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Group Connectivity of 3-Edge-Connected Chordal Graphs

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Abstract. Let A be a finite abelian group and G be a digraph. The *boundary* of a function $f : E(G) \mapsto A$ is a function $\partial f : V(G) \mapsto A$ given by $\partial f(v) = \sum_{e \text{ leaving } v} f(e) - \sum_{e \text{ entering } v} f(e)$. The graph G is A-connected if for every $b : V(G) \mapsto A$ with $\sum_{v \in V(G)} b(v) = 0$, there is a function $f : E(G) \mapsto A - \{0\}$ such that $\partial f = b$. In [J. Combinatorial Theory, Ser. B 56 (1992) 165–182], Jaeger *et al* showed that every 3-edge-connected graph is A-connected, for every abelian group A with $|A| \geq 6$. It is conjectured that every 3-edge-connected graph is A-connected, for every abelian group A with $|A| \geq 5$; and that every 5-edge-connected graph is A-connected, for every abelian group A with $|A| \geq 3$.

In this note, we investigate the group connectivity of 3-edge-connected chordal graphs and characterize 3-edge-connected chordal graphs that are A-connected for every finite abelian group A with $|A| \geq 3$.

1. Introduction

We consider finite graphs which may contain loops or multiple edges. The groups considered in this paper are finite Abelian (additive) groups. For a finite Abelian group A, the additive identity of A will be denoted by 0 (zero) throughout this paper. Let G and H be graphs. If H is a subgraph of G, then we write $H \subseteq G$.

Let G be a digraph. For a vertex $v \in V(G)$, let

$$E_G^-(v) = \{(u, v) \in E(G) : u \in V(G)\}, \quad \text{and} \quad E_G^+(v) = \{(v, u) \in E(G) : u \in V(G)\}.$$

The subscript G may be omitted when G is understood from the context.

Let A be a nontrivial Abelian group and let A^* denote the set of nonzero elements in A. Define

$$F(G, A) = \{f : E(G) \to A\} \quad \text{and} \quad F^*(G, A) = \{f : E(G) \to A^*\}.$$

For each $f \in F(G, A)$, the **boundary** of f is a function $\partial f : V(G) \to A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. Throughout this paper, we shall adopt the following convenience: if $X \subseteq E(G)$ and if $f : X \to A$ is a function, then we regard f as a function $f : E(G) \to A$ such that $f(e) = 0$ for all $e \in E(G) - X$.

Let G be an undirected graph and A be an abelian group. A function $b : V(G) \to A$ is called an **A-valued zero sum function on** G if $\sum_{v \in V(G)} b(v) = 0$ in G. The set of all A-valued zero sum functions on G is denoted by $Z(G, A)$. A graph G is **A-connected** if G has an orientation G' such that for every function $b \in Z(G, A)$, there is a function $f \in F^*(G', A)$ such that $b = \partial f$. For an Abelian group A, let $\langle A \rangle$ denote the family of graphs that are A-connected. It is observed in [7] that that $G \in \langle A \rangle$ is independent of the orientation of G.

An **A-nowhere-zero-flow** (abbreviated as A-NZF) in G is a function $f \in F^*(G, A)$ such that $\partial f = 0$. The nowhere-zero-flow problems were introduced by Tutte [14], and recently surveyed by Jaeger in [6].

The concept of A-connectivity was introduced by Jaeger *et al* in [7], where A-NZF's were successfully generalized to A-connectivities. A concept similar to the group connectivity was independently introduced in [8], with a different motivation from [7].

Tutte's 3-flow conjecture ([2], Unsolved Problem 48) states that every 4-edge-connected graph has a Z_3-NZF. It is known that planar graphs and projective planar graphs are two graph families in which Tutte's 3-flow conjecture holds ([12] and [16]). In this note, we shall show (in Section 4) that every 4-edge-connected chordal graph is A-connected, for each Abelian group A with $|A| \geq 3$. In particular, every 4-edge-connected chordal graph admits a Z_3-NZF. There are examples, presented in Section 4, showing that both being chordal and being 4-edge-connected cannot be relaxed in this result. We also investigate 3-edge-connected chordal graphs in Section 4 and characterize those 3-edge-connected chordal graphs which are not Z_3-connected.

In Section 2, former related results are presented and in Section 3, some properties on group connectivity will be developed. These properties will be used in Section 4 to prove the main results.

2. Prior Results

We present some prior results on Z_3-NZF's in this section.

Theorem 2.1 (Grötzsch [4]). *Every* 4-*edge-connected planar graph has a* Z_3-*NZF.*

Theorem 2.2 (Tutte [15]). *Let G be a* 3-*regular graph. Then G has a* Z_3-*NZF if and only if G is bipartite.*

Theorem 2.3 (Grübaum [5], Aksionov [1], Steinberg and Younger [11] and Thomassen [13]). *Every* 2-*edge-connected graph with at most three edge cuts of size* 3 *and embedable in the plane has a* Z_3-*NZF.*

Theorem 2.4 (Steinberg and Younger [11]). *Every* 2-*edge-connected graph with at most one edge cut of size* 3 *and embedable in the projective plane has a* Z_3-*NZF.*

Theorem 2.5 (Lai and Zhang [10]). *Let G be a k-edge-connected graph with t odd vertices. If $k \geq 4\lceil \log_2 t \rceil$, then G has a Z_3-NZF.*

It is conjectured in [7] *that every* 5*-edge-connected graphs is in* $\langle Z_3 \rangle$.

3. Some Properties

We need the notion of graph contraction. Let G be a graph and let $X \subseteq E(G)$ be an edge subset. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge e in X and deleting e. If $X = \{e\}$, then we write G/e for $G/\{e\}$. If H is a subgraph of G, then we write G/H for $G/E(H)$. Note that even G is a simple graph, the contraction G/X may have loops and multiple edges.

Let G be a graph and let $v \in V(G)$ be a vertex of degree $m \geq 4$. Let $N(v) = \{v_1, v_2, \ldots, v_m\}$ denote the set of vertices adjacent to the vertex v, and let $X = \{vv_1, vv_2\}$. The graph $G_{[v,X]}$ is obtained from $G - \{vv_1, vv_2\}$ by adding a new edge that joins v_1 and v_2. If $m = 2k$ is even and if

$$M = \langle \{v_1, v_{k+1}\}, \{v_2, v_{k+2}\}, \ldots, \{v_k, v_{2k}\} \rangle$$

is a way to pair the vertices in $N(v)$, then $G_{(v,M)}$ denotes the graph obtained from $G - v$ by adding k new edges e_i joining v_i and v_{k+i}, $(1 \leq i \leq k)$.

Lemma 3.1. *Let A be an Abelian group. Let G be a graph and let $v \in V(G)$ be a vertex of degree $m \geq 4$.*

(i) *If for some X of two edges incident with v in G, $G_{[v,X]} \in \langle A \rangle$, then $G \in \langle A \rangle$.*

(ii) *Let m be even, let M be a way to pair the vertices of $N(v)$ such that $G_{(v,M)} \in \langle A \rangle$, and let $b \in Z(G, A)$ be given. If $b(v) = 0 \in A$, then there is a function $f \in F^*(G, A)$ such that $\partial f = b$.*

(iii) (Corollary 2.3, [7]) *Let $e = v_1v_2$ be an edge in G. If $G - e \in \langle A \rangle$, then $G \in \langle A \rangle$.*

Proof. (i). We may assume that $N(v) = \{v_1, \ldots, v_m\}$ and $X = \{vv_1, vv_2\}$. Since being A-connected is a property independent of the orientation, we may assume that in G, the two edges in X are oriented from v_1 to v, and from v to v_2; and we may assume that in $G_{[v,X]}$, the newly added edge (denoted by v_1v_2) is oriented from v_1 to v_2.

Let $b \in Z(G, A)$. Since $V(G) = V(G_{[v,X]})$, $b \in Z(G_{[v,X]}, A)$ also. Since $G_{[v,X]} \in \langle A \rangle$, there is a function $f' \in F^*(G_{[v,X]}, A)$ such that $\partial f' = b$. Define $f \in F(G, A)$ by $f(e) = f'(e)$ if $e \in E(G) - X$, and $f(v_1v) = f(vv_2) = f'(v_1v_2)$. Then it is easy to see that $f \in F^*(G, A)$ such that $\partial f = b$. This proves (i).

(ii). The proof for (ii) is similar to that for (i), and so is omitted. $\square$

For a subgraph H of a graph G, let $A_G(H)$ denote the vertices in $V(H)$ that are adjacent to some vertices in $V(G) - V(H)$ in G. (Vertices in $A_G(H)$ are called the **vertices of attachment of H in G**.)

Proposition 3.2. *For any Abelian group A, $\langle A \rangle$ is a family of connected graphs* *satisfying each of the following holds*:

(C1) $K_1 \in \langle A \rangle$,
(C2) *if* $e \in E(G)$ *and if* $G \in \langle A \rangle$, *then* $G/e \in \langle A \rangle$, *and*
(C3) *if* $H \in \langle A \rangle$ *and if* $G/H \in \langle A \rangle$, *then* $G \in \langle A \rangle$.

(*A family of connected graphs satisfying* (C1)–(C3) *is called a* **complete family**, *first introduced by Catlin* [3]. *In fact, Catlin defined and studied complete families in a more general way. For more properties of complete families, see* [3].)

Proof. Let A be an Abelian group. By Proposition 2.1 of [7], every member in $\langle A \rangle$ is connected. Note that (C1) follows from the definition of $\langle A \rangle$ trivially.

Let $e \in E(G)$ and assume that $G \in \langle A \rangle$. Let G' be G/e with an orientation, let the two ends of e be u and v and let w denote the vertex in G' to which u and v are identified. Let G also denote the digraph with the same orientation as G' on the edges in $E(G) - \{e\}$ and with e oriented from u to v.

For any $b' \in Z(G', A)$, define a function b by

$$b(z) = \begin{cases} b'(z) & \text{if } z \in V(G') - \{w\} = V(G) - \{u, v\} \\ b'(w) & \text{if } z = u \\ 0 & \text{if } z = v. \end{cases}$$

Then $\sum_{z \in V(G)} b(z) = \sum_{z \in V(G')} b'(z) = 0$, and so $b \in Z(G, A)$. Since $G \in \langle A \rangle$, there is a function $f \in F^*(G, A)$ with $\partial f = b$. Let f' be the restriction of f to $E(G) - \{e\}$. Since

$$\begin{aligned} \partial f'(w) &= \sum_{e' \in E_G^+(v) \cup E_G^+(u) - \{e\}} f(e') - \sum_{e' \in E_G^-(v) \cup E_G^-(u) - \{e\}} f(e') \\ &= \partial f(u) + \partial f(v) = b(u) + b(v) = b'(w), \end{aligned}$$

$\partial f' = b'$ and so by definition, $G/e \in \langle A \rangle$. Therefore (C2) holds.

Suppose that both $H \in \langle A \rangle$ and $G/H \in \langle A \rangle$. We may assume that G has a fixed orientation. Thus the edges in both H and G/H are oriented by the orientation of G. By Lemma 3.1(iii), we may assume that H is an induced subgraph of G, and so $E(G)$ is the disjoint union of $E(H)$ and $E(G/H)$. Note that H is connected and so H will be contracted to a vertex v_H (say) in G/H. Let $b \in Z(G, A)$ and let $a_0 = \sum_{v \in V(H)} b(v)$. Define $b_1 : V(G/H) \to A$ by

$$b_1(z) = \begin{cases} b(z) & \text{if } z \neq v_H \\ a_0 & \text{if } z = v_H. \end{cases}$$

Then $\sum_{z \in V(G/H)} b_1(z) = \sum_{z \in V(G)} b(z)$ and so $b_1 \in Z(G/H, A)$. Since $G/H \in \langle A \rangle$, there is a function $f_1 \in F^*(G/H, A)$ such that $\partial f_1 = b_1$.

For each $z \in V(H)$, define

$$b_2(z) = \begin{cases} b(z) + \sum_{e \in E^-_{G/H}(v_H) \cap E^-_G(z)} f_1(e) - \sum_{e \in E^+_{G/H}(v_H) \cap E^+_G(z)} f_1(e) & \text{if } z \in A_G(H) \\ b(z) & \text{otherwise.} \end{cases}$$

Note that

$$\sum_{z \in V(H)} b_2(z) = \sum_{z \in V(H)} b(z) + \sum_{e \in E^-_{G/H}(v_H)} f_1(e) - \sum_{e \in E^+_{G/H}(v_H)} f_1(e) = a_0 - \partial f_1(v_H) = 0,$$

and so $b_2 \in Z(H, A)$. Since $H \in \langle A \rangle$, there is a function $f_2 \in F^*(H, A)$ such that $\partial f_2 = b_2$.

Define, for each $e \in E(G)$, $f(e) = f_1(e) + f_2(e)$. Since $E(G)$ is the disjoint union of $E(H)$ and $E(G/H)$, $f \in F^*(G, A)$; and for a vertex $z \in V(G)$,

$$\partial f(z) = \partial f_1(z) + \partial f_2(z) = \partial f_1(z) + b(z) - \partial f_1(z) = b(z).$$

Therefore $G \in \langle A \rangle$, and so (C3) follows. $\square$

The "only if" part of Lemma 3.3 was observed in [7] without a proof. We present the whole proof of Lemma 3.3 for the sake of completeness.

Lemma 3.3. *Let $n \geq 1$ denote an integer and let C_n denote the cycle of n vertices. Then for any Abelian group A, $C_n \in \langle A \rangle$ if and only if $|A| \geq n + 1$.*

Proof. Let $C_n = v_1 v_2 \cdots v_n v_1$ and assume that the edge $v_i v_{i+1}$ is oriented from v_i to v_{i+1}, for each $i = 1, 2, \ldots, n \pmod n$.

Suppose first that $C_n \in \langle A \rangle$. We shall assume $|A| = m \leq n$ to find a contradiction.

Let $A = \{a'_1, a'_2, \ldots, a'_m\}$ with $a'_m = 0$. Let $a_0 = 0$, $a_1 = a'_1$ and $a_i = a'_i - a'_{i-1}$ for $2 \leq i \leq m-1$ and $a_m = -\sum_{i=1}^{m-1} a_i$. Then $a_1, a_2, \ldots, a_{m-1}$ is a sequence of elements in A such that $\{\sum_{j=1}^{i} a_j : 1 \leq i \leq m-1\} = A - \{0\}$. Moreover, for any $x \in A$,

$$\left\{ x + \sum_{j=0}^{i} a_j : 1 \leq i \leq m-1 \right\} = A. \tag{1}$$

Define $b(v_i) = a_i$ for $1 \leq i \leq m$, and $b(v_i) = 0$ for $m+1 \leq i \leq n$. Since $|A| = m$, $\sum_{i=1}^{n} b(v_i) = \sum_{i=1}^{m-1} a_i + a_m = 0 \in A$ and so $b \in Z(C_n, A)$. Since $C_n \in \langle A \rangle$, there is a function $f \in F^*(C_n, A)$ such that $\partial f = b$. Denote $f(v_n v_1)$ by x. Then since $\partial f = b$, we must have $f(v_i v_{i+1}) = x + \sum_{j=1}^{i} a_i$ for all $i = 1, 2, \ldots, m$ and $f(v_i v_{i+1}) = x$ for all $i = m+1, \ldots, n$. By (1), one of $f(v_i v_{i+1})$, where $1 \leq i \leq n$, must be $0 \in A$, contrary to the assumption that $f \in F^*(C_n, A)$.

Conversely, assume that $|A| \geq n+1$. Let $b \in Z(C_n, A)$, and let $B = \{a \in A : a = -b(v_i) \text{ for some } 1 \leq i \leq n-1\}$. Then $|B| \leq n-1$. Since $|A| \geq n+1$, there is an $x \in A - (B \cup \{0\})$. Define $f \in F(G, A)$ by $f(v_i v_{i+1}) = b(v_i) + x$ $(1 \leq i \leq n-1)$ and $f(v_n v_1) = x$. Then $\partial f(v_i) = b(v_i)$ and $f \in F^*(C_n, A)$. Hence $C_n \in \langle A \rangle$. $\square$

Let $k \geq 1$ be an integer and let H be a subgraph of G. The **k-closure of H in G** is the (unique) maximal subgraph of G the form $H \cup \Gamma_1 \cup \cdots \cup \Gamma_n$ where for each i, $1 \leq i \leq n$, Γ_i is a cycle and $|E(\Gamma_i) - E(H \cup \Gamma_1 \cup \cdots \cup \Gamma_{i-1})| \leq k$.

Corollary 3.4 below follows from Lemma 3.3 and Proposition 3.2(C3).

Corollary 3.4 (Corollary 2.4 of [7]). *Let A be a finite Abelian group with $|A| > k$. Let H be a subgraph of G. If $H \in \langle A \rangle$, then the k-closure of H in G is also in $\langle A \rangle$.*

Corollary 3.5 *below follows from Proposition* 3.2(C3) *and Lemma* 3.3.

Corollary 3.5. *Let A be an Abelian group with $|A| \geq 3$, let $n \geq 5$ be an integer, and let $K_n - e$ denote the graph obtained from K_n by removing an edge from K_n. Each of the following holds*:

(i) $K_n - e \in \langle A \rangle$, *and*
(ii) $K_n \in \langle A \rangle$.

Proposition 3.6. *Let W_n be the wheel of $n+1$ vertices. Then $W_4 \in \langle A \rangle$, for any Abelian group A with $|A| \geq 3$.*

Proof. Let $v_1v_2v_3v_4v_1$ be the rim 4-cycle of W_4 and let v denote the center vertex of W_4. If $|A| \geq 4$, then by Lemma 3.3 and Proposition 3.2, $W_4 \in \langle A \rangle$. Hence we only need to prove the case when $A = Z_3$. Express $Z_3 = \{0, 1, -1\}$.

Let $b \in Z(W_4, Z_3)$. We shall find $f \in F^*(W_4, Z_3)$ by defining $f(e) = 1$, $\forall e \in E(W_4)$ and adjust the orientation of W_4 to meet the requirement of $\partial f = b$. In the rest of the proof, for an edge $e = xy \in E(W_4)$, we write (x, y) to mean that e is oriented from x to y. An orientation D of W_4 will be expressed by a set of oriented edges.

If $b(v) = 0$, then let $M = \langle \{v_1, v_2\}, \{v_3, v_4\} \rangle$ be a partition of $N(v)$. Then $(W_4)_{(v,M)}$ is the 3-regular graph with a 4-cycle and two disjoint 2-cycles. By Lemma 3.3 and by Proposition 3.2(C3), $(W_4)_{(v,M)} \in \langle Z_3 \rangle$, and so by Lemma 3.1(ii), there is a function $f \in F^*(W_4, Z_3)$ such that $\partial f = b$. Hence we assume that $b(v) \neq 0 \in Z_3$, and so $b(v) \in \{1, -1\}$. We only need to show that case when $b(v) = 1$, by symmetry.

Since $\sum_{z \in V(W_4)} b(z) = 0$, we may assume that $b(v_1) \neq 0 \in Z_3$.

Suppose first that $b(v_1) = -1$. Since $b \in Z(W_4, Z_3)$, either $b(v_2) = b(v_3) = b(v_4) = 1$, whence

$$D_1 = \{(v, v_1), (v_2, v), (v_3, v), (v_4, v), (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$$

is an orientation satisfying $\partial f = b$; or $b(v_2) = b(v_3) = b(v_4) = 0$, whence

$$D_2 = \{(v_1, v), (v_2, v), (v, v_3), (v_4, v), (v_2, v_1), (v_2, v_3), (v_4, v_3), (v_4, v_1)\}$$

is an orientation satisfying $\partial f = b$.

Hence we assume that $b(v_1) = 1$, and by symmetry, we assume that $b(v_i) \neq -1, \forall i$ with $2 \leq i \leq 4$. Since $b \in Z(W_4, Z_3)$ and by symmetry, either $b(v_2) = 1$ and $b(v_3) = b(v_4) = 0$, whence

$$D_3 = \{(v_1, v), (v_2, v), (v, v_3), (v_4, v), (v_1, v_2), (v_2, v_3), (v_4, v_3), (v_4, v_1)\}$$

is an orientation satisfying $\partial f = b$; or $b(v_3) = 1$ and $b(v_2) = b(v_4) = 0$, whence

$$D_4 = \{(v_1, v), (v_2, v), (v_3, v), (v, v_4), (v_2, v_1), (v_2, v_3), (v_3, v_4), (v_1, v_4)\}$$

is an orientation satisfying $\partial f = b$.

In any case, such a function $f \in F^*(W_4, Z_3)$ can be found so that $\partial f = b$, and so $W_4 \in \langle Z_3 \rangle$, by definition. □

4. Group Connectivity of Chordal Graphs

We shall prove the following Theorem 4.2, which implies that every 4-edge-connected chordal graph has a Z_3-NZF. A graph G is **chordal** if every induced cycle of G has length at most 3. Lemma 4.1 is an easy observation.

Lemma 4.1. *Every* 2*-connected simple graph* H *with* $|V(H)| \geq 4$ *has a cycle of length at least* 4.

Theorem 4.2. *Let* A *be an Abelian group of order at least* 3. *If* G *is a* 4*-edge-connected chordal graph, then* $G \in \langle A \rangle$.

Before proving Theorem 4.2, *we present some examples to show that both conditions of* G *in Theorem* 4.2 *are needed.*

Example 4.3. We shall show that being 4-edge-connected cannot be relaxed in Theorem 4.2, in the sense that there are infinitely many 3-edge-connected chordal graphs that are not in $\langle Z_3 \rangle$. The graph K_4 is a 3-edge-connected chordal graph that does not have a Z_3-NZF, and so it cannot be Z_3-connected. By Proposition 3.2(C2), any graph contractible to K_4 cannot be Z_3-connected. In particular, any graph with a block (a maximal 2-connected subgraph) isomorphic to K_4 cannot be Z_3-connected.

In fact, there is a class of 3-edge-connected, 2-connected chordal graphs that are not Z_3-connected. We shall present this class. Let $m \geq 1$ be an integer and let $G_1, G_2, \ldots, G_m$ be m disjoint copies of K_4 each of which has a distinguished edge $e_i = x_i y_i$, $(1 \leq i \leq m)$. Let $G(m)$ denote the graph obtained from the disjoint union of $G_1 - e_1, G_2 - e_2, \ldots, G_{m-1} - e_{m-1}$ and G_m by identifying $x_1, x_2, \ldots, x_m$ into a single vertex x and by identifying $y_1, y_2, \ldots, y_m$ into a single vertex y. The edge e_m, with its ends being x and y, is now an edge in $G(m)$. We shall show that $G(m)$ does not have a Z_3-NZF, and therefore cannot be Z_3-connected.

Suppose that $G(m)$ has a Z_3-NZF. Thus $G(m)$ is oriented and there is a function $f \in F^*(G(m), Z_3)$ such that $\partial f = 0$. Expressing $Z_3 = \{0, 1, -1\}$ and reversing the direction of the edges in $G(m)$ if necessary, we may assume that $f(e) = 1$, $\forall e \in E(G(m))$. For each i with $1 \leq i \leq m$, denote $V(G_i) = \{x_i, y_i, u_i, v_i\}$, where u_i and v_i are the two vertices of this subgraph G_i with degree 3 in $G(m)$. Since $f(e) = 1, \forall e \in E(G(m))$, we may assume that all three edges incident with u_i are directed out from u_i, and so all three edges incident with v_i are directed into v_i, for each i with $1 \leq i \leq m$. It follows by $\partial f = 0$ that $f(e_m) = 0$, contrary to the assumption that $f \in F^*(G(m), Z_3)$. □

Example 4.4. Being chordal cannot be relaxed in Theorem 4.2 either, in the sense that there are 4-edge-connected non-chordal graphs that are not Z_3-connected. The following class of graphs (together with its proof) is based on an example given by Jaeger *et al* in [7]. Let $m \geq 1$ be an integer and let C_1 and C_2 be two disjoint cycles of $6m$ vertices. Denote

$$C_1 = u_1u_2 \cdots u_{6m}u_1 \quad \text{and} \quad C_2 = v_1v_2 \cdots v_{6m}v_1.$$

Obtain a graph $J(m)$ from the disjoint union of C_1 and C_2 by adding these new edges: $\bigcup_{j=1}^{3m}\{u_{2j-1}v_{2j-1}, u_{2j}v_{2j}, u_{2j-1}v_{2j}, u_{2j}v_{2j-1}\}$. Note that $J(m)$ has $3m$ disjoint K_4's and every independent set of $J(m)$ has at most one vertex in each of these $3m$ K_4's. Thus every independent set in $J(m)$ has at most $3m$ vertices. Note also that $|V(J(m))| = 12m$ and $|E(J(m))| = 24m$. It is easy to see that $J(m)$ is 4-regular and 4-connected.

We argue by contradiction to show that $J(m)$ is not Z_3-connected. Let $b \equiv 1$ be a constant function defined on $V(J(m))$. Since $|V(J(m))| = 12m$, $b \in Z(J(m), Z_3)$. Assume that there is a function $f \in F^*(J(m), Z_3)$ such that $\partial f = b$. Expressing $Z_3 = \{0, 1, -1\}$ and reversing the direction of edges in $J(m)$ if necessary, we may assume that $f(e) = 1, \forall e \in E(J(m))$. Since $J(m)$ is 4-regular, each vertex in $J(m)$ has either out degree 1 or 4. Let V_i denote the number of vertices of out degree i in $J(m)$, where $i \in \{1, 4\}$. Then $V_1 + V_4 = |V(J(m))| = 12m$ and $4V_4 + V_1 = |E(J(m))| = 24m$. It follows that $V_1 = 8m$ and $V_4 = 4m$. However, since every independent set in $J(m)$ has at most $3m$ vertices, there must be two vertices in V_4 that are adjacent in $J(m)$, which is impossible since $J(m)$ is 4-regular. □

Theorem 4.2 will follow from the following stronger Theorem 4.7, which shows that the graphs presented in Example 4.3 are basically the only exceptional graphs for a 3-edge-connected chordal graph to be A-connected, for any Abelian group A with $|A| \geq 3$. We need two more easy observations.

Lemma 4.5. *If G is simple, 2-connected, 3-regular and chordal, then G is isomorphic to a K_4.*

Proof. Pick $v \in V(G)$ and denote $N(v) = \{v_1, v_2, v_3\}$. Since G is 2-connected and simple, the two edges vv_1 and vv_2 must be in a shortest cycle C of G of length at least 3. Since G is chordal, C must have length exactly 3, and so $v_1v_2 \in E(G)$. Similarly, we may assume $v_1v_3, v_2v_3 \in E(G)$. Since G is 3-regular and connected, $V(G) = \{v, v_1, v_2, v_3\}$, and so G is isomorphic to a K_4. □

Let K_4 be a given complete graph on 4 vertices $\{u, v, x, y\}$ with a distinguished edge $a = xy$, and let G be a graph disjoint from this K_4 with $|E(G)| \geq 2$ and with a distinguished edge $a' = x'y'$. Define a new graph $G \oplus K_4$ to be the graph obtained from the disjoint union of $G - e'$ and K_4 by identifying x' and x to form a new vertex, also called x, and by identifying y' and y to form a new vertex, also called y. Note that the edge $a = xy$ is now an edge of $G \oplus K_4$ and that $G = G \oplus K_4 - \{u, v\}$.

Lemma 4.6. *$G \oplus K_4 \in \langle Z_3 \rangle$ if and only if $G \in \langle Z_3 \rangle$.*

Proof. Let $G' = G \oplus K_4$. We shall use the notation in the definition of $G \oplus K_4$ and assume that $V(G') = V(G) \cup \{u, v\}$ and $V(K_4) = \{u, v, x, y\}$. In the proof below, K_4 denotes the given complete graph on 4 vertices in the definition of $G \oplus K_4$.

Assume that $G \in \langle Z_3 \rangle$. Let $X = \{xu, xv\}$. Then $G'_{[x,X]}$ has y as a cut vertex. Note that the graph $G'_{[x,X]}/G$ is spanned by a 3-cycle with one edge of this 3-cycle in a 2-cycle. By Proposition 3.2(C3) and by Lemma 3.3 (with $n = 2$), $G_{[x,X]'}/G \in \langle Z_3 \rangle$. By Proposition 3.2(C3) and by the assumption that $G \in \langle Z_3 \rangle$, $G'_{[x,X]} \in \langle Z_3 \rangle$. By Lemma 3.1(i) and by $G'_{[x,X]} \in \langle Z_3 \rangle$, $G' \in \langle Z_3 \rangle$.

Conversely, assume that $G' \in \langle Z_3 \rangle$. Let $b \in Z(G, Z_3)$. We shall show that there is a function $f \in F^*(G, Z_3)$ such that $\partial f = b$. Define $b' : V(G') \to Z_3$ by

$$b'(z) = \begin{cases} b(z) & \text{if } z \in V(G') - \{u, v\} \\ 0 & \text{if } z \in \{u, v\}. \end{cases}$$

Since $V(G') = V(G) \cup \{u, v\}$ and since $b'(u) = b'(v) = 0$, $b' \in Z(G', Z_3)$. Since $G' \in Z_3$, there is a function $f' \in F^*(G', Z_3)$ such that $\partial f' = b'$. Expressing $Z_3 = \{0, 1, -1\}$ and reversing the direction on $E(K_4)$, we may assume that $f'(e) = 1, \forall e \in E(K_4)$. Since u and v has degree 3 in G', we may assume that the three edges incident with u are all oriented away from u, and so the three edges incident with v are oriented into v. Let f be the restriction of f' on $E(G)$. Then it is easy to see that $\partial f = b$. Since $f' \in F^*(G'.Z_3)$, $f \in F^*(G, Z_3)$. It follows by definition that $G \in \langle Z_3 \rangle$. □

Remark. Lemma 4.6 provides an alternative proof for the fact that the graphs $G(m)$ in Example 4.3 are not Z_3-connected.

Theorem 4.7. *Let G be a 3-edge-connected chordal graph. Then one of the following holds*:

(i) *G is A-connected, for any Abelian group A with $|A| \geq 3$; or*
(ii) *G has a block isomorphic to a K_4; or*
(iii) *G has a subgraph G_1 such that $G_1 \notin \langle Z_3 \rangle$ and such that $G = G_1 \oplus K_4$.*

Proof. Let A be an Abelian group with $|A| \geq 3$ and let G be a counterexample such that $G \notin \langle A \rangle$ and $|E(G)|$ minimized.

Suppose first that G has a nontrivial subgraph H (a subgraph with at least one edge) such that $H \in \langle A \rangle$. Then by the definition of contraction, G/H, the graph obtained from G by contracting all edges in H, is also 3-edge-connected and chordal. Since $|E(H)| \geq 1$ and since $|E(G/H)| = |E(G)| - |E(H)|$, by the minimality of $|E(G)|$, $G/H \in \langle A \rangle$ By Proposition 3.2(C3), $G \in \langle A \rangle$, a contradiction. Therefore, G does not have any nontrivial subgraph H such that $H \in \langle A \rangle$.

Since G is chordal, G must have a 3-cycle. By Lemma 3.3, if $|A| \geq 4$, then 3-cycles are in $\langle A \rangle$. Since G must not have a nontrivial subgraph in $\langle A \rangle$, it must be the case that $A = Z_3$.

By Lemma 3.3 and since G must not have nontrivial subgraph in $\langle Z_3 \rangle$, we may assume that G has no loops and 2-cycles. Therefore by the minimality of G and by Proposition 3.6, we assume that G does not satisfy any of (i)–(iii) of Theorem 4.7, that G is simple, 2-connected and chordal, and that

$$\text{G does not contain a } W_4 \text{ as a subgraph.} \tag{2}$$

Let $v \in V(G)$ be an arbitrary vertex and let $N(v) = \{v_1, v_2, \ldots, v_m\}$. Since G is simple, $m = d(v) \geq 3$. Define $\overline{N}(v) = N(v) \cup \{v\}$ and let

$$G_v = G[N(v)] \quad \text{and} \quad \overline{G}_v = G[\overline{N}(v)]$$

be induced subgraphs of G. We first prove the following claims.

Claim 1. *G_v is connected.*

Suppose that G_v has more than one components. Since G is 2-connected, there is a shortest cycle C that contains v, one vertex in one component of G_v and a vertex in another component of G_v. Since G is a simple chordal graph and since C is a shortest cycle, $|E(C)| = 3$ and so there is an edge in G joining the two components of G_v. Since G_v is an induced subgraph, this edge should have been in G_v, a contradiction. This proves Claim 1.

Claim 2. *Either G_v is not 2-connected, or both $d(v) = 3$ and $\overline{G}_v$ is isomorphic to a K_4.*

Suppose that G_v is 2-connected. If $d(v) \geq 4$, then by Lemma 4.1, G_v must have a cycle of length at least 4. Since G is chordal, G_v must have a cycle of length 4. Therefore, $\overline{G}(v)$ has a W_4, contrary to (2). Hence we assume that $d(v) = 3$. Then G_v is a 2-connected graph with 3 vertices. Hence $\overline{G}_v$ is isomorphic to a K_4. This proves Claim 2.

Claim 3. *Every vertex $v \in V(G)$ such that G_v is connected but not 2-connected must be in a vertex cut of cardinality 2 in G.*

Assume that $v \in V(G)$ is a vertex such that G_v is not 2-connected and such that v is not in a vertex cut of cardinality 2. Let $N(v) = \{v_1, v_2, \ldots, v_m\}$ with $m = d(v) \geq 3$. Since G_v is connected but not 2-connected, G_v has a cut vertex v_m (say). It follows that there are nontrivial and connected subgraphs L_1, L_2 of G_v such that $G_v = L_1 \cup L_2$ and such that $V(L_1) \cap V(L_2) = \{v_m\}$. We may assume that v_1 is adjacent to v_m in L_1 and v_2 is adjacent to v_m in L_2.

Since $\{v, v_m\}$ is not a vertex cut of G, $G - \{v, v_m\}$ has a (v_1, v_2)-path. Let P denote a shortest (v_i, v_j)-path in $G - \{v, v_m\}$ such that $v_i \in V(L_1)$ and $v_j \in V(L_2)$. Then since G is chordal, P must be a path of length one, and so $v_i v_j \in E(G)$. It follows that v_m is not a cut vertex of G_v, a contradiction. This proves Claim 3.

If G is cubic, then Theorem 4.7(ii) follows from Lemma 4.5, contrary to the assumption that G is a counterexample. Hence we may assume $\Delta(G) \geq 4$.

A subgraph H of G is called a **2-block** if H is 2-connected and if $|A_G(H)| = 2$. A 2-block H of G is **minimal** if H is a 2-block and if H does not properly contain another 2-block of G. By Claim 1, Claim 2 and by the assumption that $\Delta(G) \geq 4$, G must have a vertex $v \in V(G)$ such that G_v is connected but not 2-connected. By Claim 3, G has a minimal 2-block H.

Since H is 2-connected and simple, $|V(H)| \geq 3$. Since $|A_G(H)| = 2$, $V(H) -$

$A_G(H) \neq \emptyset$. Since H is a minimal 2-block, every vertex in $V(H) - A_G(H)$ cannot be in a vertex cut of cardinality 2. By Claim 1 and Claim 3, G_v is 2-connected, $\forall v \in V(H) - A_G(H)$. By Claim 2, $\forall v \in V(H) - A_G(H)$, $d(v) = 3$ and $\overline{G}_v$ is isomorphic to a K_4. Thus if $|V(H)| \geq 5$, then one vertex in $\overline{G}_v$ will be a cut-vertex of G, contrary to the assumption that G is 2-connected. Hence $|V(H)| = 4$ and so H is isomorphic to a K_4. By Lemma 4.6, Theorem 4.7(iii) follows, contrary to the assumption that G is a counterexample.

This contradiction establishes the theorem. □

Note that $K_4 \notin \langle Z_3 \rangle$. By Proposition 3.2(C2) and by Lemma 4.5, graphs with a structure described in (ii) or (iii) of Theorem 4.7 cannot be in $\langle Z_3 \rangle$. Thus Theorem 4.7 can also be stated as the following characterization.

Corollary 4.8. *Let G be a* 3*-edge-connected chordal graph. The G is A-connected for every Abelian group A with* $|A| \geq 3$ *if and only if G does not have the structure described in Theorem* 4.7(ii) *and* (iii).

Proof of Theorem 4.2. When G is 4-edge-connected, neither Theorem 4.7(ii) nor Theorem 4.7(iii) will occur, and so by Corollary 4.8, $G \in \langle A \rangle, \forall A$ with $|A| \geq 3$. □

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