Group Colorability of Graphs

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Abstract

Let $G = (V, E)$ be a graph and A a non-trivial Abelian group, and let $F(G, A)$ denote the set of all functions f: $E(G) \to A$. Denote by D an orientation of $E(G)$. Then G is A-colorable if and only if for every $f \in F(G, A)$ there exists an A-coloring c: $V(G) \to A$ such that for every $e = (x, y) \in E(G)$ (assumed to be directed from x to y), $c(x) - c(y) \neq f(e)$. If G is a graph, we define its group chromatic number $\chi_1(G)$ to be the minimum number m for which G is A-colorable for any Abelian group A of order $\geq m$ under the orientation D. In this paper, we investigated the properties of the group chromatic number, proved the Brooks Type theorem for $\chi_1(G)$, and characterized all bipartite graphs with group chromatic number at most 3, among other things.

1. INTRODUCTION

Graphs in this note are simple, finite and loopless, unless otherwise stated. Undefined terms and notation are from [2]. We use $H \subseteq G$ to denote that H is a subgraph of G.

Let $G = (V, E)$ be a graph and A a non-trivial Abelian group, and let $F(G, A)$ denote the set of all functions f: $E(G) \to A$. Denote by D an orientation of $E(G)$. An oriented edge uv of G (assumed to be directed from u to v) is called an arc uv. The graph G under the orientation D is sometimes denoted by $D(G)$.

DEFINITION 1.1. For $f \in F(G, A)$, an A-**coloring** (or (A, f)-**coloring**) of G under the orientation D is a function c: $V(G) \to A$ such that for every arc $e = uv \in E(G)$, $c(u) - c(v) \neq f(e)$.

DEFINITION 1.2. G is A-**colorable** under the orientation D if for every $f \in F(G, A)$,there exists an A-coloring.

F. Jaeger, N. Linial, C. Payan, and M. Tarsi [7] proposed the definition of group colorability of graphs as the equivalence of group connectivity of M, where M is a cographic matroid. Clearly, an A-colorable graph is $|A|$-colorable (take $f = 0$) and A-colorability is the dual of local A-connectivity, in the same way that k-colorability is the dual of admitting a k-nowhere-zero flow.

DEFINITION 1.3. The **group chromatic number** of a graph G is defined to be the minimum m for which G is A-colorable for any group A of order $\geq m$ under the orientation D. The group chromatic number of graph G under the orientation D is simply denoted by $\chi_1(G)$.

Let $\chi(G)$ denote the **chromatic number**,which is the minimum k for which G is k-colorable; if $\chi(G) = k$, G is said to be k-chromatic.

Since an A-colorable graph is $|A|$-colorable, for any graph G, $\chi_1(G) \geq \chi(G)$.

F. Jaeger, N. Linial, C. Payan, and M. Tarsi [7] proved the following result:

THEOREM 1.1 ([7], Proposition 4.2). If G is a simple planar graph, then $\chi_1(G) \leq 6$.

DEFINITION 1.4. Let G be a graph, $H \subseteq G$,and A a non-trivial Abelian group. Then (G, H) is said to be **A-extendible** if for any $f \in F(G, A)$ and any A-coloring c' of H for $f|_{E(H)}$, there is an A-coloring c of G for f such that $c|_{V(H)} = c'$. G is said to be **strong A-colorable** if for any subgraph H of G, (G, H) is A-extendible.

By definition, (G, H) is A-extendible if and only if for any $f \in F(G, A)$, any A-coloring of H for $f|_{E(H)}$ can be extended to G for f.

2. ELEMENTARY PROPERTIES

LEMMA 2.1. Let D be an orientation of $E(G)$ and E_0 be a subset of $E(G)$. Let D' be the orientation of $E(G)$ obtained from D by reversing the direction of every arc in E_0. Assume that A is a non-trivial Abelian group. If G is A-colorable under the orientation D, then G is also A-colorable under the orientation D'.

Proof. Let $f' \in F(G, A)$. We consider the ordered pair (D, f), where f is defined as follows:

$$f(e) = \begin{cases} f'(e), & if\ e \notin E_0 \\ -f'(e), & if\ e \in E_0. \end{cases} \tag{1}$$

Since G is A-colorable under the orientation D, by Dfinition 1.1, there exists a function c: $V(G) \to A$ such that for every arc $e = xy \in E[D(G)]$, $c(x) - c(y) \neq f(e)$. If $e \notin E_0$, then $e \in E[D'(G)]$ and $c(x) - c(y) \neq f(e) = f'(e)$; if $e \in E_0$, then $yx \in E[D'(G)]$ and $c(x) - c(y) \neq f(e)$, namely, $c(y) - c(x) \neq -f(e) = f'(e)$. Hence, G is A-colorable under the orientation D'. □

By Lemma 2.1, it is easy to see that

THEOREM 2.1. Let G be a graph and D be an orientation of $E(G)$. Then, for any Abelian group A, graph G is A-colorable under the orientation D if and only if G is A-colorable under every orientation of $E(G)$.

THEOREM 2.2. Let A be an Abelian group. Then graph G is A-colorable if and only if each block of G is A-colorable.

Proof. If G is A-colorable, then every subgraph of G is also A-colorable, and so each block of G is A-colorable.

It clearly suffices to prove the converse for connected graphs with two blocks. Let G be a connected graph with two blocks G_1 and G_2 and assume that G_1 and G_2 are A-colorable. Let v_0 be the cut vertex of G. Then $v_0 \in V(G_1) \cap V(G_2)$.

For any $f \in F(G, A)$, we can get two functions $f|_{E(G_1)} \in F(G_1, A)$ and $f|_{E(G_2)} \in F(G_2, A)$. Let $f_1 = f|_{E(G_1)}$ and $f_2 = f|_{E(G_2)}$. Since G_1 and G_2 are A-colorable, there exist an A-coloring $c_1 : V(G_1) \to A$ for $f_1 \in F(G_1, A)$ and an A-coloring $c_2 : V(G_2) \to A$ for $f_2 \in F(G_2, A)$. Let $c_2' : V(G_2) \to A$ be defined by

$$c_2'(v) = c_2(v) - c_2(v_0) + c_1(v_0)$$

for each $v \in V(G_2)$. Obviously, c_2' is an A-coloring of G_2 for f_2. Define $c : V(G) \to A$ as follows:

$$c(v) = \begin{cases} c_1(v), & if\ v \in V(G_1) \\ c_2'(v), & if\ v \in V(G_2). \end{cases} \tag{2}$$

It is easy to see that c is an A-coloring of G for $f \in F(G, A)$.□

THEOREM 2.3. Let A be an Abelian group and $H \subseteq G$. If (G, H) is A-extendible and H is A-colorable, then G is A-colorable.

Proof. For any $f \in F(G, A)$, since H is A-colorable, there is an A-coloring $c_1 : V(H) \to A$ for $f|_{E(H)}$. Since (G, H) is A-extendible, c_1 can be extended to G for f such that $c|_{V(H)} = c_1$. Then G is A-colorable. □

THEOREM 2.4. Let A be an Abelian group and $H_2 \subseteq H_1 \subseteq G$. If (G, H_1) and (H_1, H_2) are A-extendible, then (G, H_2) is also A-extendible.

Proof. For any $f \in F(G,A)$, let $f_1 = f|_{E(H_1)}$ and $f_2 = f|_{E(H_2)}$. Since (H_1, H_2) is A-extendible, any A-coloring c_1 of H_2 for f_2 can be extended to an A-coloring c_1' of H_1 for f_1 such that $c_1'|_{V(H_2)} = c_1$. Since (G, H_1) is A-extendible, any A-coloring c_1' of H_1 for f_1 can be extended to an A-coloring c of G for f such that $c|_{V(H_1)} = c_1'$ and $c|_{V(H_2)} = c_1'|_{V(H_2)} = c_1$. Hence, (G, H_2) is A-extendible. □

Let A and A' be two Abelian groups and let $\varphi : A \to A'$ be a homomorphism. Then $im(\varphi)$, the image of A under φ, is a subgroup of A'.

THEOREM 2.5. Let $\varphi : A \to A'$ be a homomorphism and G be a graph. If G is $im(\varphi)$-colorable, then G is also A-colorable.

Proof. Let $f \in F(G,A)$. Then $\varphi f \in F(G,A')$. Since G is $im(\varphi)$-colorable, there exists an $im(\varphi)$-coloring $c' : V(G) \to im(\varphi)$ such that for every arc $e = xy$ of G, $c'(x) - c'(y) \neq \varphi f(e)$. Define $c : V(G) \to A$ as follows: for $v \in V(G)$, let $c(v) = a \in A$ such that $\varphi(a) = c'(v)$. For every arc $e = xy \in E(G)$, it is easy to see that $c(x) - c(y) \neq f(e)$. Otherwise, $\varphi(c(x) - c(y)) = \varphi f(e)$, namely, $c'(x) - c'(y) = \varphi f(e)$, a contradiction.□

COROLLARY 2.1. Let G be a graph. If φ is a homomorphism of A onto A' and G is A'-colorable, then G is also A-colorable.

Let N be a normal subgroup of A. The function $\pi : A \to A/N$ (A/N is called the quotient group of A) defined by $\pi(a) = aN$ is a homomorphism of A onto A/N. By Corollary 2.1, we have the following result.

COROLLARY 2.2. Let A be an Abelian group and N be any subgroup of A. If the graph G is A/N-colorable, then G is also A-colorable.

Suppose that A and A' are two finite cyclic groups with orders m and n, respectively. Then there exists a homomorphism of A onto A' if and only if $n|m$.

COROLLARY 2.3. For any graph G, and any positive integers k and n, if G is Z_n-colorable, then G is also Z_{kn}-colorable.

In Section 4, we show that for any graph G, $\chi_1(G) \leq \Delta(G)+1 \leq |V(G)|$, which implies that any graph G is A-colorable, where $\Delta(G)$ is the maximum degree of G and A is an Abelian group with order $\geq \Delta(G)+1$.

3. Z_2 – COLORABLE GRAPHS

LEMMA 3.1. If G is Z_2-colorable, then G is a forest.

Proof. We prove it by contradiction. Assume that $c = v_0v_1 \cdots v_kv_0$ is a directed cycle of G. Let $e_i = v_iv_{i+1}$ $(i = 0, 1, \cdots, k-1)$ and $e_k = v_kv_0$. Let $f \in F(G, Z_2)$ be defined as follows.

(i). If k is odd, then

$$f(e) = \begin{cases} 1, & if\ e = e_k, \\ 0, & otherwise. \end{cases}$$

(ii). If k is even, let $f(e) = 0$ for any $e \in E(G)$.

We only consider the case when k is odd. The other case is similar.

Assume that for the function f, there exists an A-coloring $c : V(G) \to Z_2$ such that for every arc $e = xy \in E(G)$, $c(x) - c(y) \neq f(e)$. If $c(v_0) = 1$, then $c(v_1) = 0, c(v_2) = 1, \cdots, c(v_k) = 0$; if $c(v_0) = 0$, then $c(v_1) = 1, c(v_2) = 0, \cdots, c(v_k) = 1$. Thus $c(v_k) - c(v_0) = 1 = f(e)$, a contradiction.

Hence, if G is Z_2-colorable, then G is acyclic. □

On the other hand, it is easy to use induction to show that every forest has group chromatic number at most 2. Therefore, we have:

THEOREM 3.1. For any graph G, $\chi_1(G) = 2$ if and only if G is a forest.

Furthermore we have:

THEOREM 3.2. Let G be a forest and $H \subseteq G$. Then (G, H) is Z_2-extendible if and only if any two components of H belong to two different components of G.

Proof. Without loss of generality, we may assume that G is a tree. We need to prove that (G, H) is Z_2-extendible if and only if H is a connected subgraph of G.

If H is connected, let u_0v_0 be an arc of G such that $u_0 \in E(H)$ and $v_0 \notin E(H)$. For any $f \in F(G, A)$, any Z_2-coloring c' of H for $f|_{E(H)}$ is easily extended to a Z_2-coloring c_1 of the subgraph $H_1 = H \cup \{u_0v_0\}$ by a simple extension: let $c_1(v) = c'(v)$ if $v \in V(H)$ and $c_1(v_0) = a \neq c'(u_0) - f(u_0v_0)$. Hence, any Z_2-coloring of H for $f|_{E(G)}$ can be extended to a Z_2-coloring of G for f by $|V(G)| - |V(H)|$ simple extensions, and so (G, H) is Z_2-extendible.

If H is not connected, we may assume that H has two components H_1 and H_2. Let $v_0v_1 \cdots v_k$ be a directed path of G such that $v_0 \in V(H_1)$, $v_k \in V(H_2)$ and $v_i \notin V(H)$ $(3 \leq i \leq k-1)$. Let $e_i = v_iv_{i+1}$ $(i = 0, 1, \cdots, k-1)$ and $f \in F(G, Z_2)$ be defined as follows: For any $e \in E(G)$, let $f(e) = 0$ if $e = e_{k-1}$, and let $f(e) = 1$ otherwise. Let c_1 be a Z_2-coloring of H for $f|_{E(H)}$ such that $c_1(v) = 1$ for every $v \in V(H)$. It is easy to see that c_1 cannot be extended to G for F, and so (G, H) is not Z_2-extendible. □

THEOREM 3.3. For any Abelian group A with order $|A| \geq 3$, and for any forest G, G is strong A-colorable.

Proof. We need to prove that for any subgraph H of G, (G, H) is A-extendible.

We may assume without loss of generality that G is a tree and perform the proof by induction on $\omega(H)$, the number of components of subgraph H.

From the proof of previous theorem, we easily know that the present theorem holds when $\omega(H) = 1$. Let k be a positive integer and assume that the theorem is valid when $\omega(H) \leq k$. Suppose, now, that H has $k+1$ components. Choose two components H_1 and H_2 of H such that there exists a directed path $P = v_0 v_1 \cdots v_k$ with $v_0 \in H_1$, $v_k \in H_2$ and $v_i \notin V(H)$ $(1 \leq i \leq k-1)$. For any $f \in F(G, A)$, let $c_1 : V(H) \to A$ be an A-coloring of H for $f|_{E(H)}$. Define $c : V(H \cup P) \to A$ as follows: Let $c(v) = c_1(v)$ if $v \in V(H)$, $c(v_i) = a_i \in A - \{c(v_{i-1}) + f(v_{i-1}v_i)\}$ $(1 \leq i \leq k-2)$ and $c(v_{k-1}) = a_{k-1} \in A - \{c(v_{k-2}) - f(v_{k-2}v_{k-1}), c(v_k) + f(v_{k-1}v_k)\}$. Then c is an A-coloring of $H \cup P$ for $f|_{E(H \cup P)}$ and is an extension of c_1, namely, $(H \cup P, H)$ is A-extendible. Now, $\omega(H \cup P) = k$, and by the induction hypothesis, $(G, H \cup P)$ is A-extendible. By Theorem 2.4, (G, H) is A-extendible.

Thus (G, H) is A-extendible for all subgraphs H of G. □

4. THE ANALOGUE OF BROOKS' THEOREM

Denote the maximum degree of the graph G by $\Delta(G)$. The following theorem is the well-known theorem of Brooks which relates the chromatic number of a graph to its maximum degree.

THEOREM 4.1 (Brooks [3]). For any connected graph G,

$$\chi(G) \leq \Delta(G) + 1$$

with equality if and only if either $\Delta(G) = 2$ and G is an odd cycle; or $\Delta(G) \geq 3$ and G is complete.

For the group chromatic number of the graph G, we can get the following analogue to Brooks' Theorem.

THEOREM 4.2. For any connected simple graph G,

$$\chi_1(G) \leq \Delta(G) + 1$$

with equality if and only if G is a cycle $(\Delta(G) = 2)$, or G is complete

$(\Delta(G) \geq 3)$.

We need some lemmas in the proof of Theorem 4.2.

LEMMA 4.1. Let G be a graph and suppose that $V(G)$ can be linearly ordered as $v_1, v_2, \cdots, v_n$ such that $d_{G_i}(v_i) \leq k$ $(i = 1, 2, \cdots, n)$, where $G_i = G[\{v_1, v_2, \cdots, v_i\}]$. Then for any Abelian group A of order $\geq k+1$, (G_{i+1}, G_i) $(i = 1, 2, \cdots, n-1)$ is A-extendible and so G is A-colorable.

Proof. Let D be an orientation of $E(G_{i+1})$ such that every $e = v_{j_1}v_{j_2} \in E(G_{i+1})$ is directed from v_{j_1} to v_{j_2} if $j_1 > j_2$ and from v_{j_2} to v_{j_1} otherwise. For any $f \in F(G_{i+1}, A)$ and any A-coloring c_1 of G_i for $f|_{E(G_i)}$, we define an A-coloring $c : V(G_{i+1}) \rightarrow A$ as follows: Assume that $v_{i_1}v_{i+1}, v_{i_2}v_{i+1}, \cdots, v_{i_r}v_{i+1}$ are all the edges joining v_{i+1} $(0 \leq r \leq k)$ in G_{i+1} and let $c(v) = c_1(v)$ if $v \in V(G_i)$, $c(v_{i+1}) = a'$ such that $a' \in A' = A - \{c(v_{i_p}) + f(v_{i_p}v_{i+1}) | p = 1, 2, \cdots, r\}$. Since $|A| \geq k+1$, $A' \neq \emptyset$. Hence (G_{i+1}, G_i) is A-extendible, where $i = 1, 2, \cdots, n-1$.

Since G_1 is A-colorable, by Theorem 2.3 and 2.4, G is A-colorable.□

By Lemma 4.1, we have the following lemma, which is essentially the same as the result of chromatic number due to G. Szekeres and H. S. Wilf [8].

LEMMA 4.2. $\chi_1(G) \leq \max_{H \subseteq G}\{\delta(H)\} + 1$.

Proof. Let $|V(G)| = n$, $k = \max_{H \subseteq G}\{\delta(H)\} + 1$, and v_n be a vertex of degree at most k. Put $H_{n-1} = G - \{v_n\}$. By assumption H_{n-1} has a vertex, say v_{n-1}, of degree at most k. Put $H_{n-2} = G - \{v_n, v_{n-1}\}$. Continuing in this way we enumerate all the vertices of G. Hence we get a sequence $v_1, v_2, \cdots, v_n$ such that each v_j is joined to at most k vertices preceding it. Thus Lemma 4.2 follows from Lemma 4.1. □

An immediate corollary is given below.

COROLLARY 4.1. For any graph G, $\chi_1(G) \leq \Delta(G) + 1$.

Since every nontrivial simple graph without a subdivision of K_4 has a vertex of degree at most 2, by Theorem 3.1 and Lemma 4.2, we have the following result.

COROLLARY 4.2. Let G be a nontrivial simple graph without subdivision of K_4. Then $\chi_1(G) = 3$ if and only if G has a cycle.

COROLLARY 4.3. $\chi_1(K_n) = n$ for the complete graph K_n on n vertices.

Proof. $n = \chi(K_n) \leq \chi_1(K_n) \leq \Delta(K_n) + 1 = n$. □

By modifying the proof of Brooks' Theorem in [1], we obtain the following:

Proof of Theorem 4.2. If G is connected and not regular of degree $\Delta(G)$, then $max_{H \subset G}\delta(H) \leq \Delta(G) - 1$ and so $\chi_1(G) \leq \Delta(G)$. Without loss of generality, let G be 2-connected and $\Delta(G)$-regular. If G is a complete graph, then $\chi_1(G) = |V(G)| = \Delta(G) + 1$.

If $\Delta(G) = 2$, then G is a cycle and so $\chi_1(G) = 3 = \Delta(G) + 1$. If G is 3-connected and G is not complete, then there are three vertices v_1, v_2 and v_n ($n = |V(G)|$) in G such that $v_1v_n, v_2v_n \in E(G)$ and $v_1v_2 \notin E(G)$. If G is 2-connected, let $\{v_n, v'\}$ be a cut set of G. Then there are two vertices v_1 and v_2 belonging to different endblocks of $G - v_n$. Now, we arrange the vertices of $G - \{v_1, v_2\}$ in nonincreasing order of their distance from v_n, say $v_3, v_4, \cdots, v_n$. Then the sequence $v_1, v_2, \cdots, v_n$ is such that each vertex other than v_n is adjacent to at least one vertex following it, namely each vertex other than v_n is joined to at most $\Delta(G) - 1$ vertices preceding it.

Let D be an orientation of $E(G)$ such that every $e = v_iv_j \in E(G)$ is directed from v_i to v_j if $i > j$ and from v_j to v_i otherwise. For any $f \in$

$F(G,A)$ ($|A| \geq \Delta(G)$), we define an A-coloring $c : V(G) \to A$ as follows: Assign $a_1 \in A$ to $c(v_1)$ and $a_2 \in A$ to $c(v_2)$ such that $a_1 + f(v_1v_n) = a_2 + f(v_2v_n)$; for v_j ($3 \leq j \leq n$), let $v_{i_1}v_j, v_{i_2}v_j, \cdots, v_{i_r}v_j \in E(G)$ ($r \leq \Delta(G)-1$ if $j < n$) be the edges joining v_j and having $i_p < j$ ($p = 1, 2, \cdots, r$), and assign a_j to $c(v_j)$ such that $a_j \in A_j = A - \{c(v_{i_p}) + f(v_{i_p}v_j) | p = 1, 2, \cdots, r\}$. If $j < n$, then $r \leq \Delta(G) - 1$ and so $A_j \neq \emptyset$; if $j = n$, then $A_n \neq \emptyset$, since $a_1 + f(v_1v_n) = a_2 + f(v_2v_n)$.

Hence, for every $f \in F(G, A)$ ($|A| \geq \Delta(G)$), there exists an A-coloring. □

5. χ_1(G) AND χ(G)

Following the definition of $\chi_1(G)$ and $\chi(G)$, we know that for any graph, $\chi_1(G) \geq \chi(G)$. In this section, we present a result that there exists a graph G such that $\chi_1(G) - \chi(G)$ may be arbitrarily large.

We first prove the following theorem.

THEOREM 5.1. For any complete bipartite graph $K_{m,n}$ with $n \geq m^m$, $\chi_1(K_{m,n}) = m + 1$.

Proof. Assume that $K_{m,n}$ has the vertex bipartition (X, Y) with $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$. Let A be an Abelian group of order $\leq m$ and D be an orientation of $E(K_{m,n})$ such that every $e = x_iy_j \in E(K_{m,n})$ is directed from y_j to x_i.

Denote the set of all functions $c : V(K_{m,n}) \to A$ by $C(K_{m,n}, A)$. For every function $c \in C(K_{m,n}, A)$, we can get a function $c|_X : X \to A$. Let $C(X, A) = \{c|_X \mid c \in C(K_{m,n}, A)\}$. Since $|A| \leq m$, $|C(X,A)| = |A|^m \leq m^m$. Assume that $C(X, A) = \{c_1, c_2, \cdots, c_r\}$, where $r = |A|^m$. Now we define $f_l \in F(K_{m,n}, A)$ ($l = 1, 2, \cdots, r$) as follows: If $l \neq j$, let $f_l(y_jx_i) = 0$ for every i, and otherwise let $f_l(y_lx_i) = a_{l_i} \in A$ such that $\{c_l(x_i) + a_{l_i} | i = 1, 2, \cdots, m\} = A$. Let $f = \sum_{l=1}^{r} f_l$. Then for any function $c : V(K_{m,n}) \to A$, there exists at least one arc $e = y_jx_i \in E(K_{m,n})$ such that $c(y_i) - c(x_i) = f(e)$. Hence $\chi_1(K_{m,n}) \geq m + 1$.

On the other hand, by Lemma 4.2, $\chi_1(K_{m,n}) \leq m + 1$.
Therefore $\chi_1(K_{m,n}) = m + 1$. □

THEOREM 5.2. For any positive integers m and k, there exists a graph G such that $\chi(G) = m$ and $\chi_1(G) = m + k$.

Proof. Let G be a graph with $(2m + k) + (m + k)^{m+k} - 1$ vertices formed from a complete subgraph K_m with m vertices and a complete bipartite subgraph K_{r_1,r_2} with $r_1 = m + k$ and $r_2 = (m + k)^{m+k}$ such that

$$|V(K_m) \cap V(K_{r_1,r_2})| = 1.$$

Obviously $\chi(G) = m$, and by Theorem 5.1 we easily know that $\chi_1(G) = m + k$. □

6. $\chi_1(G)$ AND $\chi_1(G^c)$

Let G^c denote the complement of a graph G. E.A.Nordhaus and J.W.Gadd (1956) proved the following theorem:

THEOREM 6.1. If G is a graph of order n, then $2\sqrt{n} \leq \chi(G) + \chi(G^c) \leq n + 1$, and $n \leq \chi(G)\chi(G^c) \leq ((n + 1)/2)^2$.

In this section, we present the following result about the group chromatic number of a graph and its complement.

THEOREM 6.2. If G is a graph of order n, then $2\sqrt{n} \leq \chi(G) + \chi(G^c) \leq \chi_1(G) + \chi_1(G^c) \leq n + 1$, and $n \leq \chi(G)\chi(G^c) \leq \chi_1(G)\chi_1(G^c) \leq ((n + 1)/2)^2$.

We need three more lemmas in the proof for Theorem 6.2.

By a simple argument which is similar to the proof of Lemma 4.1, we have the following lemma.

LEMMA 6.1. For any graph G, vertex $v_0 \in V(G)$, and any Abelian group A with $|A| \geq d_G(v_0) + 1$, $(G, G - v_0)$ is A-extendible, and so G is A-colorable if and only if $G - v_0$ is A-colorable.

LEMMA 6.2. Any simple graph G has at least $\chi_1(G)$ vertices of degree at least $\chi_1(G) - 1$.

Proof. Let $k = \chi_1(G)$. By Lemma 6.1, we may assume that each vertex of G has degree at least $k - 1$, and so $|V(G)| \geq k$. Hence, G has at least k vertices of degree $\geq k - 1$. □

LEMMA 6.3. If $d_1 \geq d_2 \geq \cdots \geq d_n$ is the degree sequence of G, then $\chi_1(G) \leq \max_i \min\{d_i + 1, i\}$.

Proof. By Lemma 6.2, we have $\chi_1(G) = \min\{d_{\chi_1(G)} + 1, \chi_1(G)\} \leq \max_i min\{d_i + 1, i\}$. □

Proof of Theorem 6.2. Since, for any graph G, $\chi_1(G) \geq \chi(G)$, we only need to show that $\chi_1(G) + \chi_1(G^c) \leq n + 1$, and $\chi_1(G)\chi_1(G^c) \leq ((n+1)/2)^2$.

Let $d_1 \geq d_2 \geq \cdots \geq d_n$ be the degree sequence of G, and $d'_1 \geq d'_2 \geq \cdots \geq d'_n$ be the degree sequence ofG^c. By Lemma 6.3, there are two integers p and q such that $\chi_1(G) \leq \min\{d_p + 1, p\}$ and $\chi_1(G^c) \leq \min\{d'_q + 1, q\}$. We consider the following cases.

Case 1. $q \geq n - p + 1$.

Then $n - 1 = d_p + d'_{n-p+1} \geq d_p + d'_q \geq (\chi_1(G) - 1) + (\chi_1(G^c) - 1)$, and so $\chi_1(G) + \chi_1(G^c) \leq n + 1$. Also $\chi_1(G)\chi_1(G^c) \leq (d_p + 1)(d'_q + 1) = d_p d'_q + d_p + d'_q + 1 \leq d_p d'_q + n \leq d_p d'_{n-p+1} + n \leq ((n-1)/2)^2 + n = ((n+1)/2)^2$.

Case 2. $q \leq n - p + 1$.

Since $\chi_1(G) \le p$ and $\chi_1(G^c) \le q$, we have $n+1 \ge p+(n-p+1) \ge p+q \ge \chi_1(G)+\chi_1(G^c)$, and also $\chi_1(G)\chi_1(G^c) \le pq \le p(n-p+1) = pn-p^2+p \le ((n+1)/2)^2 - (p-(n+1)/2)^2 \le ((n+1)/2)^2$.□

Obviously, for any graph G with order n, if $\chi(G)+\chi(G^c) = n+1$, then $\chi_1(G)+\chi_1(G^c) = n+1$; if $\chi(G)\chi(G^c) = ((n+1)/2)^2$, then $\chi_1(G)\chi_1(G^c) = ((n+1)/2)^2$. For any $n \ge 6$, we define a graph G with n vertices as follows: $G = K_{n/2,n/2}$ if n is even; $G = K_{k,k} \cup v_0$, where $k = (n-1)/2$ and v_0 is a isolated vertex. It is easily checked that $\chi(G)+\chi(G^c) < \chi_1(G)+\chi_1(G^c) < n+1$ and $\chi(G)\chi(G^c) < \chi_1(G)\chi_1(G^c) < ((n+1)/2)^2$.

7. THE GROUP CHROMATIC NUMBER OF $K_{m,n}$

For the complete bipartite graph $K_{m,n}$, if m or n is one, then $K_{m,n}$ is a tree and so its group chromatic number equals two. In this section, we consider the complete bipartite graph $K_{m,n}$ with m and $n > 1$. Let $K_{m,n}$ have two partite sets U with m vertices and V with n vertices. We may assume that each edge uv is directed from v to u, where $v \in V$ and $u \in U$.

THEOREM 7.1. If m or n is two, then $\chi_1(K_{m,n}) = 3$.

Proof. By Lemma 4.2, we have that $\chi_1(K_{m,n}) \le max_H \delta(H) + 1$, where the maximum is taken over all induced subgraphs of $K_{m,n}$. Hence $\chi_1(K_{m,n}) \le 3$. On the other hand, since $K_{m,n}$ is not a tree, $\chi_1(K_{m,n}) > 2$. Therefore, $\chi_1(K_{m,n}) = 3$.

THEOREM 7.2. $\chi_1(K_{3,n}) = 4$, if $n \ge 6$.

Proof. By Lemma 4.2, it is easily seen that $\chi_1(K_{3,n}) \le 4$. Hence, we need to show that there exists a function $f \in F(K_{3,n}, Z_3)$, where Z_3 a non-trivial Abelian group with order 3, such that there is not any Z_3-coloring of $K_{m,n}$ for f.

We need to consider only the graph $K_{3,6}$ with partite sets $U = \{u_1, u_2, u_3\}$

and $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. We define a function $f \in F(K_{3,6}, Z_3)$ as follows (Figure 1): $f(v_1u_2) = 1$, $f(v_1u_3) = 2$, $f(v_3u_3) = 1$, $f(v_4u_2) = 1$, $f(v_4u_3) = 1$, $f(v_5u_2) = 1$, $f(v_6u_2) = 2$, and $f(vu) = 0$ for any other edge vu.

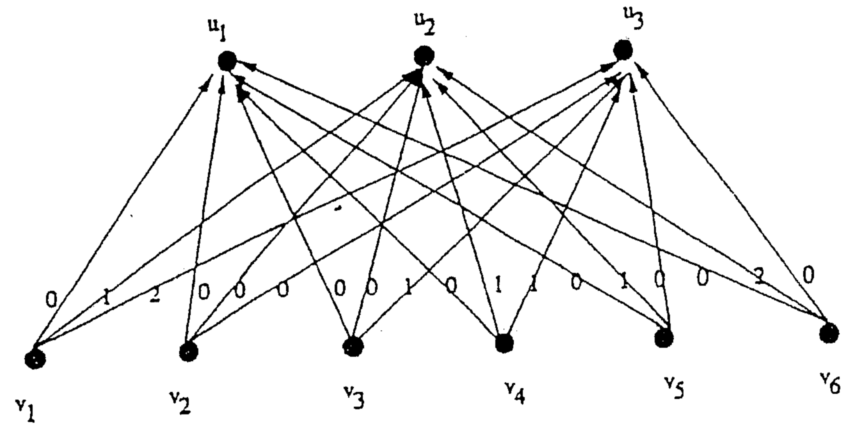

Figure 1: $K_{3,6}$

We can see that:

(7.2.1) if $(c(u_1), c(u_2), c(u_3)) \in \{ (0,0,0), (1,1,1), (2,2,2) \}$, then $\{f(u_1v_1)+ c(u_1),\ f(u_2v_1) + c(u_2),\ f(u_3v_1) + c(u_3)\}=\{0,1,2\}$;

(7.2.2) if $(c(u_1), c(u_2), c(u_3)) \in \{(1,2,0), (1,0,2), (2,1,0), (2,0,1), (0,1,2), (0,2,1)\}$, then $\{f(u_1v_2) + c(u_1), f(u_2v_2) + c(u_2), f(u_3v_2) + c(u_3)\} = \{0,1,2\}$;

(7.2.3) if $(c(u_1), c(u_2), c(u_3)) \in \{(2,0,0), (0,2,0), (0,1,1), (1,0,1), (1,2,2), (2,1,2)\}$, then $\{f(u_1v_3) + c(u_1), f(u_2v_3) + c(u_2), f(u_3v_3) + c(u_3)\} = \{0,1,2\}$;

(7.2.4) if $(c(u_1), c(u_2), c(u_3)) \in \{(0,0,1), (0,1,0), (1,2,1), (1,1,2), (2,2,0), (2,0,2)\}$, then $\{f(u_1v_4) + c(u_1), f(u_2v_4) + c(u_2), f(u_3v_4) + c(u_3)\} = \{0,1,2\}$;

(7.2.5) if $(c(u_1), c(u_2), c(u_3)) \in \{ (0,0,2), (1,1,0), (2,2,1) \}$, then $\{f(u_1v_5)+ c(u_1),\ f(u_2v_5) + c(u_2),\ f(u_3v_5) + c(u_3)\} = \{0,1,2\}$;

(7.2.6) if $(c(u_1), c(u_2), c(u_3)) \in \{ (1,0,0), (2,1,1), (0,2,2) \}$, then $\{f(u_1v_6)+ c(u_1),\ f(u_2v_6) + c(u_2),\ f(u_3v_6) + c(u_3)\} = \{0,1,2\}$;

Hence, no matter which element we assign to u_1, u_2 and u_3, we can find a vertex v_i such that $\{f(u_1v_i) + c(u_1), f(u_2v_i) + c(u_2), f(u_3v_i) + c(u_3)\} = \{0,1,2\}$ and so a proper value of $c(v_i)$ cannot be found. Hence, there is no

Z_3-coloring of $K_{3,6}$ for the function f. □

LEMMA 7.1. Suppose that c is an (A, f)-coloring of G. Then for any $a \in A$ and $\sigma \in Aut(A)$, $c + a$ is an (A, f)-coloring of G and σc is (A, f)-coloring of G.

The proof of this lemma is not included since it is quite straightforward.

LEMMA 7.2. $\chi_1(K_{4,4}) = 4$.

Proof. By Theorem 4.2, $\chi_1(K_{4,4}) \leq 4$. Hence, it suffices to show that $K_{4,4}$ is not Z_3-colorable. Suppose that $K_{4,4}$ has partite sets $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, v_3, v_4\}$. We define a function $f \in F(K_{4,4}, Z_3)$ as follows: $f(v_1u_2) = 2$, $f(v_1u_3) = 1$, $f(v_2u_1) = 1$, $f(v_2u_3) = 2$, $f(v_3u_1) = 2$, $f(v_3u_1) = 1$, and $f(vu) = 0$ for any other edge vu, as shown in Figure 2.

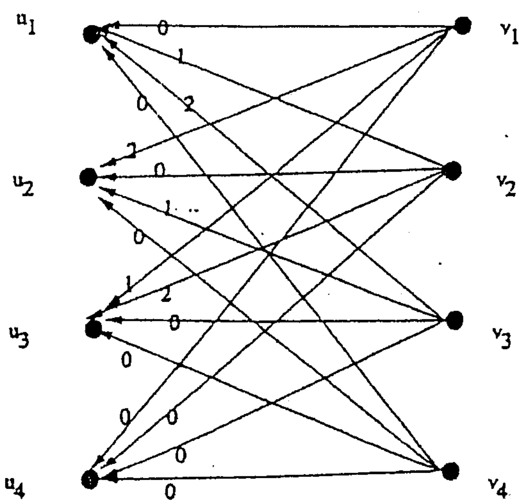

Figure 2: $K_{4,4}$

Suppose that $K_{4,4}$ has a Z_3-coloring c: $V(K_{4,4}) \to Z_3$ for f. By lemma 7.1, we may assume that $c(u_1) = 0$ and $c(v_1) = 1$. Therefore, a contradiction arises if one of the following holds:

(i) $\{c(u_1), c(u_2), c(u_3)\} = Z_3$, or

(ii) $\{c(v_1), c(v_2), c(v_3)\} = Z_3$, or

(iii) $Z_3 - \{c(u_1), c(u_2), c(u_3)\} = Z_3 - \{c(v_1), c(v_2), c(v_3)\} \neq \emptyset$.

Note that when (i) holds, no color is available for $c(v_4)$; when (ii) holds, no color is available for $c(u_4)$; when (iii) holds, since $c(u_3) \neq 0$ and $c(v_2) \neq 1$, each of the sides of (iii) has exactly one element. On the other hand, since $f(v_4u_4) = 0$, no color is available for $c(u_4)$ and $c(v_4)$ if $c(u_4) \neq c(v_4)$. If $c(u_2) = 0$, then $c(v_2) = 2$ and $c(v_3) = 0$. Hence, (ii) holds. Thus, we may assume that $c(u_2) = 1$, and so $c(v_2) \in \{0, 2\}$. To avoid (i), $c(u_3) = 1$ and so $c(v_2) \neq 0$, which implies $c(v_2) = 2$. Since $c(y_3) \neq 2$ and $c(v_3) \neq c(u_3) + 0 = 1$, $c(v_3)$ must be 0, and so (ii) holds.□

LEMMA 7.3. $\chi_1(K_{3,4}) = 3$.

Proof. Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3, v_4\}$ be the partite sets of $K_{3,4}$. For a given function $f \in F(K_{3,4}, Z_3)$ and a vertex v_i, there are at most 3!(=6) coloring possibilities $c(u_1), c(u_2)$ and $c(u_3)$ such that $\{c(u_1)+f(u_1v_i), c(u_2)+f(u_2v_i), c(u_3)+f(u_3v_i)\} = Z_3$. Since $|V| = 4$, there are at most 24 coloring possibilities for $c(u_1), c(u_2)$ and $c(u_3)$ such that there is not a coloring for this given f. However, there are $3^3 = 27$ coloring possibilities for the vertex set U. Therefore, we can find a coloring for $c(u_1), c(u_2)$ and $c(u_3)$ such that $\{c(u_1)+f(u_1v_i), c(u_2)+f(u_2v_i), c(u_3)+f(u_3v_i)\} \neq Z_3$ for each $i(= 1, 2, 3$ or $4)$.□

LEMMA 7.4. $\chi_1(K_{3,5}) = 3$.

Proof. Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3, v_4, v_5\}$ be the partite sets of $K_{3,5}$. For a given function $f \in F(K_{3,5}, Z_3)$ and a vertex v_i, if we color vertices u_1, u_2, u_3 by $c(u_1), c(u_2)$ and $c(u_3)$ such that $\{c(u_1) + f(u_1v_i), c(u_2) + f(u_2v_i), c(u_3) + f(u_3v_i)\} = Z_3$, then we say the coloring $\{c(u_1), c(u_2), c(u_3)\}$ is prohibited by v_i. At each v_i, there are 3!=6 prohibited colorings for a given f. We check total 27 cases and see that these prohibited colorings can be only one of the following nine cases for each v_i:

(7.4.1) $\{(0,0,1), (0,1,0), (1,1,2), (1,2,1), (2,0,2), (2,2,0)\}$

(7.4.2) $\{(0,0,0), (0,1,2), (1,1,1), (1,2,0), (2,0,1), (2,2,2)\}$

(7.4.3) $\{(0,0,2),(0,1,1),(1,1,0),(1,2,2),(2,0,0),(2,2,1)\}$
(7.4.4) $\{(0,0,0),(0,2,1),(1,0,2),(1,1,1),(2,1,0),(2,2,2)\}$
(7.4.5) $\{(0,0,2),(0,2,0),(1,0,1),(1,1,0),(2,1,2),(2,2,1)\}$
(7.4.6) $\{(0,0,1),(0,2,2),(1,0,0),(1,1,2),(2,1,1),(2,2,0)\}$
(7.4.7) $\{(0,1,1),(0,2,0),(1,0,1),(1,2,2),(2,0,0),(2,1,2)\}$
(7.4.8) $\{(0,1,0),(0,2,2),(1,0,0),(1,2,1),(2,0,2),(2,1,1)\}$
(7.4.9) $\{(0,1,2),(0,2,1),(1,0,2),(1,2,0),(2,0,1),(2,1,0)\}$

From these cases, we see that at most 24 colorings of the vertices u_1, u_2, u_3 are prohibited by the 5 vertices v_1, v_2, v_3, v_4, v_5 for any given function $f \in F(K_{3,5}, Z_3)$. Hence, we find a coloring of u_1, u_2, u_3 which is not prohibited by any vertex in V. Therefore, $K_{3,5}$ is Z_3-colorable. □

By the previous lemmas and theorems, we can conclude that:

THEOREM 7.3. Let $K_{m,n}$ be a complete bipartite graph with m and $n \geq 2$. Then $\chi_1(K_{m,n}) = 3$ if and only if $m = 2$ or $n = 2$ or $(m, n) \in \{(3,4),(4,3),(3,5),(5,3)\}$. □

By using the same idea of the proof of Lemma 7.3, we can similarly show the following theorem.

THEOREM 7.4. $\chi_1(K_{4,n}) = 4$ if $4 \leq n \leq 10$.

Proof. By Lemma 4.2, $\chi_1(K_{4,n}) \leq 5$, and by Lemma 7.2, we know that $\chi_1(K_{4,n}) \geq 4$ if $n \geq 4$. Let A_4 be an Abelian group with order 4. We show that $K_{4,n}$ is A_4-colorable if $4 \leq n \leq 10$. Let $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, \cdots, v_n\}$ be the partite sets of $K_{4,n}$. For a given function $f \in F(K_{4,n}, A_4)$ and a vertex v_i, if we color vertices u_1, u_2, u_3, u_4 by $c(u_1), c(u_2), c(u_3)$ and $c(u_4)$ such that $\{c(u_1)+f(u_1v_i), c(u_2)+f(u_2v_i), c(u_3)+f(u_3v_i), c(u_4)+f(u_4v_i)\} = A_4$, then we say the coloring $\{c(u_1), c(u_2), c(u_3), c(u_4)\}$ is prohibited by v_i. We see that there are 24 colorings of vertices u_1, u_2, u_3, u_4 which are prohibited by a vertex v_i for a given f, where $1 \leq i \leq n$. However, there are $4^4 = 256$ colorings for the vertex set U. If $4 \leq n \leq 10$, then $4^4 > 24n$, and so we find a coloring of u_1, u_2, u_3, u_4 which is not prohibited by any vertex in V. Hence, $K_{4,n}$ is A_4-colorable if $4 \leq n \leq 10$. □

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