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Cycle Cover Ratio of Regular Matroids

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A cycle in a matroid is a disjoint union of circuits. This paper proves that every regular matroid M without coloops has a set S of cycles whose union is $E(M)$ such that every element is in at most three of the cycles in S. It follows immediately from this that, on average, each element of M is in at most three members of the cycle cover S. This improves on a 1989 result of Jamshy and Tarsi who proved that there is a cycle cover for which this average is at most 4, and conjectured that a cycle cover exists for which the average is at most 2.

1. INTRODUCTION

Graphs and matroids considered in this paper are finite, with loops and parallel elements permitted. Terms and notations not defined here can be found in [4] for graphs and in [7] for matroids. All groups considered in this paper are abelian (additive) groups. Throughout this paper, $\mathbf{Z}$ denotes the additive group of the integers and $\mathbf{Z}_2$ denotes the field of two elements.

Let A be an abelian group. For any $a \in A$, define $(+1)a = a$, $(-1)a = a$, the additive inverse of a in A, and $0 \cdot a = 0$, where the right-hand side zero denotes the additive identity of the group A. For a given integer $n \geq 1$, let $V(n, A)$ denote the set of all n-dimensional vectors over A. When $A = \mathbf{Z}_2$, we simply write $V(n, 2)$ for $V(n, \mathbf{Z}_2) = \mathbf{Z}_2^n$. If $n = |E|$, for some set $E \neq \emptyset$, and if the components of vectors in $V(n, A)$ are labeled with elements in E (in which case we also write $V(E, A)$ for $V(n, A)$), then for each $\alpha \in V(n, A)$ and for each $x \in E$, $\alpha(x)$ denotes the x-coordinate of α. If $f : E \mapsto A$ is a function and if $D \subseteq E$ is a subset, denote $f(D) = \sum_{e \in D} f(e)$.

Let M be a matroid, and let $\mathcal{C}(M)$ denote the set of all circuits of M. A **cycle** of M is a disjoint union of circuits in M. Note that the union may be an empty one, and so the empty set (as a subset of $E(M)$) will be both a cycle and an independent set in M. A **cocycle** of M is a cycle of M^*.

In order to state the problem raised by Jamshy and Tarsi in [6] on the cycle cover ratio of regular matroids we need to introduce the concept of matroid orientations and nowhere zero flows on orientable matroids.

We basically follow [9] for the definitions of orientation of matroids, nowhere zero A-flows (A-NZFs) and nowhere zero k-flows (k-NZFs) on an oriented matroid, except that we use A to denote an abelian group, and $S(f)$ to denote the **support** of a nowhere zero A-flow f on an oriented matroid M. For notational convenience, we view a function $f : E(M) \mapsto A$ as a row vector $f \in V(E(M), A)$. Thus if D and D^* are the incidence matrix of circuits against elements in M and in M^*, respectively, and if $w(D)$ and $w(D^*)$ are signing of D and D^*, respectively, such that $w(D)(w(D^*))^T = 0$, then an A-flow in the oriented matroid M is a row vector $f \in V(E(M), A)$ such that the matrix product

$$f \cdot (w(D^*)^T) = 0. \tag{1}$$

The following result on nowhere zero flows of a regular matroid is well known.

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(1.1) (Arrowsmith–Jaeger [1], Brylawski [3] and Tutte [10]). For a regular matroid M, M has an A-NZF if and only if M has an $|A|$-NZF.

Note that if A denotes the additive group of a ring with characteristic 2, then (1) can be replaced by $f(D) = \sum_{e \in D} f(e) = 0$ for every cocycle D of M, and so the orientability of M is not needed.

In this paper, we will basically consider nowhere zero A-flows on binary matroids, where A is the additive group of a ring with characteristic 2. Thus (1) can be replaced by $f(D) = \sum_{e \in D} f(e) = 0$ for every cocycle D of M, and so the orientability of M is not needed. Such an approach allows us to discuss the problem in terms of cycles without worrying about orientations. Note that not all binary matroids are orientable. In fact, Bland and Las Vergnas [2] showed that a binary matroid is orientable if and only if it is regular.

Let M be a binary matroid. A **cycle cover** of M is a multiset S of cycles of M such that every element of M lies in at least one member of S. A **cycle 2-cover** of M is a cycle cover S such that every element of M lies in exactly two members in S. Clearly M has a cycle cover if and only if M has no coloops. If $S = \{C_1, C_2, \ldots, C_m\}$ is a cycle cover of M, then the **length** of S is

$$l(S) = \sum_{i=1}^{m} |C_i|.$$

For a coloopless regular matroid M, define

$$l(M) = \min\{l(S) : S \text{ is a cycle cover of } M\}.$$

The ratio $r(M) = l(M)/|E(M)|$ will be called **cycle cover ratio**. The parameter

$$s(k) = \sup\left\{\frac{l(M)}{|E(M)|} : M \text{ is a regular matroid admitting a } k\text{-NZF}\right\}$$

is defined and studied in [6] and [9]. One of the open problems posed in [9] asks if there exists a constant integer c such that for any $k \geq 2$, $s(k) \leq c$. This problem has been solved by Jamshy and Tarsi, who proved the following theorem.

(1.2) (Jamshy and Tarsi [6]). For any $k \geq 2$, $s(k) \leq 4$.

In [6], it is proved that the cycle 2-cover conjecture for graphs is equivalent to the cycle 2-cover conjecture for regular matroids. Tarsi [9] observed that if every regular matroid M has a cycle 2-cover, then $s(k) \leq 2$, and so it was conjectured in [6] that $s(k) \leq 2$, for any $k \geq 2$. Utilizing the famous decomposition theorem of Seymour on regular matroids, we shall show in this paper that Theorem (1.2) can be improved to $s(k) \leq 3$ for any $k \geq 2$.

2. Nowhere Zero Flows and Cycle Covers

Groups in this section are the additive group of the vector space $V(n, 2)$, and the matroids in this section are binary matroids. For a vector $\alpha \in V(n, 2)$, $\|\alpha\|$ denotes the number of nonzero coordinates of α; and $\pi_i(\alpha)$ denote the ith coordinate of α, where $1 \leq i \leq n$. Thus π_i is the projection map onto the ith coordinate.

Let M be a binary matroid and let $S = \{C_1, C_2, \ldots, C_s\}$ be a cycle cover of M. For each $e \in E(M)$, let $d_S(e)$ denote the number of cycles in S which contain e. For a positive integer l, a cycle cover S of M is a **cycle ($\leq l$)-cover** if $1 \leq d_S(e) \leq l$, for any $e \in E(M)$. Thus M has a (≤ 1)-cycle cover if and only if M is a cycle. Note that by the definition of $s(k)$, if every regular matroid admitting a k-NZF has a (≤ 3)-cycle cover, then $s(k) \leq 3$. Therefore, our aim is to show that every coloopless regular matroid has a cycle (≤ 3)-cover. The following is well known.

(2.1) ([7], Theorem 9.1.2). Let M be a binary matroid and let $X \in E(M)$. Then

(i) X is a cycle if and only if $|X \cap D|$ is even for any cocycle D of M.
(ii) X is a cocycle if and only if $|X \cap C|$ is even for any cycle C of M.

(2.2) Let M be a binary matroid and let $f : E(M) \mapsto V(n, 2)$ be a $V(n, 2)$-flow. Each of the following holds.

(i) For each i with $1 \leq i \leq n$, $S_i(f) = \{e \in E(M) : \pi_i(f(e)) = 1\}$ is a cycle of M.
(ii) If f is a $V(n, 2)$-NZF, then $S = \{S_i(f) : 1 \leq i \leq n\}$ is a cycle cover of M such that $d_S(e) = \|f(e)\|, \forall e \in E(M)$.

PROOF. For any $D \in \mathcal{C}(M^*)$, since f is a $V(n, 2)$-flow and by (1), $f(D) = 0 \in V(n, 2)$. It follows that for each i, $\pi_i \circ f(D) = \pi_i(0)$ in $\mathbf{Z}_2$. Therefore by the definition of $S_i(f)$, one has $|S_i(f) \cap D| = (\pi_i \circ f)(D) = 0$ in $\mathbf{Z}_2$, and so by (2.1), $S_i(f) \in \mathcal{C}(M)$.

When f is a $V(n, 2)$-NZF, every $e \in E(M)$ lies in a $S_i(f)$, for some $1 \leq i \leq n$, and so $S = \{S_i(f) : 1 \leq i \leq n\}$ is a cycle cover of M. By the definition of $S_i(f)$, e lies in exactly those cycles $S_i(f)$ where $\pi_i(f(e)) = 1$, and so $d_S(e) = \|f(e)\|$. □

(2.3) Let $D = \{e_1, e_2, e_3\} \in \mathcal{C}(M^*)$, and let $S = \{C_1, C_2, \ldots, C_s\}$ be a cycle ($\leq$3)-cover of M. If $d_S(e_1) \leq d_S(e_2) \leq d_S(e_3)$, then

$$(d_S(e_1), d_S(e_2), d_S(e_3)) \in \{(1, 1, 2), (1, 2, 3), (2, 2, 2), (2, 3, 3)\}.$$

PROOF. This follows from (2.1) and from the assumption that $1 \leq d_S(e) \leq 3$. □

The following fact is straightforward from linear algebra.

(2.4) For an integer m with $1 \leq m \leq n$, if $\alpha_1, \alpha_2, \ldots, \alpha_m$ are linearly independent in $V(n, 2)$ and if $\beta_1, \beta_2, \ldots, \beta_m$ are linearly independent in $V(n, 2)$, then there exists a nonsingular linear transformation $L : V(n, 2) \mapsto V(n, 2)$ such that $L(\alpha_i) = \beta_i$, $1 \leq i \leq m$.

Jaeger's 8-NZF theorem [5] indicates that every coloopless graph has an 8-NZF, which implies that every coloopless graphic matroid has a cycle ($\leq$3)-cover. To prove our result in this paper, we need a slightly stronger version of the 8-NZF theorem for both graphic matroids and cographic matroids.

(2.5) Let G be a coloopless graph and let $M = M(G)$. Each of the following holds.

(i) For any $e_0 \in E(M)$, and for any integer x_0 with $1 \leq x_0 \leq 3$, M has a cycle ($\leq$3)-cover S such that $d_S(e) = x_0$.
(ii) Let $D = \{e_1, e_2, e_3\} \in \mathcal{C}(M^*)$, and let $1 \leq x_1 \leq x_2 \leq x_3 \leq 3$ be integers such that

$$(x_1, x_2, x_3) \in \{(1, 1, 2), (1, 2, 3), (2, 2, 2), (2, 3, 3)\}. \quad (2)$$

Then M has a cycle ($\leq$3)-cover S such that $d_S(e_i) = x_i$, $(1 \leq i \leq 3)$.

PROOF. By Jaeger's 8-NZF theorem, there is a $V(3, 2)$-NZF $f : E(M) \mapsto V(3, 2)$ such that $S(f) = E(M)$.

In particular, $f(e_0) \neq 0$ and so $f(e_0)$ is linearly independent. Pick a vector $\beta \in V(3, 2)$ such that $\|\beta\| = x_0$. By (2.4), there is a nonsingular linear transformation L such that $L(f(e_0)) = \beta$. Since f is a $V(3, 2)$-NZF, by (1), for each $D \in \mathcal{C}(M^*)$, $f(D) = 0$. Since L

is a linear transformation, one has $L(f)(D) = L(f(D)) = L(0) = 0$. It follows by (1) that $L(f) = L \circ f$ is also a $V(3, 2)$-NZF of M. Thus (2.5)(i) follows from (2.2)(ii).

Next we assume that $(x_1, x_2, x_3) \in \{(1, 1, 2), (1, 2, 3), (2, 2, 2)\}$. Note that since $D \in \mathcal{C}(M^*)$, $f(e_1) + f(e_2) = f(e_3) \neq 0$, and so $f(e_1)$, $f(e_2)$ are linearly independent in $V(3, 2)$. Define

$$\begin{aligned} \beta_1 &= (1, 0, 0), \beta_2 = (0, 1, 0) && \text{if } (x_1, x_2, x_3) = (1, 1, 2), \\ \beta_1 &= (1, 0, 0), \beta_2 = (0, 1, 1) && \text{if } (x_1, x_2, x_3) = (1, 2, 3), \\ \beta_1 &= (1, 1, 0), \beta_2 = (0, 1, 1) && \text{if } (x_1, x_2, x_3) = (2, 2, 2). \end{aligned}$$

Then β_1, β_2 are linearly independent in $V(3, 2)$. By (2.4), there is a nonsingular linear transformation $L : V(3, 2) \mapsto V(3, 2)$ such that $L(f(e_i)) = \beta_i$, for each $i = 1, 2$. Note that $\|\beta_1\| = x_1$ and $\|\beta_2\| = x_2$. Moreover, since $f(e_1) + f(e_2) = f(e_3)$ and since L is linear, $L(f(e_3)) = L(f(e_1)) + L(f(e_2)) = \beta_1 + \beta_2$, and so $\|L(f(e_3))\| = \|\beta_1 + \beta_2\| = x_3$. Note that $L(f) = L \circ f$ is also a $V(3, 2)$-NZF of M, and so when $(x_1, x_2, x_3) \in \{(1, 1, 2), (1, 2, 3), (2, 2, 2)\}$, (2.5)(ii) follows from (2.2)(ii).

Finally we assume that $(x_1, x_2, x_3) = (2, 3, 3)$. Note that $f(e_2) + f(e_3) = f(e_1) \neq 0$, and so $f(e_2)$, $f(e_3)$ are linearly independent in $V(3, 2)$. By what we have just proved in the previous paragraph, we may assume that $f(e_2) = (1, 0, 0)$ and $f(e_3) = (0, 1, 0)$. Define a linear transformation $L : V(3, 2) \mapsto V(4, 2)$ as follows:

$$L(z_1, z_2, z_3) = (z_1, z_2, z_3) \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} = (z_1 + z_3, z_2 + z_3, z_1 + z_2, z_1 + z_2 + z_3).$$

Note that $\|L(f(e_i))\| = 3$, when $i = 2, 3$. Note also that the matrix defining L above has rank 3, and so for any $\alpha \in V(3, 2) - \{0\}$, $L(\alpha) \neq 0 \in V(4, 2)$. Moreover, for any $\alpha = (z_1, z_2, z_3) \in V(3, 2) - \{0\}$, one has $0 < \|L(\alpha)\| \leq 3$. Furthermore, since L is linear and since $f(e_1) = f(e_2) + f(e_3)$, $\|L(f(e_1))\| = \|L(f(e_2)) + L(f(e_3))\| = 2$. Again, since L is a linear transformation and by observations mentioned above, $L(f)$ is a $V(4, 2)$-NZF. It follows by (2.2) that the corresponding cycle cover is a cycle ($\leq$3)-cover of M. □

(2.6) Let G be a connected loopless graph on $n \geq 5$ vertices and let $M = M(G)$. Each of the following holds.

(i) For any $e_0 \in E(M)$, and for any integer x_0 with $1 \leq x_0 \leq 3$, M has a cocycle ($\leq$3)-cover S such that $d_S(e) = x_0$.

(ii) Let $D = \{e_1, e_2, e_3\} \in \mathcal{C}(M)$, and let $1 \leq x_1 \leq x_2 \leq x_3 \leq 3$ be integers such that

$$(x_1, x_2, x_3) \in \{(1, 1, 2), (1, 2, 3), (2, 2, 2), (2, 3, 3)\}.$$

Then M has a cocycle ($\leq$3)-cover S such that $d_S(e_i) = x_i$, $(1 \leq i \leq 3)$.

PROOF. This follows from the fact that the set of edges incident with a vertex in a graph G is a circuit in $M^*(G)$, and all such sets constitute a cycle 2-cover of $M^*(G)$. Details of the proof will then be left to the reader. □

(2.7) Let $R_{10} = M[A]$ be a vector matroid over the real numbers, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Then R_{10} is a cycle. Consequently, for any element $e \in E(M)$ and for any integer x_0 with $1 \leq x_0 \leq 3$, M has a cycle ($\leq$3)-cover S with $d_S(e) = x_0$.

PROOF. Let e_i denote the ith column of A, $1 \leq i \leq 10$. Then one can directly verify that $\{e_1, e_3, e_5, e_7, e_8, e_9\}$ and $\{e_2, e_4, e_6, e_{10}\}$ are circuits. □

3. CYCLE ($\leq$3)-COVER OF REGULAR MATROIDS

We shall apply Seymour's decomposition theorem for regular matroids to prove our main results in this section. Therefore we need the notion of 1-sum, 2-sum and 3-sum. Given two sets X and Y, the **symmetric difference** of X and Y, is

$$X \Delta Y = (X \cup Y) - (X \cap Y).$$

Let M_1 and M_2 be two binary matroids where $E(M_1)$ and $E(M_2)$ may intersect. Define $M_1 \Delta M_2$ to be the binary matroid on $E = E(M_1)\Delta E(M_2)$ whose cycles are all subsets of E of the form $C_1 \Delta C_2$, where C_1 is a cycle of M_1 and C_2 is a cycle of M_2. In particular,

(i) when $E(M_1) \cap E(M_2) = \emptyset$, then $M_1 \Delta M_2$ is the 1-sum (also called the direct sum) of M_1 and M_2;
(ii) when $E(M_1) \cap E(M_2) = \{e_0\}$, where, for each $i \in \{1, 2\}$, the element e_0 is neither a loop nor a coloop of M_i, then $M_1 \Delta M_2$ is the 2-sum of M_1 and M_2 (about the element e_0);
(iii) when $E(M_1) \cap E(M_2) = C$, where C is a 3-circuit of both M_1 and M_2, where C includes no cocircuit of either M_1 and M_2, and where for $i \in \{1, 2\}$, $|E(M_i)| \geq 7$, then $M_1 \Delta M_2$ is the 3-sum of M_1 and M_2 (about the 3-circuit C).

When M is a k-sum of M_1 and M_2, for $k = 1, 2, 3$, we say that M is a **proper** k-sum of M_1 and M_2 if M_1 and M_2 are both isomorphic to proper minors of M.

(3.1) (Seymour, [8]). Every regular matroid M can be constructed by means of direct sums, 2-sums and 3-sums starting with matroids each of which is isomorphic to a minor of M, and each of which is either graphic, cographic, or isomorphic to R_{10}.

(3.2) (Seymour [8], Proposition 13.2.1 of [7]). Every regular matroid can be obtained from copies of R_{10} and from 3-connected regular matroids without R_{10}-minors by a sequence of direct sums and 2-sums.

For application purpose, we need to state Seymour's decomposition theorem slightly differently as in (3.3) below.

(3.3) (Seymour, [8]). Let M be a coloopless regular matroid. Then one of the following holds.

(i) M is coloopless and graphic.
(ii) M is coloopless and cographic.
(iii) M is isomorphic to R_{10}.
(iv) $M = M_1 \Delta M_2$ is the proper 1-sum or 2-sum of two coloopless regular matroids M_1 and M_2, where one of them is either graphic, or cographic, or isomorphic to R_{10}.
(v) $M = M_1 \Delta M_2$ is the proper 3-sum of two coloopless regular matroids M_1 and M_2, where one of them is either graphic or cographic.

PROOF. We may assume that M is connected. For otherwise we can argue by induction on a component.

Assume that (3.3)(i), (ii) and (iii) do not hold. If M does not have an R_{10}-minor, then, by (3.1), we may assume that M_2 is the matroid last added in the construction of M, as described in (3.1). Therefore, $M = M_1 \Delta M_2$ is a proper i-sum ($2 \le i \le 3$), and M_2 must be either graphic, or cographic, or isomorphic to R_{10}. Since M does not have a minor isomorphic to R_{10}, M_2 must be either graphic or cographic, or so (3.3)(iv) or (v) must hold.

Therefore we assume that M has a minor isomorphic to R_{10}. By (3.2), and since (3.3)(i), (ii) and (iii) do not hold, M can be expressed as a 2-sum $N_1 \Delta N_2$, where N_2 is the matroid last added in the construction of M, as described in (3.2). If N_2 is isomorphic to R_{10}, or if N_2 is graphic or cographic, then (3.3)(iv) must hold. Therefore, we assume that N_2 is a 3-connected regular matroid without a minor isomorphic to R_{10}, and is neither graphic nor cographic. By (3.1), $N_2 = N_3 \Delta M_2$ is a 3-sum such that M_2 must be either graphic or cographic. It follows that $M = M_1 \Delta M_2$ is a 3-sum satisfying (3.3)(v). □

(3.4) Seymour [8] indicated that when M is a proper 3-sum of M_1 and M_2, M^* cannot be the 3-sum of M_1^* and M_2^*. However, one has (remark on p. 319 of [8])

$$(M_1 \Delta M_2)^* = M_1^* \Delta M_2^*.$$

(3.5) Note that M is regular if and only if M^* is regular. Applying (3.3) to M^* and noting that $R_{10}^* \cong R_{10}$, Theorem (3.3)(v) can be restated as follows:

$M^* = M_1^* \Delta M_2^*$ is the proper 3-sum of two coloopless regular matroids M_1^* and M_2^*, where one of them is either graphic or cographic.

(3.6) Let M_1 and M_2 be two binary matroids, and let $M = M_1 \Delta M_2$ be a 2-sum of M_1 and M_2 about the element e_0. Let S_1 and S_2 be cycle ($\le$3)-covers of M_1 and M_2, respectively. If $d_{S_1}(e_0) = d_{S_2}(e_0)$, then M has a cycle ($\le$3)-cover.

PROOF. Let $x = d_{S_1}(e_0) = d_{S_2}(e_0)$. Then, for $i = 1, 2$, there are cycles $C_1^{(i)}, \ldots, C_x^{(i)}$ in S_i which contain e_0. Define $C_j = C_j^{(1)} \Delta C_j^{(2)}$, $(1 \le j \le x)$. By the definition of a 2-sum, each C_j is a cycle of M. It follows that

$$S = (S_1 - \{C_1^{(1)}, \ldots, C_x^{(1)}\}) \cup (S_2 - \{C_1^{(2)}, \ldots, C_x^{(2)}\}) \cup \{C_j : 1 \le j \le x\}$$

is a cycle ($\le$3)-cover of M. □

(3.7) Let M be a binary matroid and $D = \{e_1, e_2, e_3\} \in \mathcal{C}(M^*)$. Let S be a cycle ($\le$3)-cover of M such that $d_S(e_i) = x_i$, $1 \le i \le 3$, where $x_1 \le x_2 \le x_3$ (relabeling if necessary). Then exactly one of the following holds.

- (i) $(x_1, x_2, x_3) = (1, 1, 2)$ and there exist distinct $C_1, C_2 \in S$ such that $e_1, e_3 \in C_1$ and e_2, $e_3 \in C_2$.
- (ii) $(x_1, x_2, x_3) = (1, 2, 3)$ and there exist distinct $C_1, C_2, C_3 \in S$ such that $e_1, e_3 \in C_1$, and $e_2, e_3 \in C_2 \cap C_3$.
- (iii) $(x_1, x_2, x_3) = (2, 2, 2)$ and there exist distinct $C_1, C_2, C_3 \in S$, such that $e_1, e_3 \in C_1$, $e_1, e_3 \in C_2$ and $e_2, e_3 \in C_3$.
- (iv) $(x_1, x_2, x_3) = (2, 3, 3)$ and there exist distinct $C_1, C_2, C_3, C_4 \in S$ such that $e_1, e_2 \in C_1$, $e_1, e_3 \in C_2$ and $e_2, e_3 \in C_3 \cap C_4$.

PROOF. Since $D = \{e_1, e_2, e_3\} \in \mathcal{C}(M^*)$, and by (2.1), for any $C \in \mathcal{C}(M)$, one has $|C \cap D| \in \{0, 2\}$. Let $S_D = \{C_1, C_2, \ldots, C_m\} \in S$ be the collection of all members in S such that $|C_i \cap D| = 2$.

By (2.3), (2) must hold, and so one needs only to consider the possible values of (x_1, x_2, x_3), as indicated in (2).

If $(x_1, x_2, x_3) = (1, 1, 2)$, then since $|C_i \cap D| = 2$ for each $C_i \in S_D$, one must have $m = |S_D| = 2$ and so (i) follows. The proofs for the other cases are similar, and will be omitted. □

(3.8) Let M_1 and M_2 be two binary matroids, and let $M^* = M_1^* \Delta M_2^*$ be a 3-sum of M_1^* and M_2^* about the 3-cocircuit $D = \{e_1, e_2, e_3\} \in \mathcal{C}(M_1^*) \cap \mathcal{C}(M_2^*)$. Let S_1 and S_2 be cycle ($\leq$3)-covers of M_1 and M_2, respectively. If $d_{S_1}(e_j) = d_{S_2}(e_j)$, for each $j = 1, 2, 3$, then M has a cycle ($\leq$3)-cover.

PROOF. Let $x_i = d_{S_1}(e_j) = d_{S_2}(e_j)$, for each $j = 1, 2, 3$. Relabeling the elements in D if needed, we may assume that $1 \leq x_1 \leq x_2 \leq x_3 \leq 3$. By (2.3), one needs only to consider the possible values of (x_1, x_2, x_3), as indicated in (2).

Assume that $(x_1, x_2, x_3) = (1, 1, 2)$. Then by (3.7)(i), for each $i = 1, 2$, there exists $C_1^{(i)}$, $C_2^{(i)} \in S_i$ such that $e_1, e_3 \in C_1^{(i)}$ and $e_2, e_3 \in C_2^{(i)}$. By (3.4), $M = M_1 \Delta M_2$, and so for $j = 1$, 2, $C_j = C_j^{(1)} \Delta C_j^{(2)}$ is a cycle of M. It follows by (3.4) that

$$S = (S_1 - \{C_1^{(1)}, C_2^{(1)}\}) \cup (S_2 - \{C_1^{(2)}, C_2^{(2)}\}) \cup \{C_1, C_2\}$$

is a cycle ($\leq$3)-cover of M in this case.

The proof for other cases are similar and so will be omitted. □

(3.9) Every coloopless regular matroid has a cycle ($\leq$3)-cover.

PROOF. Note that R_{10} is a cycle. Therefore, if M is graphic or cographic, or is R_{10}, then M has a cycle ($\leq$3)-cover.

We proceed by induction on $|E(M)|$. By applying (3.3) to M^*, we may assume that either $M = M_1 \Delta M_2$ is a proper 2-sum such that M_2 is either graphic, cographic, or $M_2 = R_{10}$, or $M^* = M_1^* \Delta M_2^*$ is a proper 3-sum such that M_2 is either graphic or cographic.

Assume first that $M = M_1 \Delta M_2$ is a proper 2-sum about the element e_0 and M_2 is either graphic, cographic, or $M_2 = R_{10}$. By the induction hypothesis, M_1 has a cycle ($\leq$3)-cover S_1. Let $x_0 = d_{S_1}(e_0)$. Then by (2.5) (if M_2 is graphic), or by (2.6) (if M_2 is cographic) or by (2.7) (if $M_2 = R_{10}$), M_2 has a cycle ($\leq$3)-cover S_2 such that $d_{S_2}(e_0) = x_0$. It follows by (3.6) that M has a cycle ($\leq$3)-cover.

Assume next that $M^* = M_1^* \Delta M_2^*$ is a proper 3-sum about the cocircuit $D = \{e_1, e_2, e_3\} \in \mathcal{C}(M_1^*) \cap \mathcal{C}(M_2^*)$, such that M_2 is either graphic or cographic. Note that if M_2 is a connected cographic matroid with $r(M_2) \leq 3$, then it cannot contain a minor isomorphic to $M^*(K_5)$ or $M^*(K_{3,3})$, as each of $M^*(K_5)$ and $M^*(K_{3,3})$ has rank at least 4. Therefore, by a theorem of Tutte [11] that a regular matroid is graphic if and only if it does not have a minor isomorphic to $M^*(K_5)$ or $M^*(K_{3,3})$, M_2 must also be graphic. Therefore, we assume that if M_2 is cographic, then M_2 is connected and $r(M_2) \geq 4$.

By the induction hypothesis, M_1 has a cycle ($\leq$3)-cover S_1. Let $d_{S_1}(e_j) = x_j$, for $j = 1$, 2, 3, where $1 \leq x_1 \leq x_2 \leq x_3 \leq 3$ (relabeling the e_1's if needed). By (2.3), (2) must hold. By (2.5)(ii) (if M_2 is graphic) or by (2.6)(ii) (if M_2 is cographic), M_2 has a cycle ($\leq$3)-cover S_2 such that for each $j = 1, 2, 3$, $d_{S_2}(e_j) = x_j$. Thus by (3.8), M has a cycle ($\leq$3)-cover, and so (3.9) follows by induction. □

As a corollary, one immediately has the following improvement of (1.2).

(3.10) For any integer $k \geq 2$, $s(k) \leq 3$.

It is natural to consider the existence of cycle ($\leq$2)-covers for coloopless regular matroids. Clearly any cycle 2-cover of M is a cycle ($\leq$2)-cover of M. On the other hand, if $S = \{C_2, C_2, \ldots, C_t\}$ is a cycle ($\leq$2)-cover of M, then $S \cup \{\Delta_{i=1}^{t} C_i\}$ is a cycle 2-cover of M. Therefore, one has the following observation.

(3.11) The existence of a cycle ($\leq$2)-cover is equivalent to the existence of a cycle 2-cover, and either of them can imply that $s(k) \leq 2$ for any $k \geq 2$.

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