T)$ is in a subgraph $H_e \in \langle A \rangle$, and so by Lemma 2.1, $G \in \langle A \rangle$. ■

This result is best possible in the sense that there exists an infinite family of 2-edge-connected, locally 2-edge-connected graphs none of which is in $\langle Z_3 \rangle$. We construct such an example below.

Let K_4 be a given complete graph on 4 vertices $\{u, v, x, y\}$ with a distinguished edge $a = xy$, and let L be a graph disjoint from this K_4 with $|E(L)| \geq 2$ and with a distinguished edge $a' = x'y'$. Define a new graph $L \oplus K_4$ to be the graph obtained from the disjoint union of $L - a'$ and K_4 by identifying x' and x to form a new vertex, also called x, and by identifying y' and y to form a new vertex, also called y. Note that the edge $a = xy$ is now an edge of $L \oplus K_4$ and that $L = L \oplus K_4 - \{u, v\}$.

Lemma 3.4 *Let L be a connected graph with at least two edges. Each of the following holds.*

(i) (*Lemma* 4.6 *of* [7]) $L \oplus K_4$ *has a* 3-*NZF, if and only if, L has a* 3-*NZF.*
(ii) *If L is* 2-*edge-connected, locally* 2-*edge-connected, then so is* $L \oplus K_4$.

Proof. Part (ii) follows from the fact that if two edge-disjoint graphs are k-edge-connected and if they share a common vertex, then their union is also k-edge-connected. ■

To obtain an infinite family of 2-edge-connected, locally 2-edge-connected graphs that are not in $\langle Z_3 \rangle$, one can start with any graph L that does not admit a 3-NZF (for example, take $L \cong K_4$), and construct bigger and bigger graphs by utilizing Lemma 3.4. Note that if a graph G does not have a 3-NZF, then G cannot be in $\langle Z_3 \rangle$. Note also that in such a construction, if one starts with a K_4, then the resulting graphs would be 3-edge-connected.

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