Extremal size of graphs without a nowhere-zero 3-flow

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Abstract

An asymptotically best possible bound for the size of a simple graph that does not admit a nowhere-zero 3-flow is determined.

1. Introduction

Graphs in this note are finite and may have loops and parallel edges. Groups in this note are finite Abelian groups. Throughout this note, A denotes an Abelian (additive) group with 0 as the additive identity. For integer $n \geq 2$, Z_n denotes the cyclic group of order n.

For a subset $X \subseteq E(G)$, the **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the edges in X. Note that even when G is simple, G/X may have loops or multiple edges. For convenience, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G, then G/H denotes $G/E(H)$.

For a vertex $v \in V(G)$, let

$$E_D^-(v) = \{(u,v) \in E(G) : u \in V(G)\}, \text{ and } E_D^+(v) = \{(v,u) \in E(G) : u \in V(G)\}.$$

The subscript D may be omitted when $D(G)$ is understood from the context. Let $E_G(v)$ denote the subset of edges incident with v in G.

Fix an orientation D of G. Let A be a nontrivial Abelian group and let A^* denote the set of nonzero elements in A. Define $F(G,A) = \{f :$

$E(G) \to A\}$ and $F^*(G, A) = \{f : E(G) \to A^*\}$. For each $f \in F(G, A)$, the **boundary** of f is a function $\partial f : V(G) \to A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. Unless otherwise stated, we shall adopt the following convenience: if $X \subseteq E(G)$ and if $f : X \to A$ is a function, then we regard f as a function $f : E(G) \to A$ such that $f(e) = 0$ for all $e \in E(G) - X$. We also use that notation (D, f) for a function $f \in F(G, A)$ to emphasize the orientation D.

Let G be an undirected graph and A be an Abelian group. Let $Z(G, A)$ denote the set of all functions $b : V(G) \mapsto A$ such that $\sum_{v \in V(G)} b(v) = 0$. A graph G is **A-connected** if G has an orientation D such that for every function $b \in Z(G, A)$, there is a function $f \in F^*(G, A)$ such that $b = \partial f$. For an Abelian group A, let $\langle A \rangle$ denote the family of graphs that are A-connected.

Suppose that $f \in F^*(G, A)$ with a fixed orientation D such that some edge $e_0 = (u, v) \in E(G)$ is oriented from u to v. If a new orientation D' of G is obtained by reversing the direction of e_0, then one can redefine $f'(e) = f(e)$ for $e \in E(G) - \{e\}$ and $f'(e_0) = -f(e_0)$. It is easy to see that $f' \in F^*(G, A)$ and ∂f in D equals $\partial f'$ in D'. Therefore, the property $G \in \langle A \rangle$ is independent of the orientation of G.

The concept of A-connectivity was introduced by Jaeger *et al* in [6], where nowhere-zero A-flows were successfully generalized to A-connectivities. A concept similar to the group connectivity was independently introduced in [7], with a different motivation from [6].

A **nowhere-zero A-flow** (abbreviated as A-NZF) in G is a function $f \in F^*(G, A)$ such that $\partial f = 0$. When A is the additive group of all integers, an A-NZF f is a **nowhere-zero k-flow** (or just k-NZF) if $\forall e \in E(M)$, $0 < |f(e)| < k$. The following is well known.

(1.1) (Arrowsmith-Jaeger [1], Brylawski [3] and Tutte [11]) For a regular matroid M, M has an A-NZF if and only if M has an $|A|$-NZF.

The nowhere-zero-flow problems were introduced by Tutte [10]. (See Jaeger's survey in [5] for more in the literature on nowhere-zero flows.) For an integer $k \geq 2$, F_k denotes the collection of all graphs admitting a nowhere zero Z_k-flow. By definition,

$$\langle Z_k \rangle \subseteq F_k. \tag{1}$$

The containment is in fact proper as can be seen in (2.2) below. Tutte [5] has several conjectures concerning what graph G admits a k-NZF. Jaeger (see [5]) showed that every 2-edge-connected graph admits a 8-NZF. This result was improved by Seymour (see [5]) who proved that every 2-edge-connected graph admits a 6-NZF. Tutte (see [5]) conjectured that every 2-edge-connected graph admits a 5-NZF. It is well known that the Petersen graph P_{10} does not have a 4-NZF, and when n is odd, W_n, the wheel with $n+1$ vertices does not have a 3-NZF.

It is natural to consider, for $k \in \{3,4\}$, the existence of a function $f(k,n)$ such that every 2-edge-connected simple graph G with n vertices and with at least $f(k,n)$ edges must have a k-NZF. The following was proved in [8].

(1.2) ([8]) Let G be a 2-edge-connected simple graph with $n \geq 18$ vertices. If

$$|E(G)| \geq \frac{(n-17)(n-18)}{2} + 34,$$

then either $G \in F_4$, or G is contractible to the Petersen graph. The bound is asymptotically best possible.

The main result of this note is the following which determined the extremal size of simple 2-edge-connected graph which does not have a 3-NZF.

(1.3) Let G be a 2-edge-connected simple graph with $n \geq 6$ vertices. If

$$|E(G)| \geq \binom{n-5}{2} + 46, \tag{2}$$

then either $G \in F_3$ or G can be contracted to a K_4.

The bound given in (1.3) is asymptotically best possible. Let W_5 denote the wheel which consists of a 5-cycle $z_1z_2z_3z_4z_5z_1$ and a center z together with the spoke edges zz_i $(1 \leq i \leq 5)$. Let $G(n)$ be a graph obtained from W_5 by replacing exactly one of its six vertices by a clique K_{n-5}, where $n \geq 8$ is an integer. Then since $G(n)$ can be contracted onto W_5 which does not have a nowhere zero 3-flow, $G(n) \notin F_3$. Note that $n = |V(G(n))|$

and $|E(G(n))| = \frac{(n-5)(n-6)}{2} + 10$, and so

$$\lim_{m\to\infty} \frac{|E(G(n))|}{\frac{(n-5)(n-6)}{2} + 49} = 1.$$

2. Proof of the main result

The following results are known.

(2.1) (Proposition 3.2 of [9]) Let H be a subgraph of G and let A be an Abelian group. Each of the following holds.

(i) If $H \in \langle A \rangle$ and if $e \in E(H)$, then $H/e \in \langle A \rangle$.

(ii) If $H \in \langle A \rangle$, then $G/H \in \langle A \rangle \iff G \in \langle A \rangle$.

(iii) If $H \in \langle Z_k \rangle$, then $G/H \in F_k \iff G \in F_k$.

(Catlin called nonempty graph families satisfying (i), (ii) and (iii) of (2.1) **complete families**. See [4].)

(2.2) (Lemma 3.3 of [9]) Let $n \geq 1$ be an integer and let C_n denote the cycle of n vertices. Then for an Abelian group A, $|A| \geq n+1$.

(2.3) (Corollary 3.5 of [9]) Let $m \geq 5$ an integer. Then both $K_m \in \langle A \rangle$ and $K_m - e \in \langle A \rangle$ for any abelian group A with $|A| \geq 3$.

(2.4) (Proposition 3.6 of [9]) Let M_2 be a matching of K_5. Then $K_5 - M_2 \in \langle A \rangle$ for any abelain group with $|A| \geq 3$.

(2.5) Every 2-edge-connected graph with at most 5 vertices is either in F_3 or is contractible to K_4.

Proof Since the 3-cycle is in F_3, we may assume that $|V(G)| \geq 4$ and that G is simple. If G has a vertex v of degree 2, then by induction, we can pick $e' \in E_G(v)$ and conclude that either $G/e' \in F_3$, whence $G \in F_3$, or G/e' is contractible to K_4, whence G is contractible to K_4.

Thus we assume that $\delta(G) \geq 3$. If $|V(G)| = 4$, then $G = K_4$. Therefore we assume that $|V(G)| = 5$. By (2.3), we may assume that $G \neq K_5$. By $\delta(G) \geq 3$, either $G = K_5 - e$, for some $e \in E(K_5)$, or $G = K_5 - M_2$ for some maximal matching M_2 of K_5. It follows by (2.3) or (2.4) that $G \in \langle Z_3 \rangle \subseteq F_3$. □

Proof of (1.3) We argue by contradiction and assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \tag{3}$$

(2.6) G does not contain any nontrivial subgraph H that is in $\langle Z_3 \rangle$.

Proof: Suppose not, and G has a subgraph H with $H \in \langle Z_3 \rangle$ and $|V(H)| \geq 2$. We may assume that H is a maximal subgraph of G that is in $\langle Z_3 \rangle$. By the maximality of H, G/H is simple. If H spans G, then by (2.1)(iii) and by (2.2), $G \in \langle Z_3 \rangle \subset F_3$, a contradiction. Hence we assume that G/H is nontrivial also.

Let $n = |V(G)|$, $m = |V(H)|$ and $l = |V(G/H)| = n - m + 1$. Since G is simple, $|E(H)| \leq m(m-1)/2$, and so by (2), we have

$$\begin{aligned} |E(G/H)| &= |E(G)| - |E(H)| \\ &\geq \binom{n-5}{2} + 46 - \binom{m}{2} \\ &= \frac{n^2 - 11n + 30 - m^2 + m}{2} + 46 \\ &= \frac{(l-5)(l-6)}{2} + 46 + (m-1)n - m^2 - 4m + 5. \end{aligned} \tag{4}$$

Thus $|E(G/H)| \geq (l-5)(l-6)/2 + 46$ if and only if $(m-1)n - m^2 - 4m + 5 < 0$. Since $l = n - m + 1$, since $m \geq 2 > 1$ and since

$$(m-1)n - m^2 - 4m + 5 = (m-1)(n-m-5) = (m-1)(l-6),$$

$(m-1)n - m^2 - 4m + 5 < 0$ if and only if $l \leq 5$.

By (2.5), every 2-edge-connected graph with at most 5 vertices is either in F_3 or is contractible to K_4. Thus if $l \leq 5$, then either $G/H \in F_3$, whence by (2.1)(ii), $G \in F_3$; or G is contractible to K_4, contrary to (3).

Hence we may assume that $l \geq 6$ and so by (4), G/H satisfies that hypothesis of (1.3). Therefore by the minimality of G, G/H is either in F_3, whence by (2.1)(ii), $G \in F_3$; or G/H is contractible to K_4, whence G is contractible to K_4, contrary to (3). This proves (2.6). □

Proof of (1.3), continued By (2.6) and by (2.3), G does not have K_5 as a subgraph. However, by (2)

$$|E(G)| \geq \binom{n-5}{2} + 46 \geq \frac{3n^2}{8}. \tag{5}$$

By Turán's Theorem ([2], page 109), G must have a K_5, and so a contradiction obtains. This proves (1.3). □

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