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Eulerian subgraphs containing given edges

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Dedicated to Paul A. Catlin

Abstract

For an integer $l \geqslant 0$, define $\mathscr{SE}(l)$ to be the family of graphs such that $G \in \mathscr{SE}(l)$ if and only if for any edge subset $X \subseteq E(G)$ with $|X| \leqslant l$, G has a spanning eulerian subgraph H with $X \subseteq E(H)$. The graphs in $\mathscr{SE}(0)$ are known as *supereulerian* graphs. Let $f(l)$ be the minimum value of k such that every k-edge-connected graph is in $\mathscr{SE}(l)$. Jaeger and Catlin independently proved $f(0) = 4$. We shall determine $f(l)$ for all values of $l \geqslant 0$. Another problem concerning the existence of eulerian subgraphs containing given edges is also discussed, and former results in [J. Graph Theory 1 (1977) 79–84] and [J. Graph Theory 3 (1979) 91–93] are extended.

Keywords: Eulerian subgraph; Supereulerian graph; Collapsible graph

1. Introduction

Graphs in this note are finite and loopless. Undefined terms and notation are from [2]. We use $H \subseteq G$ to denote the fact that H is a subgraph of G. For a graph G, $O(G)$ denotes the set of all vertices of odd degree in G. A graph G with $O(G) = \emptyset$ is an *even graph*, and a connected even graph is an *eulerian graph*. A graph is *supereulerian* if it has a spanning eulerian subgraph. The collection of all supereulerian graphs will be denoted by $\mathscr{SL}$. For the literature on the subject of supereulerian graphs, see Catlin's excellent survey [4]. As indicated by the authors in [1], characterizing supereulerian graphs appears very difficult. Pulleyblank in [8] pointed out that the problem of determining if a graph G is supereulerian is NP-complete.

A *bond* is a minimal edge-cut. A bond X of G is an *odd bond* if $|X|$ is odd. In [1] Boesch et al. proved Theorem 1.1 below, and in [7], Jaeger presented an elegant simple proof.

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Theorem 1.1 (Boesch et al. [1] and Jaeger [7]). *Let H'' be a subgraph of a graph H. The following are equivalent*:

(i) *There is an Eulerian subgraph H' such that $H'' \subseteq H' \subseteq H$;*
(ii) *$E(H'')$ contains no odd bond of H.*

Theorem 1.2 (Jaeger [7]). *If G has two edge-disjoint spanning trees, then G is in $\mathscr{SL}$.*

Let $F(G)$ be the minimum number of edges that must be added to G so that the resulting graph has 2 edge-disjoint spanning trees. Thus Theorem 1.2 says that if $F(G)=0$, then $G \in \mathscr{SL}$. In [3], Catlin showed that Theorem 1.2 can be improved.

Theorem 1.3 (Catlin [3]). *If $F(G) \leqslant 1$, then either G is in $\mathscr{SL}$, or G can be contracted to K_2.*

Each of Theorems 1.2 and 1.3 implies the corollary below.

Corollary 1.4. *If G is 4-edge-connected, then G is in $\mathscr{SL}$.*

Both Theorems 1.2 and 1.3 are recently extended in [6], (see Theorem 2.2).

In this note, we consider the problem of finding spanning eulerian subgraphs that contain given edge subsets. For an integer $l \geqslant 0$, define $\mathscr{SE}(l)$ to be the family of graphs such that $G \in \mathscr{SE}(l)$ if and only if for any edge subset $X \subseteq E(G)$ with $|X| \leqslant l$, G has a spanning eulerian subgraph H with $X \subseteq E(H)$. Thus $\mathscr{SE}(0)=\mathscr{SL}$. Let $f(l)$ be the minimum value of k such that every k-edge-connected graph is in $\mathscr{SE}(l)$. As there are 3-edge-connected graphs that are not in $\mathscr{SL}$, Corollary 1.4 says $f(0)=4$. In Section 2, we shall display some preliminaries and in Section 3, we shall determine $f(l)$ for all values of l. The main result in Section 4, Theorem 4.1, is an extension of Theorem 1.1, which was developed by Catlin and the author in their earlier attempts to prove Theorem 2.2 below.

2. Preliminaries

In [3], Catlin defined the collapsible graphs. Let $R \subseteq V(G)$. A subgraph Γ of G is called an *R-subgraph* if both $G-E(\Gamma)$ is connected and $v \in R$ if and only if v has odd degree in Γ. A graph G is *collapsible* if for any even subset R of $V(G)$, G has an R-subgraph. Catlin showed [3] that every vertex of G is lying in a unique maximal collapsible subgraph of G. The collection of all collapsible graphs is denoted by $\mathscr{CL}$. Clearly $\mathscr{CL} \subset \mathscr{SL}$.

The contraction G/H is obtained from G by contracting each edge of H and deleting the resulting loops. The *reduction* of G is obtained from G by contracting all maximal collapsible subgraphs, and is denoted by G'. A graph G is *reduced* if G is the reduction of some graph.

Theorem 2.1 (Catlin [3]). *Let G be a graph. Each of the following holds.*

(i) [3, Theorem 5] *G reduced iff G has no nontrivial collapsible subgraph.*
(ii) [3, Theorem 8] *If G is reduced, then G is simple, and contains no K_3, and $\delta(G) \leqslant 3$, and G can be covered by at most two edge-disjoint forests.*
(iii) [3, Theorem 8] *If G is reduced, then for any $H \subseteq G$, either $H \in \{K_1, K_2\}$ or $|E(H)| \leqslant 2|V(H)| - 4$.*
(iv) [3, Theorem 3] *If H is a collapsible subgraph of G, then $G \in \mathscr{CL}$ if and only if $G/H \in \mathscr{CL}$.*
(v) [3, Theorem 3] *If H is a collapsible subgraph of G, then $G \in \mathscr{SL}$ if and only if $G/H \in \mathscr{SL}$.*

The following result proves a conjecture of Catlin in [5], and generalizes Theorems 1.2 and 1.3.

Theorem 2.2 (Catlin et al. [6]). *Let G be a connected graph. If $F(G) \leqslant 2$, then either $G \in \mathscr{CL}$, or the reduction of G is in $\{K_2, K_{2,t}, (t \geqslant 1)\}$.*

Let $\tau(G)$ denote the maximum number of edge-disjoint spanning trees of G. Catlin recently proved a relationship between $\tau(G)$ and $\kappa'(G)$, the edge-connectivity.

Theorem 2.3 (Catlin [5]). *Let G be a graph and let $p \geqslant 1$ be an integer. The following are equivalent*:

(i) $\kappa'(G) \geqslant 2p$.
(ii) *For any $X \subseteq E(G)$ with $|X| \leqslant p$, $\tau(G - X) \geqslant p$.*

Let G be a graph and let $X \subseteq E(G)$. The graph G_X is obtained from G by replacing each edge $e \in X$ with ends u_e and v_e by a (u_e, v_e)-path P_e of length 2, where the internal vertex $w(e)$ of the path P_e is newly added.

Lemma 2.4. *Let $p \geqslant 2$ be an integer, let G be a graph and let $X \subseteq E(G)$. Each of the following holds*:

(i) *G has a spanning eulerian subgraph H such that $X \subseteq E(H)$ if and only if $G_X \in \mathscr{SL} = \mathscr{SE}(0)$.*
(ii) *$G \in \mathscr{SE}(l)$ if and only if for any $X \subseteq E(G)$ with $|X| \leqslant l$, $G_X \in \mathscr{SL}$.*
(iii) *$\tau(G - X) \geqslant 2$ if and only if $\tau(G_X) \geqslant 2$.*
(iv) *If $F(G) = 0$ (that is, $\tau(G) \geqslant 2$) and if $X = \{e_1, e_2\} \subseteq E(G)$, then $F(G_X) \leqslant 2$.*

(v) *If* $F(G)=0$ (*that is*, $\tau(G)\geqslant 2$) *and if* $X=\{e\}\subseteq E(G)$, *then* $F(G_X)\leqslant 1$.
(vi) *If* $\tau(G)\geqslant p$, *then for any* $X\subseteq E(G)$ *with* $|X|\leqslant p$, $F(G_X)\leqslant 2$.

Proof. (i) and (ii) follow from the definitions. To prove (iii), it suffices to prove the case when $|X|=1$ and proceed induction. Suppose $X=\{e\}$ and e has ends u and v. Then $G_{\{e\}}$ is obtained from $G-e$ by adding a new vertex w_e which is adjacent to u and v in $G_{\{e\}}$. If $G-e$ has edge-disjoint spanning trees T_1' and T_2', then $T_1=T_1'+uw_e$, $T_2=T_2'+vw_e$ are edge-disjoint spanning trees of $G_{\{e\}}$. The converse holds easily.

(iv) Let e_i' be an edge not in $E(G)$ but e_i' is parallel to e_i, for $i=1,2$. Then, $G\equiv(G+\{e_1',e_2'\})-X$. Thus by (iii) and by $\tau(G)\geqslant 2$, $\tau((G+\{e_1',e_2'\})_X)=\tau(G_X+\{e_1',e_2'\})\geqslant 2$. Hence, $F(G_X)\leqslant 2$.

(v) The proof for (v) is similar to that for (iv).

(vi) Let $T_1,\dots,T_p$ be disjoint spanning trees of G. Let $e_1,e_2\in X$ and $X_1=X-\{e_1,e_2\}$. Since $|X|\leqslant p$, X_1 can meet at most $p-2$ of the T_i's, and so $\tau(G-X_1)\geqslant 2$. It follows by Lemma 2.4(iii) that $F(G_{X_1})=0$. Then by Lemma 2.4(iv), $F(G_X)=F((G_{X_1})_{\{e_1,e_2\}})\leqslant 2$. □

3. The values of the function $f(l)$

For an edge subset $X\subseteq E(G)$, recall that G_X is the graph obtained from G by subdividing each edge in X into a path of length 2. Let $W(G;X)=V(G_X)-V(G)$ denote the set of newly added vertices in the process of subdividing edges in X. For any integer $i\geqslant 0$,

$$D_i(G)=\{v\in V(G)\colon v \text{ has degree } i \text{ in } G\}.$$

Let G' denote the reduction of G. A vertex $v'\in V(G')$ is *nontrivial* if v' is the contraction image of a nontrivial maximal collapsible subgraph H of G. Otherwise v' is a *trivial* vertex in the reduction G'.

Lemma 3.1. *Let* G *be a graph with* $\kappa'(G)\geqslant 3$, *and let* $X\subset E(G)$. *Suppose that the reduction of* G_X, (*denoted by* G_X'), *is a* $K_{2,t}$, *for some* $t\geqslant 3$. *Then each of the following holds.*

(i) *Every vertex in* $D_2(G_X')$ *is a trivial vertex.*
(ii) $D_2(G_X')\subseteq W(G;X)$.
(iii) $t\geqslant\kappa'(G)$.

Proof. Let $v'\in D_2(G_X')$ and let e_1,e_2 be the two edges incident with v' in G_X'. If v' is nontrivial, then $\{e_1,e_2\}$ would be an edge-cut of G, contrary to the assumption of $\kappa'(G)\geqslant 3$. Therefore, v' must be trivial and in $W(G;X)$. This proves (i) and (ii).

By Lemma 3.1(ii), the t vertices of degree 2 in G_X' are obtained from subdividing t edges $e_1,\dots,e_t\in X$, and so $\{e_1,\dots,e_t\}$ forms an edge cut of G. Therefore, $t\geqslant\kappa'(G)$. □

Example 3.2. Let $l \geqslant 3$ be an integer. Let $G(l)$ denote an l-edge-connected graph with $\delta(G(l)) = l$. Let $v \in V(G(l))$ be a vertex of degree l and let

$$E(v) = \{e \in E(G(l)) \colon e \text{ is incident with } v\}.$$

If l is odd, then $G(l)$ cannot have an eulerian subgraph containing $X = E(v)$ by Theorem 1.1. Therefore for $l \geqslant 3$, we have

$$f(l) \geqslant \begin{cases} l+1, & l \geqslant 3 \text{ and } l \text{ is odd}, \\ l, & l \geqslant 4 \text{ and } l \text{ is even}. \end{cases} \tag{1}$$

Theorem 3.3. *Let $l \geqslant 0$ be an integer. Then*,

$$f(l) = \begin{cases} 4, & 0 \leqslant l \leqslant 2, \\ l+1, & l \geqslant 3 \textit{ and } l \textit{ is odd}, \\ l, & l \geqslant 4 \textit{ and } l \textit{ is even}. \end{cases}$$

Proof. By Corollary 1.4 and by the fact that there exist 3-edge-connected graphs not in $\mathscr{SL}$, $f(l) = 4$ for $0 \leqslant l \leqslant 2$. Thus, we only need to prove the theorem for $l \geqslant 3$. Suppose first that $l = 2p+1$ and that $p \geqslant 1$ is an integer. By (1), it suffices to show in this case that

$$f(l) \leqslant l+1, \tag{2}$$

Let G be an $(l+1)$-edge-connected graph, and let $X \subseteq E(G)$ with $|X| = l$. By Lemma 2.5(i), it suffices to show

$$G_X \in \mathscr{SL}. \tag{3}$$

Choose $X_1 \subset X$ with $|X_1| = p+1$. By Theorem 2.3, $\tau(G - X_1) \geqslant p+1$. Therefore, by Lemma 2.4(vi), $F(G_X) \leqslant 2$. If G_X is collapsible, then by $\mathscr{CL} \subset \mathscr{SL}$, (3) holds. Thus, we assume that G_X is not collapsible. By Theorem 2.2, the reduction of G_X, denoted by G'_X, is in $\{K_2, K_{2,t}, (t \geqslant 1)\}$. Since $\kappa'(G) \geqslant l+1 \geqslant 2p+2$, G_X is also 2-edge-connected, and so $G'_X \notin \{K_2, K_{1,2}\}$. Hence $G'_X = K_{2,t}$ for some $t \geqslant 2$. By Lemma 3.1(iii), $t \geqslant \kappa'(G) \geqslant l+1$. By Lemma 3.1(ii), $t \leqslant |X| = l$. This contradiction shows that G_X must be collapsible. Hence (2) holds when $l = 2p+1$ is odd.

Now, assume that $l = 2p$ for some $p \geqslant 2$, and we want to show in this case

$$f(l) \leqslant l. \tag{4}$$

Let G be a graph with $\kappa'(G) \geqslant l$, and let $X \subseteq E(G)$ with $|X| = l$. Choose $X_1 \subset X$ with $|X_1| = p$. By Theorem 2.3, $\tau(G - X_1) \geqslant p$. By Lemma 2.4(vi), $F(G_X) \leqslant 2$. As before, we may assume that $F(G_X) = 2$ and $G'_X = K_{2,t}$ for some $t \geqslant 2$. By Lemma 3.1(ii) and (iii), $l = |X| \geqslant t \geqslant \kappa'(G) \geqslant l$, and so $t = l$. However, $l = 2p \geqslant 4$ is an even number, and so $G'_X = K_{2,t}$ is eulerian. By Theorem 2.1(v), $G_X \in \mathscr{SL}$. Therefore (4) holds when $l = 2p$ is even.

This completes the proof for Theorem 3.3. □

4. An extension of Theorem 1.1

Let G be a graph. For each bond $B \subseteq E(G)$ and each even subset $S \subseteq V(G)$, we define $p(B,S,G)=1$ if each component of $G-B$ has an odd number of vertices in S; and $p(B,S,G)=0$ otherwise.

Theorem 4.1. *Let G'' be a subgraph of G and let $S \subseteq V(G'')$ be an even subset. The following are equivalent*:

(i) *There is a subgraph G' such that $G'' \subseteq G' \subseteq G$ and such that $S = O(G')$.*
(ii) *$E(G'')$ contains no bond B of G such that $|B| + p(B,S,G)$ is odd.*
(iii) *G has an $(O(G) \oplus S)$-subgraph Γ such that $E(\Gamma) \cap E(G'') = \emptyset$.*

Proof. Suppose that Theorem 4.1(iii) holds. Let $G' = G - E(\Gamma)$. Then, G' satisfies Theorem 4.1(i). Conversely, we assume Theorem 4.1(i). Choose G' to be a maximal subgraph of G satisfying Theorem 4.1(i), and so by the maximality of G', $\Gamma = G - E(G')$ is a forest. Note that $v \in (O(G) \oplus S)$ if and only if $v \in (O(G) \cup O(G')) - (O(G) \cap O(G'))$, and so if and only if v has odd degree in Γ. Therefore, Γ is an $(O(G) \oplus S)$-subgraph, and so we have shown that Theorem 4.1(i) and (iii) are equivalent.

Next, we shall show that Theorem 4.1(i) and (ii) are equivalent. Assume first that $S = \emptyset$. Then Theorem 4.1(i) and (ii) are precisely the same as Theorem 1.1(i) and (ii). Therefore, we may assume that $s = |S| > 0$ and let $S = \{u_1, u_2, \ldots, u_s\}$. Let H be the supergraph of G obtained by adding to G a set $\{v_0, v_1, \ldots, v_s\}$ of $s+1$ vertices and a set of $2s$ new edges such that the following conditions hold in H:

(H1) $N(v_0) = \{v_1, v_2, \ldots, v_s\}$.
(H2) Each vertex $u_i \in S$ is joint by a new edge to v_i, $(1 \leqslant i \leqslant s)$.
(H3) Each v_i has degree 2 in H, $(1 \leqslant i \leqslant s)$.

Similarly, for any subgraph G' of G with $S \subseteq V(G')$, construct a supergraph H' of G' by adding to G' the same $s+1$ new vertices and the same $2s$ new edges that were added to G to form H. Thus $H' \subseteq H$. In particular, we denote by H'' such a supergraph obtained from G''.

Note that $G = H - \{v_0, v_1, \ldots, v_s\}$, $G' = H' - \{v_0, v_1, \ldots, v_s\}$ and $G'' = H'' - \{v_0, v_1, \ldots, v_s\}$.

Thus (i) of Theorem 4.1 holds, if and only if Theorem 1.1(i) holds, and so by Theorem 1.1, if and only if Theorem 1.1(ii) holds. That is, H'' contains no odd bond of H. Let X be a bond of H that is contained in H''. Then X can be partitioned into two parts: $B = X \cap E(G)$ and $B' = X - B$. By the definition of H'', B must be a bond of G. Since $|B'| \equiv p(B,S,G) \pmod 2$, we have $|X| \equiv |B| + p(B,S,G) \pmod 2$. Therefore, Theorem 1.1(ii) is equivalent to Theorem 4.1(ii). This establishes the equivalence between Theorem 4.1(i) and (ii), and so the proof of Theorem 4.1 is completed. □

Corollary 4.2. *Theorem* 4.1 *implies Theorem* 1.1.

Proof. Let $S = \emptyset$. Then the equivalence of (i) and (ii) of Theorem 4.1 implies Theorem 1.1. $\square$

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