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Group Chromatic Number of Graphs without K_5-Minors

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Abstract. Let G be a graph with a fixed orientation and let A be a group. Let $F(G,A)$ denote the set of all functions $f\colon E(G) \mapsto A$. The graph G is **A-colorable** if for any function $f \in F(G,A)$, there is a function $c\colon V(G) \mapsto A$ such that for every directed $e = uv \in E(G)$, $c(u) - c(v) \neq f(e)$. The **group chromatic number** $\chi_1(G)$ of a graph G is the minimum m such that G is A-colorable for any group A of order at least m under a given orientation D.

In [J. Combin. Theory Ser. B, 56 (1992), 165–182], Jaeger *et al.* proved that if G is a simple planar graph, then $\chi_1(G) \leq 6$. We prove in this paper that if G is a simple graph without a K_5-minor, then $\chi_1(G) \leq 5$.

1. Introduction

Graphs in this note are finite and simple. We follow Bondy and Murty [1] for undefined terms. Thus $\chi(G)$, $\delta(G)$ and $\kappa(G)$ denote the chromatic number, the minimum degree, and the connectivity of a graph G, respectively. We use $H \subseteq G$ to denote the fact that H is a subgraph of G. Let G be a graph and let X be a set of edges with ends in $V(G)$, then $G \cup X$ denotes the simple graph with vertex set $V(G)$ and edges set $E(G) \cup X$.

For a subset $X \subseteq E(G)$, the **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and deleting the resulting loops and deleting all but one edge in each equivalence class of parallel edges. Thus the contraction of a simple graph is again simple.

Let $G = (V,E)$ be a graph and A a non-trivial group, and let $F(G,A)$ denote the set of all functions $f\colon E(G) \to A$. Denote by D an orientation of G. An oriented edge directed from u to v is called an **arc** uv. The graph G under the orientation D is sometimes denoted by $D(G)$.

For a function $f \in F(G,A)$, an **(A,f)-coloring** of G under the orientation D is a function $c\colon V(G) \mapsto A$ such that for any arc $e = uv \in E(G)$, $c(u) - c(v) \neq f(e)$; the graph G is **A-colorable** under the orientation D if for any $f \in F(G,A)$, G has an (A,f)-coloring. The **group chromatic number** $\chi_1(G)$ of a graph G is the minimum integer m such that G is A-colorable for any group A of order at least m under the orientation D.

It is easy to see that $\chi(G) \leq \chi_1(G)$, for any graph G. However, $\chi_1(G) - \chi_1(G)$ can be arbitrarily large, as noted in [5]. In [4], the following is proved.

Theorem 1.1 (Jaeger, Linial, Payan, and Tarsi, [4]). *If G is a simple planar graph, then $\chi_1(G) \leq 6$.*

Let K be a graph. We say that a graph G has an K**-minor** if G has a subgraph H which can be contracted to K. A graph G is K**-minor free** if it does not have a K-minor. By Kuratowski's theorem [1] (also see [3] and [8]), planar graphs are K_5-minor free graphs. In this paper, we modify the techniques of Thomassen [7] and Škrekovski [6] to prove that Theorem 1.1 can be generalized to K_5-minor free graphs with a better bound.

Theorem 1.2. *Let G be a K_5-minor free simple graph. Then $\chi_1(G) \leq 5$.*

The proof of the main result has two steps. Following similar ideas of Thomassen [7] and Škrekovski [6], a stronger version of Theorem 1.2 is first proved for planar graphs in the next section, which is applied to prove the K_5-minor free case in the last section.

2. Planar Graphs

Let G be a plane graph which has no loops or multiple edges. Call G a **near-triangulation** if every face of G other than the exterior face is a 3-cycle. A cycle C in a plane graph G is **separating** if both the interior and exterior of C contain vertices of G.

Let ϕ_1 and ϕ_2 be two maps with domains D_1 and D_2, respectively. Assume that $\phi_1(x) = \phi_2(x)$ for any $x \in D_1 \cap D_2$. If $D_1 \subseteq D_2$, then ϕ_2 is an **extension** of ϕ_1; and we also write $\phi_2|_{D_1} = \phi_1$. Note that if $\phi_1(x) = \phi_2(x)$ for every $x \in D_1 \cap D_2$, then the map

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } x \in D_1 \\ \phi_2(x) & \text{if } x \in D_2 \end{cases}$$

is a well defined map with domain $D_1 \cup D_2$. In this case, we say that ϕ **is obtained by combining** ϕ_1 **and** ϕ_2.

Theorem 2.1. *Assume that G is a near-triangulation with an orientation D and an exterior directed cycle $C = v_1 v_2 \cdots v_p v_1$. Let A be a group with $|A| \geq 5$ and $f \in F(G, A)$. Let $a_1, a_2 \in A$ such that $a_1 - a_2 \neq f(v_1 v_2)$, and let $A_3, A_4, \ldots, A_p$ be subsets of A such that $|A_3| \leq |A| - 3$, and $|A_i| = 2$ for $4 \leq i \leq p$. Then there is an (A, f)-coloring $c : V(G) \mapsto A$ such that $c(v_1) = a_1$, $c(v_2) = a_2$ and $c(v_i) \notin A_i$, $3 \leq i \leq p$.*

Proof. We argue by induction on $|V(G)|$. Assume first that $|V(G)| = 3$ and $C = v_1v_2v_3v_1$ is a directed 3-cycle. Since $|A| \geq 5$ and $|A_3| = 2$, there is an element $a_3 \in A - \{a_1 + f(v_3v_1), a_2 - f(v_2v_3)\} \cup A_3$. Thus the map $c : V(G) \mapsto A$ given by $c(v_i) = a_i$ $(1 \leq i \leq 3)$ is the desired (A,f)-coloring.

Now we assume that $|V(G)| \geq 4$. If exterior cycle C has a chord v_iv_j, where $2 \leq i \leq j-2 \leq p-1$, (regarding $v_{p+1} = v_1$), then G has two cycles $C_1 = v_1v_2\cdots v_iv_jv_{j+1}\cdots v_1$ and $C_2 = v_jv_iv_{i+1}\cdots v_{j-1}v_j$. Let G_1 denote the plane graph which is the interior of C_1 (together with C_1), and G_2 denote the plane graph which is the interior of C_2 (together with C_2). Note that $G_1 \cap G_2$ is the edge v_iv_j and $G = G_1 \cup G_2$.

Let $c_0 : \{v_1, v_2\} \mapsto A$ be given by $c_0(v_i) = a_i$, $(1 \leq i \leq 2)$. Applying induction to G_1, c_0 can be extended to $c_1 : V(G_1) \mapsto A$, such that $c_1(v_l) \notin A_l$, $3 \leq l \leq i$ and $j \leq l \leq p$. Then applying induction to G_2, $c_1|_{\{v_i,v_j\}}$ can be extended to $c_2 : V(G_2) \mapsto A$, such that $c_2(v_l) \notin A_l$, $i+1 \leq l \leq j-1$.

Therefore, a desired (A,f)-coloring may be obtained by combining c_1 and c_2.

Assume now that C has no chord. Let $v_1, u_1, u_2, \ldots, u_m, v_{p-1}$ be the neighbors of v_p in that clockwise order around v_p. Since G is a near triangulation, and since C has no chord, $C' = v_1v_2\cdots v_{p-1}u_mu_{m-1}\cdots u_2u_1v_1$ is the exterior cycle of the plane graph $G' = G - v_p$.

Without loss of generality, we assume that $C' = v_1v_2\cdots v_{p-1}u_mu_{m-1}\cdots u_2u_1v_1$ is a directed cycle and each edge u_iv_p are oriented from u_i to v_p, for each i with $1 \leq i \leq m$. Choose $b_1, b_2 \in A - (\{f(v_pv_1) + a_1\} \cup A_p)$ with $b_1 \neq b_2$, and let $A_i' = \{f(u_iv_p) + b_1, f(u_iv_p) + b_2\}$, for $i = 1, 2, \ldots, m$.

By induction, there exists an $(A, f|_{E(G')})$-coloring $c_1 : V(G') \mapsto A$ such that

$$c_1(v_1) = a_1, \quad c_1(v_2) = a_2, \quad c(v_i) \notin A_i, \ 3 \leq i \leq p-1, \text{ and} \\ c(u_j) \notin A_j', \ 1 \leq j \leq m. \tag{1}$$

Since $\{b_1, b_2\} - \{c_1(v_{p-1}) - f(v_{p-1}v_p)\} \neq \emptyset$, we may assume that $b_1 \neq c_1(v_{p-1}) - f(v_{p-1}v_p)$. Define $c : V(G) \mapsto A$ as follows:

$$c(x) = \begin{cases} c_1(x) & \text{if } x \neq v_p, \\ b_1 & \text{if } x = v_p. \end{cases} \tag{2}$$

Then by (1), (2) and by the choice of b_1, c is indeed an (A,f)-coloring of G satisfying the conclusions of Theorem 2.1. This completes the proof. □

Let $H \subseteq G$ be graphs, and A be a group. Given an $f \in F(G,A)$, if for an $(A, f|_{E(H)})$-coloring c_0 of H, there is an (A,f)-coloring c of G such that c is an extension of c_0, then we say that c_0 is **extended to** c. If any $(A, f|_{E(H)})$-coloring c_0 of H can be extended to an (A,f)-coloring c, then we say that (G,H) is **(A,f)-extensible**. If for any $f \in F(G,A)$, (G,H) is (A,f)-extensible, then (G,H) is **A-extensible**.

Corollary 2.2. *Let G be a simple planar graph and let $H \subseteq G$ be a subgraph isomorphic to a K_2 or a K_3. Then for any group A with $|A| \geq 5, (G,H)$ is A-extensible.*

Proof. Let $f \in F(G,A)$ and $c_0 : V(H) \mapsto A$ be given such that c_0 is an $(A, f|_{E(H)})$-coloring. We shall show that c_0 can be extended to an (A,f)-coloring $c : V(G) \mapsto A$.

If $H \cong K_2$, then we may assume that $V(H) = \{v_1, v_2\}$, and that in an planar embedding of G, the only edge in $E(H)$ is on the exterior cycle of G, and without loss of generality, assume that G is a near triangulation. By Theorem 2.1, with the A_i's arbitrarily chosen, c_0 can be extended to an (A,f)-coloring of G.

Thus we assume that $H \cong K_3$. If H is a separating cycle of G, then we can apply induction to extend c_0 to the interior of H, and then to the exterior of H, thereby obtaining an (A,f)-extension of c_0 by combining these two extensions of c_0.

Hence we assume that $H = K_3$ is not a separating cycle. Without loss of generality, we may assume that G has an orientation such that H is the exterior directed cycle $C = v_1v_2v_3v_1$.

Let $a_i = c_0(v_i)$, where $i \in \{1,2,3\}$. Since c_0 is an $(A, f|_{E(H)})$-coloring, $a_3 \notin \{a_1 + f(v_3v_1), a_2 - f(v_2v_3)\}$. Therefore $A_3 = A - \{a_3, a_1 + f(v_3v_1),\ a_2 - f(v_2\ v_3)\}$ has exactly $|A| - 3$ elements. By Theorem 2.1, c_0 can be extended to an (A,f)-coloring of G. This completes the proof. □

Corollary 2.3. *Let G be a simple planar graph and let $H \subseteq G$ be a subgraph isomorphic to a K_4. Then for any group A with $|A| \geq 5, (G,H)$ is A-extensible.*

Proof. Embed G on the plane so that this K_4 partition that plane into 4 regions. Corollary 2.3 obtains by applying Corollary 2.2 to each of these regions. □

Corollary 2.4. $\chi_1(G) \leq 5$ *for every simple planar graph G.*

3. K_5-Minor-Free Graphs

In this section, we shall prove a stronger version of Theorem 1.2. We need two former results in the proof.

Theorem 3.1 (Wagner [8], Kuratowski [1], Harary and Tutte [3]). *Graph G is planar if and only if G has no K_5-minor or $K_{3,3}$-minor.*

Theorem 3.2 (Halin [2]). *Every 4-connected non-planar graph contains K_5 as a minor.*

Theorem 3.3. *Let G be a connected K_5-minor free graph and let A be a group with $|A| \geq 5$. Suppose that H is a subgraph of G isomorphic to K_2 or K_3. Then (G,H) is A-extensible.*

Proof. Let $f \in F(G,A)$ and c_0 be an $(A, f|_{E(H)})$-coloring be given. We shall show by induction on $|V(G)|$ that c_0 can be extended to an (A,f)-coloring of G.

By Corollary 2.2, we may assume that G is not planar. Since G is K_5-minor free, we may assume that $|V(G)| \geq 6$; and by Theorem 3.2, $\kappa(G) \leq 3$.

We argue by induction. Assume first that $|V(G)| = 6$. Since G is not planar and is K_5-minor free, by Theorem 3.2, G is spanned by a $K_{3,3}$. Let $\{u_1, u_2, u_3\}$ and $\{w_1, w_2, w_3\}$ be the bipartition of this $K_{3,3}$. Without loss of generality, we assume that $V(H) = \{u_1, w_1\}$ or $V(H) = \{w_1, w_2\}$ if $H \cong K_2$, and $V(H) = \{u_1, w_1, w_2\}$ if $H \cong K_3$. If no vertex in $V(G) - V(H)$ has degree at most 4 in G, then $G[\{u_1, u_2, u_3\}] \cong K_3$ and so G contain a K_5, contrary to the assumption that G is K_5-minor free. Thus $G - V(H)$ has a vertex of degree at most 4 in G. Label this vertex as v_6. Since G is simple, we can label $V(G)$ as $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ such that $\delta(G[\{v_1, v_2, \ldots, v_i\}]) \leq 4$, for each i with $2 \leq i \leq 6$, and such that $V(H) = \{v_1, v_2\}$ if $|V(H)| = 2$, and $V(H) = \{v_1, v_2, v_3\}$ if $|V(H)| = 3$. Now it is easy to see that for $f \in F(G, A)$, we can always color vertices one by one in the order they appear so that the resulting coloring is an (A, f)-coloring.

Assume then that $|V(G)| > 6$, and that the theorem holds for graphs with smaller values of $|V(G)|$.

Let $t = \kappa(G)$ and $T = \{x_1, \ldots, x_t\}$ be a minimum vertex cut of G. By the assumption that G is connected and by $\kappa(G) \leq 3$, $1 \leq t \leq 3$.

Let G_1 and G_2 be induced subgraphs of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = T$ and $V(G_i) - V(G_{3-i}) \neq \emptyset$, $i = 1, 2$. Assume that $H \subseteq G_1$. For $i = 1$ and 2, let $G'_i = G_i$ if $t = 1$; let $G'_i = G_i \cup \{x_1x_2\}$ if $t = 2$; and let $G'_i = G_i \cup \{x_1x_2, x_2x_3, x_3x_1\}$ if $t = 3$. Let $G' = G'_1 \cup G'_2$ and extend f to $f' \in F(G', A)$ arbitrarily.

Claim 1. *We may assume that $t = 3$.*

If $t = 1$ or 2, then G'_1 and G'_2 are K_5-minor-free. Since $|V(G'_1)| < |V(G)|$ and $|V(G'_2)| < |V(G)|$, and by induction (in case that $t = 1$, for each $i = 1, 2$, we color another vertex u_i adjacent to x_1 in G_i with color $c_0(u_i)$ such that $c_0(u_i) - c_0(x_1) \neq f(u_ix_1)$), there is an $(A, f'|_{E(G'_1)})$-coloring $c_1 : V(G'_1) \mapsto A$ which extends c_0; and an $(A, f'|_{E(G'_2)})$-coloring $c_2 : V(G'_2) \mapsto A$ which extends $c_1|_T$. It follows that an (A, f)-coloring $c : V(G) \mapsto A$ which extends c_0 is obtained by combining c_1 and c_2. This proves Claim 1.

Claim 2. *For each $i \in \{1, 2\}$, if $G_i \ncong K_{1,3}$, then G_i has a subgraph which can be contracted to a K_3 such that the contraction images of the three vertices of this K_3 contain x_1, x_2 and x_3, respectively.*

We only prove that case when $i = 1$. The case when $i = 2$ is similar. Since each graph in Figure 1 can be contracted to a K_3 satisfying the conclusion of Claim 2, we shall show that G_1 contains a subgraph which is a subdivision of a graph isomorphic to one of the graphs in Figure 1.

Let $L_1, L_2, \ldots, L_k$ be the components of $G_1 - T$. By Claim 1, $\kappa(G) = 3$, and so there are at least three edges from each component L_i $(1 \leq i \leq k)$ to T which are incident with x_1, x_2 and x_3 respectively.

If $E[G[T]] \neq \emptyset$, we may assume that $x_2x_3 \in E(G)$. Then $G_1/E(L_1)$ contains a subgraph isomorphic to (a) of Figure 1.

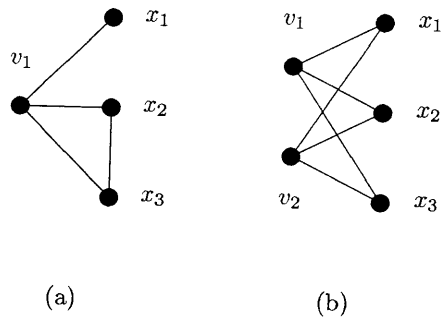

Fig. 1. The two graphs in the proof of Claim 2

Suppose that $E[G[T]] = \emptyset$ and $k \geq 2$. Then $G_1/E(L_1 \cup L_2)$ contains a subgraph L isomorphic to (b) of Figure 1.

Suppose that $E[G[T]] = \emptyset$ and $k = 1$. Since $G_1 \not\cong K_{1,3}$, $|V(G_1 - T)| \geq 2$. Pick a vertex $v_1 \in V(G_1 - T)$. By $\kappa(G) = 3$, G_1 has a (v_1, x_1)-path P_1, a (v_1, x_2)-path P_2 and a (v_1, x_3)-path P_2 such that $V(P_i) \cap V(P_j) = \{v_1\}$ whenever $i \neq j$.

Case 1. Assume first that there is a vertex $v_1 \in V(G_1) - T$ with the three (v_1, x_i)-paths P_i's, and an edge u_1u_2 such that $u_1 \in V(P_1)$, $u_2 \in V(P_2)$ and $u_1, u_2 \neq v_1$. Let $L = G_1[\cup_{i=1}^3 E(P_i) \cup \{u_1u_2\}]$. Then L is a subdivision of a graph isomorphic to (a) of Figure 1.

Case 2. Assume Case 1 does not hold. Since $|V(G_1) - T| \geq 2$, there is a $v_2 \in V(G_1) - (\cup_{i=1}^3 V(P_i))$ and $v_2 \neq v_1$. By $\kappa(G) = 3$, G_1 has a (v_2, x_1)-path P_1', a (v_2, x_2)-path P_2' and a (v_2, x_3)-path P_3' such that $V(P_i') \cap V(P_j') = \{v_1\}$ whenever $i \neq j$.

Since Case 1 does not hold, $(E(P_1') \cup E(P_2')) \cap (\cup_{i=1}^3 E(P_i)) \subset T$. It follows that $G_1[(\cup_{i=1}^3 E(P_i)) \cup (E(P_1') \cup E(P_2'))]$ is a subdivision of a graph isomorphic to (a) of Figure 1. This completes the proof of Claim 2.

Claim 3 below follows easily by Claim 2.

Claim 3. *If $G_i \not\cong K_{1,3}$, then G_{3-i}' is K_5-minor free, where $i = 1$ or 2.*

Claim 4. *We may assume either $G_1 \cong K_{1,3}$ or $G_2 \cong K_{1,3}$.*

If for both $i = 1$ and 2, $G_i \not\cong K_{1,3}$, then by Claim 3, G_{3-i}' is K_5-minor-free. Since $|V(G_i) - V(G_{3-i})| > 0$, $|V(G_i')| = |V(G_i)| < |V(G)|$. By induction, c_0 can be extended to an $(A, f'|_{E(G_1')})$-coloring c_1; and then $c_1|_T$ can be extended to an $(A, f'|_{E(G_2')})$-coloring c_2. Thus, an (A, f')-coloring c of G' extending from c_0 can be obtained by combining c_1 and c_2. Since G is a subgraph of G' and since $f'|_{E(G)} = f$, c is also an (A, f)-coloring of G. This proves Claim 4.

If $G_2 \cong K_{1,3}$, let $V(G_2) = \{x_1, x_2, x_3, v\}$, $G'' = G \cup \{x_1x_2, x_2x_3\}$ and $G_1'' = G_1 \cup \{x_1x_2, x_2x_3\}$. (See (a) of Figure 2). Then G_1'' is K_5-minor-free since G_2 can be contracted to a path $P = x_1x_2x_3$. Extend f to $f'' \in F(G'', A)$ arbitrarily. By induction, c_0 can be extended from H to G_1'', then from G_1'' to G'' since

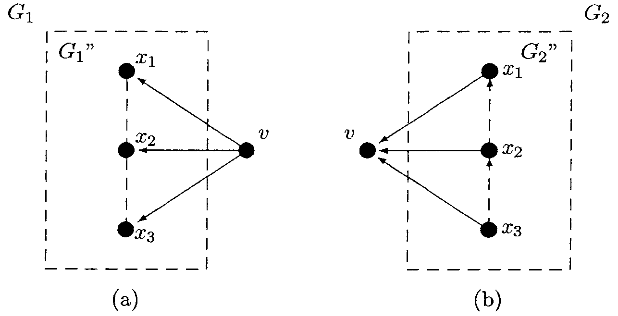

Fig. 2. $G_2 \cong K_{1,3}$ or $G_1 \cong K_{1,3}$

$d(v) = 3 < |A|$. Thus an (A, f'')-coloring c of G'' extending from c_0 is obtained. Since G is a subgraph of G'', c is also an (A, f)-coloring of G.

If $G_1 \cong K_{1,3}$, we assume that $V(G_1) = \{v, x_1, x_2, x_3\}$. Let $G'' = G \cup \{x_1x_2, x_2x_3\}$, $G_1'' = G_1 \cup \{x_1x_2, x_2x_3\}$ and $G_2'' = G_2 \cup \{x_1x_2, x_2x_3\}$. (See (b) of Figure 2.)

Then G_2'' is K_5-minor free since G_1 can be contracted to a path $P = x_1x_2x_3$ and since G is assumed to be K_5-minor free. Since $H \subseteq G_1$ and $G_1 \cong K_{1,3}$, H must be a K_2, and so we may assume that $H = G[\{v, x_1\}]$ and that $c_0(x_1) - c(v) \neq f(x_1v)$. Extend c_0 to $c_1 : V(G_1 - x_3) \mapsto A$ and extend f to $f'' \in F(G'', A)$ by the following steps (i)–(iii).

(i) Define $f''(x_2x_1) \in A$ arbitrarily.
(ii) Extend c_0 from H to $c_1 : \{v, x_1, x_2\} \mapsto A$ in $G''[\{v, x_1, x_2\}]$ by choosing $c_1(x_2) \in A - \{c_0(v) + f(x_2v),\ c_0(x_1) + f(x_2x_1)\}$.
(iii) Define $f''(x_3x_2) = c_0(v) + f(x_3v) - c_0(x_2)$.

By induction, $c_1|_{\{x_1,x_2\}}$ can be extended to an $(A, f''|_{E(G_2'')})$-coloring c_2 in G_2''. Obtain c by combining c_1 and c_2. Since $c_2(x_3) - c_2(x_2) = c_2(x_3) - c_1(x_2) \neq c_0(v) + f(x_3v) - c_1(x_2)$, and since $c_2(x_2) = c_1(x_2)$, we have $c_2(x_3) - c_1(v) \neq f(x_3v)$. Combine c_1 and c_2 to get an (A, f)-coloring c of G which extends c_0. This proves Theorem 3.3. □

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