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Nowhere zero flows in line graphs

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Abstract

Cai an Corneil (Discrete Math. 102 (1992) 103–106), proved that if a graph has a cycle double cover, then its line graph also has a cycle double cover, and that the validity of the cycle double cover conjecture on line graphs would imply the truth of the conjecture in general. In this note we investigate the conditions under which a graph G has a nowhere zero k-flow would imply that $L(G)$, the line graph of G, also has a nowhere zero k-flow. The validity of Tutte's flow conjectures on line graphs would also imply the truth of these conjectures in general.

1. Introduction

Graphs in this note are finite and loopless but it may have parallel edges. A collection $\mathscr{C}$ of cycles of a graph G is called a *cycle double cover* of G, if every edge of G is in exactly two members in the collection $\mathscr{C}$. Groups in this note are finite abelian groups. Throughout this note, A denotes a finite abelian group. For integer $n \geqslant 2$, Z_n denotes the cyclic group of order n.

Given a graph G with $E(G) \neq \emptyset$, the *line graph* of G, denoted by $L(G)$, has $E(G)$ as the vertex set, where two vertices e_1, e_2 in $L(G)$ are linked by exactly one edge in $L(G)$ if and only if the corresponding edges e_1, e_2 are adjacent but not parallel edges, and by exactly two edges in $L(G)$ if and only if the corresponding edges e_1, e_2 are parallel edges in G. Note that our definition for line graphs is slightly different from the one in [1] (called edge graph there) only when G is not simple. The following is proved in [2].

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Theorem 1.1 (Cai and Corneil [2]). *If the graph G has a cycle double cover, then its line graph also has a cycle double cover.*

The prominent *cycle double cover conjecture* was posed by Szekeres and Seymour ([9,10]), which states that every 2-edge-connected graph admits a cycle double cover. Cai and Corneil showed that it suffices to verify this conjecture for line graphs.

Theorem 1.2 (Cai and Corneil [2]). *The cycle double cover conjecture holds for all 2-edge-connected graphs if and only if it holds for all 2-edge-connected line graphs.*

Let $D = D(G)$ be an orientation of an undirected graph G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let $tail(e) = u$ and $head(e) = v$. For a vertex $v \in V(G)$, let

$$E_D^-(v) = \{e \in E(D) : v = \text{tail}(e)\}, \quad \text{and} \quad E_D^+(v) = \{e \in E(D) : v = \text{head}(e)\}.$$

The subscript D may be omitted when $D(G)$ is understood from the context. Let $E_G(v)$ denote the subset of edges incident with v in G. For an integer $k \geqslant 2$, a *nowhere-zero k-flow* (abbreviated as a k-NZF) of G is an orientation D of G together with a map $f : E(D) \mapsto \{-(k-1), -(k-2), \ldots, -2, -1, 1, 2, \ldots, k-1\}$ such that at each vertex $v \in V(G)$,

$$\sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e) = 0.$$

As noted in [5], the existence of a nowhere-zero k-flow of a graph G is independent of the choice of the orientation D.

The problem of nowhere-zero flows of a graph is closely related to the problem of cycle double covers [5]. Thus, we in this note try to answer this question: if, for some integer k, G has a nowhere-zero k-flow, does $L(G)$ also have a nowhere-zero k-flow? A motivation of this study, as in [2], is that the validity of Tutte's flow conjectures on line graphs would also imply the truth of these conjectures in general. The following results are obtained.

Theorem 1.3. *Let $k \geqslant 4$ be an integer and let G be a 2-edge-connected graph. If G has a nowhere-zero k-flow, then $L(G)$ has a nowhere-zero k-flow.*

Theorem 1.4. *If G has a nowhere-zero 3-flow and if the minimum degree of G is at least 4, then $L(G)$ has a nowhere-zero 3-flow.*

Tutte has several conjectures on the nowhere-zero flow problem (to be introduced in Section 2). We shall also show that it suffices to verify these flow conjectures for line graphs.

Section 2 gives some basic facts about group connectivity of a graph, which will be needed in the proof. The main results are proved in Section 3.

2. Group connectivity of a graph

The proofs of the main results in this note need the help of group connectivity of a graph. Fix an orientation D of G. Let A be a nontrivial abelian group with identity 0, and let A^* denote the set of nonzero elements in A. Define $F(G,A)=\{f : E(G)\to A\}$ and $F^*(G,A)=\{f : E(G)\to A^*\}$. For each $f\in F(G,A)$, define $\partial f : V(G)\to A$ by

$$\partial f(v)=\sum_{e\in E^+(v)} f(e)-\sum_{e\in E^-(v)} f(e),$$

where '$\sum$' refers to the addition in A. Unless otherwise stated, we shall adopt the following convention: if $X\subseteq E(G)$ and if $f : X\to A$ is a function, then we regard f as a function $f : E(G)\to A$ such that $f(e)=0$ for all $e\in E(G)-X$. We also use notation (D,f) for a function $f\in F(G,A)$ to emphasize the orientation D.

Let G be an undirected graph and A be an abelian group. Let $Z(G,A)$ denote the collection of all functions $b : V(G)\to A$ satisfying $\sum_{v\in V(G)} b(v)=0$. A graph G is *A-connected* if G has an orientation D such that for every function $b\in Z(G,A)$, there is a function $f\in F^*(G,A)$ such that $b=\partial f$. For an abelian group A, let $\langle A\rangle$ denote the family of graphs that are A-connected. As noted in [6], that $G\in\langle A\rangle$ is independent of the orientation D of G.

An *A-nowhere-zero-flow* (abbreviated as an A-NZF) in G is a function $f\in F^*(G,A)$ such that $\partial f=0$. The nowhere-zero-flow problems were introduced by Tutte [11], and recently surveyed by Jaeger in [5].

Theorem 2.1 (Tutte [12]). *Let A be an abelian group with $|A|=k$. Then a graph G has an A-NZF if and only if G has a k-NZF.*

Following Jaeger [5], for an integer $k\geqslant 2$, F_k denotes the collection of all graphs admitting a k-NZF. By definition, $\langle Z_k\rangle\subseteq F_k$.

The concept of A-connectivity was introduced by Jaeger et al. in [6], where A-NZF's were successfully generalized to A-connectivities. A concept similar to the group connectivity was independently introduced in [7], with a different motivation from [6].

Tutte has three fascinating conjectures on nowhere-zero flows.

3-Flow Conjecture (Jaeger [5]). *Every 4-edge-connected graph is in F_3.*

4-Flow Conjecture (Jaeger [5]). *Every 2-edge-connected cubic graph either is in F_4 or has a subgraph contractible to the Petersen graph (such a subgraph is called a Petersen minor).*

A generalized version of Tutte's 4-flow conjecture states that every 2-edge-connected (not necessarily cubic) graph without a Petersen minor is in F_4.

5-Flow Conjecture (Jaeger [5]). *Every 2-edge-connected graph is in F_5.*

We need the notion of contractions. For a subset $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the edges in X. Note that even when G is simple, G/X may have multiple edges. For convenience, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G, then G/H denotes $G/E(H)$. The following two propositions are known (see [6] or [8]).

Proposition 2.2. *Let H be a subgraph of G, let A be an abelian group and let Z_k denote the cyclic group of order k. Then each of the following holds*:

(i) *If $H \in \langle A \rangle$ and if $e \in E(H)$, then $H/e \in \langle A \rangle$.*
(ii) *If $H \in \langle A \rangle$, then $G/H \in \langle A \rangle \Leftrightarrow G \in \langle A \rangle$.*
(iii) *If $H \in \langle Z_k \rangle$, then $G/H \in F_k \Leftrightarrow G \in F_k$.*

(Catlin called nonempty graph families satisfying (i) and (ii) of Proposition 2.2 *complete families*. See [3].)

Proposition 2.3. *Let $n \geqslant 2$ be integers, let A be an abelian group and let $C_n = z_1 z_2 \ldots z_n z_1$ be a cycle of length n. Then, $C_n \in \langle A \rangle \Leftrightarrow |A| \geqslant n+1$.*

Let G be a connected loopless graph with minimum degree $\delta(G) \geqslant 2$. By the definition of a line graph, we note that for each vertex $v \in V(G)$ with degree d, $E_G(v)$ induces a subgraph spanned by a complete graph of order d in $L(G)$, and that each edge $e \in E(G)$ is incident with exactly two vertices. Therefore, we have the following observations.

Lemma 2.4. *For each $v \in V(G)$, let G_v denote the subgraph induced by the vertices $E(v)$ in $L(G)$. Then each of the following holds*:

(i) *Each G_v is spanned by a complete graph K_d, where d is the degree of v in G.*
(ii) *$L(G) = \bigcup_{v \in V(G)} G_v$ is an edge-disjoint union.*
(iii) *Every $e \in V(L(G))$ is in exactly two of these G_v's.*

Having observed these, we in the next two corollaries investigate the group connectivity of complete graphs. We shall call a cycle of length n an *n-cycle*.

Corollary 2.5. *Let A be an abelian group. Assume that $|A| \geqslant 4$ and $m \neq 2$. Let G be a graph spanned by a K_m. Then $G \in \langle A \rangle$. For $m = 2$, if $b \in Z(K_2, A)$ is not the zero map, then there is a function $f \in F^*(K_2, A)$ such that $\partial f = b$.*

Proof. Since $K_1 \in \langle A \rangle$ for any abelian group A by definition, we assume that $m \geqslant 2$. Since $K_1 \in \langle A \rangle$ for any A and by Proposition 2.2(ii), we may assume that $G = K_m$. Let A be an abelian group with $|A| \geqslant 4$ and let $m \geqslant 3$ be an integer. Since $m \geqslant 3$, each K_m contains a 3-cycle C. By Proposition 2.3 with $n = 3$, $C \in \langle A \rangle$. By Proposition 2.2(ii) with $H = C$, $K_m/C \in \langle A\rangle$ if and only if $K_m \in A$. Note either $m \in \{3,4\}$ and K_m/C is a vertex or a 2-cycle, whence, $K_m/C \in \langle A\rangle$ by definition or by Proposition 2.2; or $m \geqslant 5$ and K_m/C is spanned by a complete graph K_{m-2}, whence by induction, $K_m/C \in \langle A\rangle$. Therefore, in any case, $K_m/C \in \langle A\rangle$ and so by Proposition 2.2(ii), $K_m \in \langle A\rangle$. When $m = 2$, we may assume that $V(K_2) = \{u, v\}$ and the only edge of K_2 is directed from u to v. Since b is not identically zero, $b(u) = -b(v) \neq 0$ in A. Let $f : E(K_2) \mapsto \{b(u)\}$. Then $\partial f = b$, as desired. □

Corollary 2.6. *Assume that A is an abelian group with $|A| \geqslant 3$ and that $m \geqslant 5$ is an integer. Let G be a graph spanned by a K_m. Then $G \in \langle Z_3\rangle$. For $m = 4$, if $b \in Z(K_4, A)$ satisfies $b(v) \neq 0$ for all $v \in V(K_4)$, then there exists an $f \in F^*(K_4, A)$ such that $\partial f = b$.*

Proof. Since $K_1 \in \langle A\rangle$ and by Proposition 2.2(ii), we assume that $G = K_m$. First, let $m \geqslant 5$ be an integer. By Corollary 2.5, $G \in \langle A\rangle$ if $|A| \geqslant 4$. Therefore, we may assume that $A = Z_3$. It is proved in [8] that $K_m \in \langle Z_3\rangle$, for any integer $m \geqslant 5$. Hence, it suffices to prove the later half of the corollary.

Let $m = 4$ and $b \in Z(K_4, A)$ be a function such that $b(v) \neq 0$ for all $v \in V(K_4)$. By Corollary 2.5, we may assume $A = Z_3 = \{\bar{0}, \bar{1}, \bar{2}\}$. By symmetry, we may assume that $V(K_4) = \{u_1, u_2, u_3, u_4\}$ and that $b(u_1) = b(u_2) = \bar{1}$ and $b(u_3) = b(u_4) = \bar{2}$. Assume further that the edge $e' = u_1u_3$ is directed from u_1 to u_3 and the edge $e'' = u_2u_4$ is directed from u_2 to u_4, and that the orientation of $K_4 - \{e', e''\}$ is a directed 4-cycle C'. Define $f \in F^*(K_4, Z_3)$ by $f \equiv \bar{1}$. Then it is easy to see that $\partial f = b$, as desired. □

3. Nowhere-zero flows in line graphs

We start with two examples, which indicate that $L(G) \in F_k$ may not imply that $G \in F_k$, even when G satisfies the necessary connectivity condition.

Example 3.1. The graph $K_{2,3} \in F_3$ but $L(K_{2,3})$, being contractible to K_4, is not in F_3. On the other hand, $L(K_4) \in F_2 \subset F_3$ but $K_4 \notin F_3$. Thus, even we require both G and $L(G)$ be 2-edge-connected, $G \in F_3$ may not imply that $L(G) \in F_3$.

Example 3.2. Let G be a connected 3-regular graph that is not in F_4. (For example, let G be the Peterson graph.) Then $L(G) \in F_2 \subset F_4$.

Proposition 3.1. *Let G be a connected graph. Then each of the following holds*:

(i) *If $\delta(G) \geqslant 3$, then $L(G) \in \langle A\rangle$ for any abelian group A with $|A| \geqslant 4$.*
(ii) *If $\delta(G) \geqslant 5$, then $L(G) \in \langle Z_3\rangle$.*

Proof. Recall that $L(G)$ is the edge-disjoint union of subgraphs each of which is spanned by a complete graph of order at least $\delta(G)$. We shall prove an equivalent claim: If G is connected and if G is an edge-disjoint union of subgraphs each of which is spanned by a complete graph of order at least 3, then $G \in \langle A \rangle$. We argue by induction on the number of such subgraphs. If G is spanned by one complete subgraph with at least 3 vertices, then by Corollary 2.5, $G \in \langle A \rangle$. Assume that G is the edge-disjoint union of subgraphs $H_1, H_2, \dots, H_m$, where each H_i is spanned by a complete graph of order at least 3. By induction, $G/H_m \in A$. By Corollary 2.5, $H_m \in \langle A \rangle$, and so by Proposition 2.2(ii), $G \in \langle A \rangle$. This proves Part (i).

For Part (ii) of Proposition 3.1, we argue similarly, using Corollary 2.6 in place of Corollary 2.5. □

By Theorems 2.1, 3.2 and 3.3 below are equivalent to Theorems 1.3 and 1.4, respectively.

Theorem 3.2. *Let A be an abelian group with $|A| \geqslant 4$. If G has an A-NZF, then $L(G)$ has an A-NZF.*

Proof. Let (D, ϕ) be an A-NZF of G. For each $v \in V(G)$, define $b_v : V(G_v) \mapsto A$ as follows: for $e \in V(G_v)$,

$$b_v(e) = \begin{cases} -\phi(e) & \text{if } v = \text{head}(e); \\ \phi(e) & \text{if } v = \text{tail}(e). \end{cases} \tag{1}$$

Since $\phi \in F^*(G, A)$, $b_v(e) \neq 0$ for any $e \in V(G_v)$, and $\sum_{e \in V(G_v)} b_v(e) = \partial\phi(v) = 0$. Thus, $b_v \in Z(G_v, A)$. By Lemma 2.4(i), each G_v is spanned by a complete graph, and so by Corollary 2.5, G_v an A-NZF (D_v, ϕ_v) such that $\partial\phi_v = b_v$. By Lemma 2.4(ii) $L(G)$ is the edge-disjoint union of all these G_v's, it makes sense to define an orientation $\hat{D} = \bigcup_{v \in V(G)} D_v$ of $L(G)$ as the disjoint union of all the D_v's. Let $\hat{f} = \sum_{v \in V(G)} \phi_v$. Since each $\phi_v \in F^*(G_v, A)$ and by Lemma 2.4(ii), $\hat{f} \in F^*(G, A)$.

Let $e \in V(L(G))$ be an arbitrary vertex. We shall verify that $\partial\hat{f}(e) = 0$. By Lemma 2.4(iii), we may assume that $e \in V(G_u) \cap V(G_v)$, and in D, e is directed from u to v. Then

$$\begin{aligned} \partial\hat{f}(e) &= \partial f_u(e) + \partial f_v(e) \\ &= b_u(e) + b_v(e) = \phi(e) - \phi(e) = 0, \end{aligned} \tag{2}$$

and so $(\hat{D}, \hat{f})$ is an A-NZF of $L(G)$. This proves the theorem. □

Theorem 3.3. *If G has an Z_3-NZF and if $\delta(G) \geqslant 4$, then $L(G)$ has an Z_3-NZF.*

Proof. Let $Z_3 = \{\bar{0}, \bar{1}, \bar{2}\}$ and let (D, ϕ) be a Z_3-NZF of G. We shall imitate the proof of Theorem 3.2 to find a Z_3-NZF $(\hat{D}, \hat{f})$ of $L(G)$. For each $v \in V(G)$, define $b_v \in Z(G_v, Z_3)$ as in (1). By Lemma 2.4(i), by the assumption that $\delta(G) \geqslant 4$ and by Corollary 2.6, G_v an A-NZF (D_v, ϕ_v) such that $\partial\phi_v = b_v$. By Lemma 2.4(ii), one can define an orientation $\hat{D} = \bigcup_{v\in V(G)} D_v$ of $L(G)$ as the disjoint union of all the D_v's. Define $\hat{f} = \sum_{v\in V(G)} \phi_v$. Then, one can imitate the same argument in the proof of Theorem 3.2 to verify that $(\hat{D}, \hat{f})$ is a Z_3-NZF of $L(G)$. □

Theorem 3.4. *Each of the following holds*:

(i) *If every* 2*-edge-connected line graph has a* 5*-NZF, then every* 2*-edge-connected graph has* 5*-NZF.*
(ii) *If every* 4*-edge-connected line graph has a* 3*-NZF, then every* 4*-edge-connected graph has* 3*-NZF.*
(iii) *If every* 2*-edge-connected cubic line graph without a Petersen minor has a* 4*-NZF, then every* 2*-edge-connected cubic graph without a Petersen minor has a* 4*-NZF.*

We need an auxiliary graph introduced by Harary and Nash-Williams [4]. Let G be a graph and let $S(G)$, the *subdivided graph* of G, be the graph obtained from G by replacing each edge e of G by a path length 2 with a newly added internal vertex v_e. Lemma 3.5 below follows immediately from the definitions.

Lemma 3.5. *Let G be a graph with $E(G) \neq \emptyset$ and let $e \in E(G)$ such that the two ends of e are u and v. Let G_e be the graph obtained from G by replacing e by a (u, v)-path uv_ev of length* 2*. Let e' denote the edge in $L(G_e)$ that has uv_e and v_ev as its ends. Then*

$$L(G_e)/\{e'\} = L(G).$$

Note that the correspondence $e \leftrightarrow e'$ defined in Lemma 3.5 is a bijection between $E(G)$ and $\{e' \mid e \in E(G)\} \subset E(L(S(G)))$.

Proof of Theorem 3.4. Since 2-cycles are in $\langle A\rangle$ for any abelian group A (Proposition 2.3) and by Proposition 2.2, it suffices to prove Theorem 4.8 for simple graphs.

We shall use the bijection $e \leftrightarrow e'$ defined in Lemma 3.5. Let G be a 2-edge-connected simple graph and let $E' = \{e' \in E(L(S(G))) \mid e \in E(G)\}$. Then,

$$E(L(S(G))) - E' = \bigcup \left(\bigcup_{v\in V(G)} E(L(E_{S(G)}(v))) \right) \text{ and}$$

$$L(S(G))/[E(L(S(G))) - E'] = G. \tag{3}$$

For if e is incident with u and v in G, then e' is incident with u' and v' in $L(S(G))/[E(L(S(G))) - E']$, where u' is the contraction image of $L(E_{S(G)}(u))$ and v' is the contraction image of $L(E_{S(G)}(v))$. Thus, we have proved the following claim. □

Claim 1. *If $L(S(G))$ has a Z_k-NZF, then $G \in F_k$.*

Claim 2. *If G is k-edge-connected, then the edge connectivity of $L(S(G))$ is not less than k.*

Suppose that X is a minimum edge cut of $L(S(G))$. If $X \subseteq E'$, then using the bijection in Lemma 3.5, let $Y = \{e \in E(G) \mid e' \in X\}$. Since X is an edge cut of $L(S(G))$, Y is an edge cut of G, and so Claim 2 will be proved. Therefore, it suffices to show that $X \subseteq E'$.

Let $X_1 = X \cap E'$ and $X_2 = X - X_1$, and assume the choice of X minimizes $|X_2|$. Let H_1 and H_2 be the two components of $L(S(G)) - X$. Assume by contradiction that $X_2 \neq \emptyset$, and so there is an edge $e \in X_2$. Since $X_2 \cap E' = \emptyset$, $e \in E(L(E_{S(G)}(v)))$ for some vertex $v \in V(G)$. Let $d = |E_G(v)|$, $H_3 = L(E_{S(G)}(v))$. For a subgraph H of $L(S(G))$, let $\partial(H)$ denote the set of edges in $L(S(G))$ that are incident with exactly one vertex in $V(H)$. Thus, $\partial(H_3) \subset E'$, $|\partial(H)| = d$ and H is a complete graph of order d that contains e. Note that $\partial(H_1) = \partial(H_2) = X$. Since H_3 is a K_d and since $X \cap E(H_3)$ separates H,

$$|X \cap E(H_3)| \geqslant d - 1. \tag{4}$$

Since each edge in $\partial(H_3)$ is incident with exactly one vertex in $V(H_3)$, it follows that

$$\partial(H_3) \cap E(L_1) \neq \emptyset \quad \text{and} \quad \partial(H_3) \cap E(L_2) \neq \emptyset. \tag{5}$$

If $X \subset (E(H_3) \cup \partial(H_3))$, then by (4) and (5), $\partial(H_3) \cap E(L_1)$ is also an edge cut with at most $|\partial(H_3)| - 1 = d - 1 \leqslant |X|$ edges, contrary to the assumption that X is a minimum edge cut with $|X_2|$ minimized. Thus,

$$X - (E(H_3) \cup \partial(H_3)) \neq \emptyset. \tag{6}$$

Let $X' = \partial(H_3)$. Then $|X'| = |\partial(H_3)| = d$ and X' is an edge-cut of $L(S(G))$. By (4) and (6), $|X| \geqslant d = |X'|$, and so X' is also a minimum edge cut of $L(S(G))$, contrary to the choice of X again. Therefore, we must have $X \subset E'$, and so Claim 2 follows.

We shall now prove Parts (i) and (ii) of Theorem 3.4. Let G be a 2-edge-connected graph. Then by Claim 2, $L(S(G))$ is also a 2-edge-connected graph. By the assumption of Theorem 3.4(i), $L(S(G))$ has a 5-NZF. By Claim 1, G has 5-NZF, and so Theorem 3.4(i) obtains. The proof of Theorem 3.4(ii) is similar.

Let G be a simple 2-edge-connected cubic graph. Assume that $L(S(G))$ has a Petersen minor H. Let $D_3(H)$ denote the set of vertices of degree 3 in H and let P_{10} denote the Petersen graph.

For each $v \in V(G)$, let $K_v = L(E_{S(G)}(v))$ denote a complete subgraph in $L(S(G))$. Note that if $X \subset E(P_{10})$ is an edge cut such that both sides of $P_{10} - X$ has at least two vertices, then $|X| \geqslant 4$. Therefore, if $X \subset E(H)$ is an edge cut of H such that both sides of $H - X$ has at least two vertices of $D_3(H)$, then

$$|X| \geqslant 4. \tag{7}$$

For each $v \in V(G)$, by the definition of $L(S(G))$, each K_v is adjacent to vertices in $L(S(G)) - V(K_v)$ via three edges in E' (these three edges are the images of the three edges in $E_G(v)$ under the bijection of Lemma 3.5). This, together with (7), implies that for each $v \in V(G)$,

$$|V(K_v) \cap D_3(H)| \leqslant 1. \tag{8}$$

Since H is a Petersen minor, $|D_3(H)| = 10$ and all vertices in $V(H) - D_3(H)$ has degree 2 in H. Let u_1 and u_2 be two vertices in $D_3(H)$ and let P be a (u_1, u_2)-path in H such that all vertices in $V(P) - \{u_1, u_2\}$ have degree 2 in H. By the definition of $L(S(G))$ and by (3),

$$V(L(S(G)) = \bigcup_{v \in V(G)} V(K_v) \text{ is a vertex disjoint union,} \tag{9}$$

and so by (9), there are two distinct vertices $v_1, v_2 \in V(G)$ such that $u_i \in V(K_{v_i})$ for $i \in \{1, 2\}$. By (9) and by the assumption that $v_1 \neq v_2$, $E(P) \neq \emptyset$. Therefore, by (3) and (8), and regarding $E(P) \cap E'$ as an edge subset of $E(G)$, one can identify $P/(E(P) - E') = G[E(P) \cap E']$, the edge induced subgraph of G, which is a (v_1, v_2)-path in G. Apply these identifications for each of the 15 paths of H representing the 15 edges of the Petersen graph, one concludes that $H/[E(H) \cap (E(H) - E')]$ corresponds to a Petersen minor of G. Hence, if G has no Petersen minor, nor does $L(S(G))$. Therefore, Theorem 3.4(iii) follows by Claims 1 and 2. □

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