Extending a Partial Nowhere-Zero 4-Flow

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Abstract: In [J Combin Theory Ser B, 26 (1979), 205–216], Jaeger showed that every graph with 2 edge-disjoint spanning trees admits a nowhere-zero 4-flow. In [J Combin Theory Ser B, 56 (1992), 165–182], Jaeger et al. extended this result by showing that, if A is an abelian group with $|A| = 4$, then every graph with 2 edge-disjoint spanning trees is A-connected. As graphs with 2 edge-disjoint spanning trees are all collapsible, we in this note improve the latter result by showing that, if A is an abelian group with $|A| = 4$, then every collapsible graph is A-connected. This allows us to prove the following generalization of Jaeger's theorem: Let G be a graph with 2 edge-disjoint spanning trees and let M be an edge cut of G with $|M| \leq 4$. Then either any partial nowhere-zero 4-flow on M can be extended to a nowhere-zero 4-flow of the whole graph G, or G can be contracted to one of three configurations, including the wheel of 5 vertices, in which cases certain partial nowhere-zero 4-flows on M cannot be extended. Our results also improve a theorem of Catlin in [J Graph Theory, 13 (1989), 465–483]. J Graph Theory 30: 277–288, 1999

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1. INTRODUCTION

We consider finite graphs, which may contain loops or multiple edges. See [1] for undefined terminology and notations in graph theory and see [8] for those in

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algebra. The groups considered in this article are finite abelian (additive) groups. For a finite abelian group A, the additive identity of A will be denoted by 0 (zero) throughout this article. Let G and H be graphs. If H is a subgraph of G, then we write $H \subseteq G$; and if H and G are isomorphic, we write $H \cong G$. A graph G is nontrivial if G is loopless with $|E(G)| > 0$.

Let G be a digraph. For a vertex $v \in V(G)$, let

$$\begin{aligned} E_G^-(v) &= \{(u,v) \in E(G): u \in V(G)\}, \text{ and} \\ E_G^+(v) &= \{(v,u) \in E(G): u \in V(G)\}. \end{aligned}$$

The subscript G may be omitted when G is understood from the context. Let $E(v) = E^+(v) \cup E^-(v)$.

Let A be a nontrivial abelian group and let A^* denote the set of nonzero elements in A. Define

$$F(G,A) = \{f: E(G) \to A\} \text{ and } F^*(G,A) = \{f: E(G) \to A^*\}.$$

For each $f \in F(G,A)$, the boundary of f is a function ∂f: $V(G) \to A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. In this article, An empty sum has value zero. Let S be a nonempty set and let A be a group. Throughout this article, we shall adopt the following convenience: if $X \subseteq S$ and if f: $X \to A$ is a function, then we regard f as a function f: $S \to A$ such that $f(e) = 0$ for all $e \in S - X$. For any function f: $S \mapsto A$, the set $supp(f) = \{e \in S\colon f(e) \neq 0\}$ is called the support of f.

Let G be an undirected graph and A be an abelian group. Denote

$$Z(G,A) = \left\{ b: V(G) \to A \text{ such that } \sum_{v \in V(G)} b(v) = 0 \right\}.$$

A graph G is A-connected if G has an orientation G' such that, for every function $b \in Z(G,A)$, there is a function $f \in F^*(G',A)$ such that $b = \partial f$. For an abelian group A, let $\langle A \rangle$ denote the family of graphs that are A-connected. It is observed in [11] that $G \in \langle A \rangle$ is independent of the orientation of G.

An A-nowhere-zero-flow (abbreviated as A-NZF) in G is a function $f \in F^*(G,A)$ such that $\partial f = 0$. The nowhere-zero-flow problems were introduced by Tutte [16], and recently surveyed by Jaeger in [9]. Tutte in [16] showed that if A_1 and A_2 are two abelian groups with $|A_1| = |A_2|$, then a graph G has an A_1-NZF if and only if it has an A_2-NZF. Thus, an A-NZF is also called a k-NZF, where $k = |A|$. Following Jaeger [9], let F_k denote the collection of graphs that have k-NZF's.

The concept of A-connectivity was introduced by Jaeger et al. in [11], where A-NZF's were successfully generalized to A-connectivities. A concept similar to the group connectivity was independently introduced in [12], with a different motivation from [11].

For a graph G, let $\tau(G)$ denote the maximum number of edge-disjoint spanning trees contained in G. The following was proved by Jaeger.

Theorem 1.1 (Jaeger, [10]). *If $\tau(G) \geq 2$, then $G \in F_4$.* ∎

By a theorem of Tutte [16], for any abelian group A, if $|A| = k$, then $\langle A \rangle \subset F_k$. Thus, Theorem 1.2 below generalizes Theorem 1.1.

Theorem 1.2 (Jaeger, Linial, Payan, and Tarsi, [11]). *If $\tau(G) \geq 2$, and if A is an abelian group with $|A| = 4$, then $G \in \langle A \rangle$.* ∎

Let G be a digraph and let $M \subseteq E(G)$. We shall also use M to denote $G[M]$, the subgraph of G induced by the edge set M. Let $W \subset V(G)$ be a nonempty subset and let $\bar{W} = V(G) - W$. The oriented edges with tail in W and head in $\bar{W}$ is denoted by $[W, \bar{W}]$, and the union (viewed as a set of undirected edges) $[W, \bar{W}] \cup [\bar{W}, W]$ is called an edge cut. A function $f \in F^*(M, A)$ is called a partial A-NZF of G on M if, for any edge cut $[W, \bar{W}] \cup [\bar{W}, W] \subseteq M$ of G,

$$\sum_{e \in [W,\bar{W}]} f(e) - \sum_{e \in [\bar{W},W]} f(e) = 0. \tag{1}$$

Call an A-NZF $f' \in F^*(G, A)$ an extension of a partial A-NZF f on M if $f'(e) = f(e)$, for all $e \in M$. A partial A-NZF is extendable if it has an extension.

It has been known that the NZF problem is the dual problem of graph vertex colorings. Several classical coloring theorems, such as Grötzsch 3-coloring theorem of planar graphs and the 5-coloring theorem of planar graphs, have been generalized to the version of extending a given k-coloring of a subgraph to a k-coloring of the whole graph. (See, for example, [14] and [15], among others.) Therefore, it is natural to ask the following question:

(P1) When can a partial 4-NZF be extended to a 4-NZF of G?

Collapsible graphs are first introduced by Catlin [3]. Let G be a graph and let $O(G)$ denote the set of vertices of G that have odd degree in G. A graph G is collapsible if, for any subset $X \subseteq V(G)$ with $|X|$ even, G has a spanning connected subgraph Γ with $O(\Gamma) = X$. (Such a subgraph Γ is called an X-subgraph of G.) Note that K_1, the edgeless graph of order 1, is collapsible. Collapsible graphs have been found very useful in several applications. See Catlin's survey [4] and its update [7] for the literature of collapsible graphs.

Catlin showed that every graph with 2 edge-disjoint spanning trees is collapsible (Theorem 2.1 in Section 2). Thus, the following problem arises naturally:

(P2) If A is an abelian group with $|A| = 4$, is every collapsible graph A-connected? In this article, we shall investigate problems P1 and P2.

Even when $\tau(G) \geq 2$, a partial A-NZF may not be extendable. Let us consider the following examples.

Example 1.1. Let $C = v_1 v_2 v_3 \cdots v_n v_1$ denote an n-cycle. A wheel of $n+1$ vertices, denoted by W_n, has vertex set V and edge set E as follows:

$$V = V(C) \cup \{v_0\} \text{ and } E = E(C) \cup \{v_0 v_i : 1 \leq i \leq n\}.$$

The vertex v_0 is called the center of the wheel, and the edges in $\{v_0v_i\colon 1 \le i \le n\}$ are called the spokes of the wheel. Note that $\tau(W_n) = 2$, for $n \ge 2$. Orient W_4 so that v_0v_i is directed from v_0 to v_i, for each $1 \le i \le 4$; and orient the edges in $E(C)$ so that v_iv_{i+1} is directed from v_i to v_{i+1}, for $1 \le i \le 3$, and v_4v_1 from v_4 to v_1. Let $A = Z_4$, the cyclic group of order 4, let $f = 1$ be a constant function in $F^*(E(v_0), Z_4)$. Then $\partial f(v_0) = 0$, and it is easy to see that f satisfies (1). If f could be extended to an A-NZF $f_1 \in F^*(W_4, Z_4)$, then let $x = f(v_4v_1)$. It follows by $\partial f_1 = 0$ that $f_1(v_iv_{i+1}) = x + i$, for each i with $1 \le i \le 3$. However, since $|Z_4| = 4$, and since $\{x, x+1, x+2, x+3\} \subseteq Z_4$, one of $x, x+1, x+2, x+3$ must be zero, and so $f_1 \notin F^*(W_4, Z_4)$, a contradiction. Therefore, f cannot be extended.

Example 1.2. Let K_3' be the loopless graph spanned by a K_3 with two additional edges. Denote $V(K_3') = \{u, v, w\}$ and $E(K_3') = \{e_1, e_2, e_3, e_4, e_5\}$ such that e_1, e_2 are incident with u and v, e_3 and e_4 with u and w, and e_5 with v and w. Then $\tau(K_3') = 2$. Assume that K_3' is so oriented that u has indegree zero. Let A be a group of order 4 and let $a \in A$ be an element with order 2. Let $f \in F(K_3', A)$ be such that $f(e_1) = f(e_2) = f(e_3) = f(e_4) = a$. Since $\partial f = 0, f$ cannot be extended to a A-NZF of K_3'.

Example 1.3. Let $K_{3,3}'$ be the simple graph obtained from $K_{3,3}$ by adding a new edge e'. Then $\tau(K_{3,3}') = 2$. Let u and v denote the two vertices of degree 4 in $K_{3,3}'$, let v_1, v_2, and v_3 be the vertices that are adjacent to both u and v, and let w denote the sixth vertex.

Assume that $K_{3,3}'$ is so oriented that u has indegree zero. Let A be a group of order 4 and let $a \in A$ be an element with order 2. Let $f \in F(E(u), A)$ be such that $f(e) = a$ for each edge e incident with u. Assume that e_1, e_2, e_3 are the edges in $K_{3,3}'$ joining v with $v_i, 1 \le i \le 3$, and that e_1, e_2, and e_3 are oriented with tail v. If f can be extended to an A-NZF f_1, then since $\partial f_1(v) = 0, f_1(e_1) + f_1(e_2)$ $+ f_1(e_3) = f(e') = a$. It follows from the assumption that $|A| = 4$ that one of $f_1(e_1), f_1(e_2)$ and $f_1(e_3)$ is a (say $(f_1(e_1) = a)$. Let e_1' denote the edge joining v_1 and u. Then $f_1(e_1) + f(e_1') = a + a = 0$, and so in order for $\partial f_1(v_1) = 0, f_1$ must take 0 as a value at the third edge incident with v_1, contrary to the assumption that f_1 is an A-NZF. Therefore, f cannot be extended.

It turns out that these are exactly the forbidden contraction configurations. Let G be a graph and let $R \subseteq E(G)$ be an edge subset. The contraction G/R is the graph obtained from G by identifying the ends of each edge in R and by deleting the resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$, and we define $G/\emptyset = G$.

The following result is obtained.

Theorem 1.3. *If $\tau(G) \ge 2$ and if $M \subseteq E(G)$ is an edge cut of G with $|M|$ at most 4, then, for any partial 4-NZF f on M, exactly one of the following holds*:

(*i*) *f can be extended to a 4-NZF of G*;

(*ii*) *G can be contracted to a wheel W_4 in such a way that the spoke edges of this wheel are exactly the edges in M*;

(*iii*) *G can be contracted to a K_3' in such a way that the edges incident with the vertex of degree* 4 *in this K_3' are exactly the edges in M;*

(*iv*) *G can be contracted to a $K_{3,3}'$ in such a way that the edges incident with a vertex of degree* 4 *in this $K_{3,3}'$ are exactly the edges in M.* ∎

When G is 4-edge-connected, $\tau(G) \geq 2$ and none of (ii), (iii), and (iv) of Theorem 1.3 will occur. Therefore, we obtain the following corollary.

Corollary 1.1. *If G is* 4*-edge-connected and if $M \subseteq E(G)$ is an edge cut of G with $|M|$ at most* 4*, then, for any partial* 4*-NZF f on M, f can be extended to a* 4*-NZF of G.*

In [5], Catlin investigated the following collection of graphs:

$$F_4^o = \{H\colon \text{for any graph } G \text{ with } H \subseteq G,\ \text{if } G/H \in F_4 \text{ then } G \in F_4\}.$$

Denoting the collection of collapsible graphs by $\mathcal{CL}$, Catlin in [5] proved the following.

Theorem 1.4 (Catlin [5]). $\mathcal{CL} \subseteq F_4^o$. ∎

Catlin in [5] showed that the 4-cycle is in $F_4^o - \mathcal{CL}$, and so the containment in Theorem 1.4 is indeed strict. One can routinely verify that, if A is an abelian group with $|A| = 4$, then $\langle A \rangle \subset F_4^o$ (Lemma 2.1 in Section 2). Also, the 4-cycle is not in $\langle A \rangle$, when $|A| = 4$. Therefore, Theorem 1.5 below shows that, if A is an abelian group with $|A| = 4$, then $\mathcal{CL} \subseteq \langle A \rangle \subset F_4^o$, improving both Theorem 1.2 and Theorem 1.4.

Theorem 1.5. *Let G be a collapsible graph and let A be an abelian group with $|A| = 4$. Then $G \in \langle A \rangle$.* ∎

In Section 2, we shall prove Theorem 1.3, assuming Theorem 1.5; and in Section 3, Theorem 1.5 will be proved.

2. PROOF OF THEOREM 1.3

We start with the observations below on graphs in $\langle A \rangle$.

Proposition 2.1 ([13]). *Let A be an abelian group with $|A| \geq 3$. Then each of the following holds.*

(*C*1) $K_1 \in \langle A \rangle$;

(*C*2) *if $G \in \langle A \rangle$ and if $e \in E(G)$, then $G/e \in \langle A \rangle$; and*

(*C*3) *if $H \in \langle A \rangle$ is a subgraph of G and if $G/H \in \langle A \rangle$, then $G \in \langle A \rangle$.* ∎

(Collections of graphs satisfying (C1)–(C3) are called complete families of connected graphs in [2] by Catlin. See [2] for more on this topic.)

In fact, in [13], we proved a result (Lemma 2.1 below) slightly more general than (C3) in Proposition 2.1. Let H be a connected subgraph of G, A be an abelian group, and let $b \in Z(G, A)$. Let v_H denote the vertex in G/H onto which H is contracted. Define $b_H \in Z(G/H, A)$ by

$$b_H(z) = \begin{cases} b(z) & \text{if } z \in V(G/H) - \{v_H\} = V(G) - V(H) \\ \sum_{v \in V(H)} b(v) & \text{if } z = v_H. \end{cases}$$

Lemma 2.1 ([13]). *Let A be an abelian group and let H be a subgraph of G such that $H \in \langle A \rangle$. For any $b \in Z(G, A)$, and for any $f_H \in F^*(G/H, A)$ with $\partial f_H = b_H$, there is an $f \in F^*(G, A)$ such that $\partial f = b$ and $f(e) = f_H(e)$, for any $e \in E(G) - E(G[E(H)])$.* ∎

In [3], Catlin showed that every graph G has a unique collection of disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$, and the graph $G' = G/(\cup_{i=1}^{c} E(H_i))$ is called the reduction of G. A graph G is reduced if G is the reduction of itself. The edge arboricity of a graph G, denoted by $a(G)$, is the smallest number of forests of G whose union is G. Let $F(G)$ denote the minimum number of edges that must be added to the graph G so that the resulting graph has 2 edge-disjoint spanning trees. The following were proved by Catlin and Catlin et al.

Theorem 2.1. *Let G be a connected graph. Each of the following holds.*

(*i*) (*Catlin, Theorem* 2 *in* [3]) *If $\tau(G) \geq 2$, then G is collapsible.*

(*ii*) (*Catlin, Theorem* 8 *in* [3]) *If G is reduced, then $a(G) \leq 2$.*

(*iii*) (*Catlin, Theorem* 7 *in* [3]) *If $F(G) \leq 1$, then either G is collapsible or the reduction of G is K_2.*

(*iv*) (*Catlin, Han and Lai*, [6]) *If $F(G) \leq 2$, the either G is collapsible, or the reduction of G is a K_2 or a $K_{2,t}$, for some integer $t \geq 1$.*

Lemma 2.2. *Let C_n denote the cycle of length n and let A be an abelian group with $|A| \geq 4$. Let w_1, w_2 be the two nonadjacent vertices of degree t in a $K_{2,t}$, where $t \geq 1$. Let $e_0 = w_1 w_2$ be an edge not in $K_{2,t}$. Each of the following holds.*

(*i*) ([11]) $C_2, C_3 \in \langle A \rangle$.

(*ii*) $K_{2,t} + e_0 \in \langle A \rangle$.

(*iii*) *Let $t \geq 2$ and let v be a vertex of degree* 2 *in $K_{2,t}$. If $b \in Z(K_{2,t}, A)$ such that $v \notin supp(b)$, then there is a function $f \in F^*(K_{2,t}, A)$ such that $\partial f = b$.*

(*iv*) *Let $v \in V(K_{2,4})$ be a vertex of degree* 4. *If $b \in Z(K_{2,4}, A)$ such that $b(v) = 0$, then there is a function $f \in F^*(K_{2,4}, A)$ such that $\partial f = b$.*

(*v*) *Let u, v be the two vertices of degree* 3 *in $K_{2,3}$. If $b \in Z(K_{2,3}, A)$ such that $b(u) = b(v) = 0$, then then there is a function $f \in F^*(K_{2,3}, A)$ such that $\partial f = b$.*

Proof. Lemma 2.2(ii) follows from Lemma 2.2(i) and Proposition 2.1(C3).

For Lemma 2.2(iii), let v and b satisfy the hypothesis of Lemma 2.2(iii), and let e', e'' be the edges incident with v in $K_{2,t}$. Then $K_{2,t}/\{e'\} \cong K_{2,t-1} + e_0$. Regard $V(K_{2,t}/\{e'\}) = V(K_{2,t} - v)$, and let b' denote the restriction of b to $V(K_{2,t} - v)$. Since $v \notin supp(b), b' \in Z(K_{2,t}/\{e'\}, A)$. By Lemma 2.2(ii), there is a function $f' \in F^*(K_{2,t}/\{e'\}, A)$ such that $\partial f' = b'$. Define $f \in F^*(K_{2,t}, A)$ by $f(e) = f'(e)$, if $e \neq e'$ and $f(e') = f'(e'')$. Then $\partial f = b$, as desired.

For Lemma 2.2(iv), let v and b satisfy the hypothesis of Lemma 2.2(iv), and let v_1, v_2, v_3, v_4 be the 4 vertices of degree 2 in $K_{2,4}$. Obtain a new graph G from

$K_{2,4} - v$ by adding two new edges $e' = v_1v_2$ and $e'' = v_3v_4$. Since $b(v) = 0, b \in Z(G, A)$. Since $|A| \geq 4$ and by Lemma 2.2(i), $G \in \langle A \rangle$. Thus, there is a function $f' \in F^*(G', A)$ such that $\partial f' = b$. Assume that e' is oriented from v_1 to v_2, and e'' is oriented from v_3 to v_4. And the edges in $E(K_{2,4}) - E(G)$ are oriented so that these edges are oriented from v_1 to v, v to v_2, v_3 to v, and v to v_4. Let $f \in F^*(K_{2,4}, A)$ be defined as follows:

$$f(e) = \begin{cases} f'(e') & \text{if } e \in \{v_1v, vv_2\} \\ f'(e'') & \text{if } e \in \{v_3v, vv_4\} \\ f'(e) & \text{otherwise.} \end{cases}$$

Then $\partial f = b$, as desired.

For Lemma 2.2(v), let u, v, and b satisfy the hypothesis of Lemma 2.2(v). Let v_1, v_2, v_3 be the vertices of degree 2 in $K_{2,3}$ and denote $a_1 = b(v_1), a_2 = b(v_2)$ and $a_3 = b(v_3)$. By Lemma 2.2(iii), we may assume that $a_i \neq 0, 1 \leq i \leq 3$. The edges of $K_{2,3}$ are oriented so that both u and v have indegree zero. Define $f \in F^*(K_{2,3}, A)$ as follows: $f(uv_1) = f(vv_2) = a_3, f(uv_3) = f(vv_1) = a_2$, and $f(vv_3) = f(uv_2) = a_1$. Then, since $b \in Z(K_{2,3}, A), a_1 + a_2 + a_3 = 0$, and so $\partial f = b$. ∎

Lemma 2.3. *if $\tau(G) \geq 2$ and if $v \in V(G)$ is a vertex of degree at most 4 in G, then for any partial 4-NZF f on $E(v)$, exactly one of the following holds*:

(*i*) *f can be extended to a 4-NZF of G*;

(*ii*) *G can be contracted to a wheel W_4 in such a way that the spoke edges of this wheel are exactly the edges in $E(v)$*;

(*iii*) *G can be contracted to a K_3' in such a way that the edges incident with the vertex of degree 4 in this K_3' are exactly the edges in $E(v)$*;

(*iv*) *G can be contracted to a $K_{3,3}'$ in such a way that the edges incident with a vertex of degree 4 in this $K_{3,3}'$ are exactly the edges in $E(v)$.*

Proof. Let A be an abelian group of order 4 and let G be a counterexample to Theorem 1.3 with as few vertices as possible. Let $v \in V(G)$ with $|E(v)| \leq 4$. Without loss of generality, we may assume that edges in $E(v)$ are all directed out from v. Let $v_1, \ldots, v_d$ be the vertices in G that are adjacent to v. Then $d \leq |E(v)| \leq 4$. Since G is a counterexample, there is a partial A-NZF $f \in F^*(E(v), A)$ such that neither (i) nor (ii) of Lemma 2.3 holds. Let b: $V(G - v) \mapsto A$ by $b(u) = 0$ if $u \in V(G) - \{v_1, \ldots, v_d\}$, and $b(v_i) = \sum_j f(e_{ij})$ for each i with $1 \leq i \leq d$, where the e_{ij}'s are the edges in $E(v)$ that join v and v_i. By (1), $0 = \partial f(v) = -\sum_{i=1}^{d} \sum_j f(e_{ij})$, and so $b \in Z(G - v, A)$.

Case 1. *$G - v$ is disconnected*.

Since $\tau(G) \geq 2, G$ is 2-edge-connected, and so each of the following must hold:

(A) $G - v$ has two components G_1 and G_2;

(B) $|E(v)| = 4$, and there are exactly 2 edges in $E(v)$ joining v to G_i, for each i with $1 \leq i \leq 2$; and

(C) for $i = 1, 2, G_i$ is either a K_1, or $\tau(G_i) \geq 2$.

If both $G_1 = G_2 = K_1$, then $E(G) = E(v)$ and so $f \in F^*(G, A)$ already. Assume that some $G_i \neq K_1$. By (B), $\tau(G_i) \geq 2$. By Theorem 2.1(i) and Theorem 1.5, $G_i \in \langle A \rangle$. Assume that $e_{i_1}, e_{i_2} \in E(v)$ are the two edges joining v to $v_{i_1}, v_{i_2} \in V(G_i)$, respectively. (Note that $v_{i_1} = v_{i_2}$ is possible). Since f is a partial A-NZF, $f(e_{i_1}) + f(e_{i_2}) = 0$. Let b_i: $V(G_i) \mapsto A$ by $b_i(v_{i_1}) = f(e_{i_1}), b_i(v_{i_2}) = f(e_{i_2})$ if $v_{i_1} \neq v_{i_2}, b_i(v_{i_1}) = f(e_{i_1}) + f(e_{i_2})$ if $v_{i_1} = v_{i_2}$, and $b_i(z) = 0$ for each $z \in V(G) - \{v_{i_1}, v_{i_2}\}$.

Then by the assumption that f is a partial A-NZF, $b_i \in Z(G_i, A)$. Since $G_i \in \langle A \rangle$, there is a function $f_i \in F^*(G_i, A)$ such that $\partial f_i = b_i$. Hence, $f + f_1 + f_2$ is a desired extension of f.

Case 2. *$G - v$ is connected, but not reduced.*

Then $G - v$ has a nontrivial collapsible subgraph H. Since $H \subseteq G - v, H \subset G$. Since $\tau(G) \geq 2, \tau(G/H) \geq 2$ also. Since H is nontrivial, $|E(G)| > |E(G/H)|$. By the minimality of G, either G/H can be contracted to a W_4 with v being the center of this W_4, whence G can be contracted to a W_4 with v being the center of the W_4; or the partial A-NZF f may be extended to an A-NZF of G/H, whence by Theorem 1.5 and by Lemma 2.1, f can be extended to an A-NZF of G. In either case, a contradiction obtains, since G is supposed to be a counterexample.

Case 3. *$G - v$ is connected, nontrivial, and reduced.*

Since $\tau(G) \geq 2$ and since $d \leq 4, F(G - v) \leq 2$. By Theorem 2.1(iv) and by the assumption of Case 3, $G - v$ is a K_2 or a $K_{2,t}$, for some integer $t \geq 1$.

If $G - v = K_2$, then by $\tau(G) \geq 2$, there must be at least three edges in $E(v)$ joining v to the two ends of the only edge e (say) in this K_2.

If there are exactly two edges in $E(v)$ joining v to each end of e, then Lemma 2.3(ii) holds. Therefore, we assume that there is only one edge in $E(v)$ joining v to one end of e and the other edges in $E(v)$ joining v with the other end of e. Since f is a partial A-NZF, it is trivial to see that f can be extended to an A-NZF of G. The case when $G - v = K_{2,1}$ is similar. Hence, we assume that $G - v = K_{2,t}$ for some $t \geq 2$.

If $t \geq 5$, then, since $|E(v)| \leq 4$, there is a vertex w of degree 2 in $G - v$ such that $b(w) = 0$. By Lemma 2.2(iii), there is a function $f' \in F^*(G - v, A)$ such that $\partial f' = b$. It follows that $f + f'$ is a desired extension of f.

If $t = 4$, then, since $|E(v)| \leq 4$, either there is a vertex w of degree 2 in $G - v$ such that $b(w) = 0$; or there is a vertex w' of degree 4 in $G - v$ such that $b(w') = 0$. By Lemma 2.2(iii) or Lemma 2.2(iv), there is always a function $f' \in F^*(G - v, A)$ such that $\partial f' = b$. It follows that $f + f'$ is a desired extension of f.

Assume that $t = 3$. Let u and w denote the vertices of degree 3 and v_1, v_2, and v_3 denote the vertices of degree 2 in $K_{2,3}$. By Lemma 2.2(iii) and (v), we may assume that v_1, v_2, v_3, u are all adjacent to v; but by $|E(v)| \leq 4, w$ must not be adjacent to v. Thus, G is contractible to a $K'_{3,3}$, whence Lemma 2.3(iv) holds.

Therefore, we assume that $t = 2$. Since $\tau(G) \geq 2$, we must have $|E(v)| = 4$. If G has a vertex z (say) of degree 2, then $b(z) = 0$, and so by Lemma 2.2(iii), there is a function $f' \in F^*(G - v, A)$ such that $\partial f' = b$, which means $f + f'$ is an A-NZF of G extending f. Therefore, $G = W_4$, whence Lemma 2.3(ii) holds. This contradicts the assumption that G is a counterexample. ∎

Proof of Theorem 1.3. Let M be an edge-cut of G with $|M| \leq 4$, and let f be a partial 4-NZF on M. We assume that (ii)–(iv) of Theorem 1.3 do not hold to prove that f can be extended to a 4-NZF of G. Let G_1 and G_2 be two disjoint subgraphs of G such that $G - M = G_1 \cup G_2$. If one of G_1 and G_2 is a K_1, then Theorem 1.3 reduces to Lemma 2.3, and is proved. Hence, we assume that both G_1 and G_2 are nontrivial.

For $i = 1, 2$, let N_i denote the graph obtained from G by identifying all the vertices in $V(G_{3-i})$ into a single vertex v_i. Note that $E(v_i) = M$. By Lemma 2.3 and by the assumption that (ii)–(iv) of Theorem 1.3 do not hold, f can be extended to a 4-NZF f_i on N_i, for $i = 1, 2$. Since $f_1(e) = f_2(e) = f(e)$ for any $e \in M$, and since $supp(f_1) \cap supp(f_2) = M$, the function $f_1 + f_2 - f$ is a 4-NZF of G, which extends f. ∎

3. PROOF OF THEOREM 1.5

Throughout this section (with the exception of Lemma 3.1), A denotes an abelian group of order 4 and A' denotes a subgroup of A with $|A'| = 2$. We start with some easy observations, stated in Lemma 3.1 and Lemma 3.2 below, whose proofs are outlined or omitted.

Lemma 3.1. *Let H be a nontrivial connected graph and let A be an abelian group with $|A| \geq 2$. Then, for any function $f \in F^*(G, A), \sum_{v \in V(H)} \partial f(v) = 0$.*

Proof. In the sum

$$\sum_{v \in V(H)} \partial f(v) = \sum_{v \in V(H)} \left\{ \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) \right\},$$

for each edge $e \in E(G), f(e)$ appears exactly once with a plus sign and exactly once with a negative sign and so this sum is zero. ∎

Lemma 3.2. *Let A be an abelian group of order* 4 *and let A' be a subgroup of A with $|A'| = 2$. Let G be a nontrivial connected graph. Then each of the following holds*:

(i) *For any $a, a' \in A - A'$, both $a + a'$ and $a - a'$ are in A'.*
(ii) *If $b \in Z(G, A)$, then $|b^{-1}(A - A')|$ is even.*

Lemma 3.3. *Let H be a nontrivial connected graph and let $b_1 \in Z(H, A)$ be such that $b_1^{-1}(A - A') = O(H)$. Then there is a function $f_1 \in F^*(H, A)$ such that $\partial f_1 = b_1$.*

Proof. The lemma is trivial if $|E(H)| = 1$. We shall argue by contradiction and assume that H is a counterexample with $|E(H)|$ minimized. Let $X = b_1^{-1}(A - A')$.

Case 1. *H has a cut vertex v (say).*

Then H has nontrivial connected subgraphs H_1 and H_2 such that $E(H_1) \cap E(H_2) = \emptyset, E(H_1) \cup E(H_2) = E(H)$ and $V(H_1) \cap V(H_2) = \{v\}$. For each $i = 1, 2$, define c_i: $V(H_i) \mapsto A$ as follows:

$$c_i(z) = \begin{cases} b_1(z) & \text{if } z \in V(H_i) - \{v\} \\ \sum_{w \in V(H_{3-i})} b_1(w) & \text{if } z = v. \end{cases} \tag{2}$$

Since $b_1 \in Z(G, A), \sum_{z \in V(H_i)} c_i(z) = \sum_{z \in V(H)} b_1(z) = 0$, and so $c_i \in Z(H_i, A)$. Let $d_i(v)$ denote the degree of v in H_i, and assume, without loss of generality, that

$$\text{if } d(v) = d_1(v) + d_2(v) \text{ is odd, (that is } v \in X), \text{ then } d_1(v) \text{ is odd.} \tag{3}$$

Note that if $v \notin X = O(H)$, then $d_1(v)$ and $d_2(v)$ must have the same parity.

If $v \in X$, then by (3), $O(H_1) = X \cap V(H_1)$. Since $c_1 \in Z(H_1, A), |c^{-1}(A - A')|$ is even by Lemma 3.2(ii). It follows that $v \in c_1^{-1}(A - A') = X \cap V(H_1) = O(H_1)$. By the minimality of H, there is a function $g_1 \in F^*(H_1, A)$ with $\partial g_1 = c_1$. Note that by (3), $d_2(v)$ is even, and so by Lemma 3.2(ii) and by $X \cap V(H_1) = O(H_1), c_2^{-1}(A - A') = (X - \{v\}) \cap V(H_2) = O(H_2)$. By the minimality of H, there is a function $g_1 \in F^*(H_1, A)$ with $\partial g_1 = c_1$. Note that by (3), $d_2(v)$ is even, and so by Lemma 3.2(ii) and by $X \cap V(H_1) = O(H_1), c_2^{-1}(A - A') = (X - \{v\}) \cap V(H_2) = O(H_2)$. By the minimality of H, there is a function $g_2 \in F^*(H_2, A)$ with $\partial g_2 = c_2$. Let $f_1 = g_1 + g_2$. Clearly $\partial f_1(z) = b_1(z)$ if $z \neq v$. At v,

$$\begin{aligned} \partial f_1(v) &= c_1(v) + c_2(v) = \sum_{w \in V(H_1)} b_1(w) + \sum_{w \in V(H_2)} b_1(w) \\ &= b_1(v) + \sum_{w \in V(H)} b_1(w) = b_1(v). \end{aligned} \tag{4}$$

Therefore, $\partial f_1 = b_1$, as desired.

Assume then both $d_1(v)$ and $d_2(v)$ are even. Then for each $i \in \{1, 2\}, O(H_i) = O(H) \cap V(H_i)$; and so by Lemma 3.2(ii), $c_i(v) \in A'$. Therefore, $c_i^{-1}(A - A') = X \cap V(H_i) = O(H_i)$. By the minimality of H, there is a function $g_i \in F^*(H_i, A)$ with $\partial g_i = c_i$, for each $i \in \{1, 2\}$. Define $f_1 = g_1 + g_2$ as before. Then (4) holds at v and so $\partial f_1 = b_1$.

Now assume that both $d_1(v)$ and $d_2(v)$ are odd. Then, since $v \notin O(H)$, for both $i = 1$ and $i = 2, |O(H) \cap V(H_i)|$ is odd, and so by Lemma 3.2(ii), $c_i(v) \in A - A'$. Thus, $c_i^{-1}(A - A') = (X \cap V(H_i)) \cup \{v\} = O(H_i)$. Therefore, by the minimality of H, there is a function $g_i \in F^*(H_i, A)$ with $\partial g_i = c_i$, for each $i \in \{1, 2\}$. Define $f_1 = g_1 + g_2$ as before. Then (4) holds at v and so $\partial f_1 = b_1$.

This precludes Case 1.

Case 2. *H has no cut vertices.*

Pick $e = uv \in E(H)$ and assume that e is directed from u to v. Choose an element $a \in A - A'$ in the following way: if $u, v \in V(H) - O(H)$ or if $u, v \in O(H)$, then a is any element in $A - A'$, and if $u \in O(H)$ and $v \in V(H) - O(H)$, then $a = b_1(u)$. Note that when $u, v \in O(H)$, both $b_1(u)$ and $b_1(v)$ are in $A - A'$, and so by Lemma 3.2(i), $b_1(u) - a, b_1(v) + a \in A'$. In the same way, if $u, v \in V(H) - O(H)$, both $b_1(u)$ and $b_1(v)$ are in A', and so $b_1(u) - a, b_1(v) + a \in A - A'$. Define a function f_e: $\{e\} \mapsto \{a\}$, and let $b_2 = b_1 - \partial f_e \in Z(H - e, A)$. Note that $O(H - e) = (O(H) \cup \{u, v\}) - (O(H) \cap \{u, v\}) = b_2^{-1}(A - A')$, and so, by the minimality of H, there is a function $f_2 \in F^*(H - e, A)$ such that $\partial f_2 = b_2$. Let $f_1 = f_e + f_2$. Then $\partial f_1 = \partial f_e + \partial f_2 = \partial f_e + b_2 = b_1$. This precludes Case 2.

Therefore, in any case, a function $f_1 \in F^*(H, A)$ can be found with $\partial f_1 = b_1$, contrary to the assumption that H is a counterexample. This proves the lemma. ∎

Proof of Theorem 1.5. Let G be a collapsible graph, A be an abelian group of order 4 and let A' be a subgroup of A with $|A'| = 2$. Let $b \in Z(G, A)$. We shall show that there is a function $f \in F^*(G, A)$ such that $\partial f = b$. Let $X = b^{-1}(A - A')$. Note that as $|A| = 4$ and $|A'| = 2, |X|$ is even.

Since G is collapsible, G has a spanning connected subgraph H with $O(H) = X$. Let $f_2 \in F^*(E(G) - E(H), A')$ and let $b_2 = \partial f_2$. By Lemma 3.1, $b_1 = b - b_2 \in Z(G, A) = Z(H, A)$. Since A' is a subgroup, and since $b_2 \in Z(G, A'), b_1^{-1}(A - A') = b^{-1}(A - A')$. By Lemma 3.3, there is a function $f_1 \in F^*(H, A)$ such that $\partial f_1 = b_1$. Let $f = f_1 + f_2$. Since $supp(f_1) \cap supp(f_2) = E(H) \cap (E(G) - E(H)) = \emptyset$ and $supp(f_1) \cup supp(f_2) = E(G), f \in F^*(G, A)$. Also $\partial f = \partial f_1 + \partial f_2 = b_1 + b_2 = b$, and so the theorem is established. ∎

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References

[1] J. A. Bondy and U. S. R. Murty, Graph theory with applications. American Elsevier, New York, 1976.

[2] P. A. Catlin, The reduction of graph family closed under contraction, Discrete Math 160 (1996), 67–80.

[3] P. A. Catlin, A reduction method to find spanning eulerian subgraphs, J Graph Theory 12 (1988), 29–45.

[4] P. A. Catlin, Supereulerian graphs: a survey, J Graph Theory 16 (1992), 177–196.

[5] P. A. Catlin, Double cycle covers and the Petersen graph, J Graph Theory 13 (1989), 465–483.

[6] P. A. Catlin, Z. Y. Han, and H.–J. Lai, Graphs without spanning closed trails, Discrete Math 160 (1996), 81–91.

[7] Z.–H. Chen and H.–J. Lai, ‘‘Reduction techniques for supereulerian graphs and related topics–-an update,’’ Combinatorics and graph theory ’95, Ku Tung–Hsin (Editor), World Scientific, Singapore, 1996, pp. 53–69.

[8] T. W. Hungerford, Algebra, Springer–Verlag, New York, 1974.

[9] F. Jaeger, ‘‘Nowhere-zero flow problems,’’ Selected topics in graph theory, L. Beineke and R. Wilson (Editors), Vol. 3, Academic, London/New York, 1988, pp. 91–95.

[10] F. Jaeger, Flows and generalized coloring theorem in graphs, J Combin Theory Ser B 26 (1979), 205–216.

[11] F. Jaeger, N. Linial, C. Payan, and M. Tarsi, Group connectivity of graphs–-a nonhomogeneous analogue of nowhere-zero flow properties, J Combin Theory Ser B 56 (1992), 165–182.

[12] H.–J. Lai, ‘‘Reduction towards collapsibility,’’ Graph theory, combinatorics, and applications, Y. Alavi and A. Schwenk (Editors), John Wiley, New York, 1995, pp. 661–670.

[13] H.–J. Lai, Group connectivity of 3-edge-connected chordal graphs, Graphs and Combinatorics, accepted.

[14] C. Thomassen, Grötzsch’s 3-color theorem and its counterparts for the torus and the projective plane, J Combin Theory Ser B 62 (1994), 268–279.

[15] C. Thomassen, 5-coloring graphs on the torus, J Combin Theory Ser B 62 (1994), 11–33.

[16] W. T. Tutte, A contribution to the theory of chromatic polynomials, Canad J Math 6 (1954), 80–91.

[17] W. T. Tutte, On the imbedding of linear graph into surfaces, Proc London Math Soc Ser 2 51 (1949), 464–483.

[18] C. Q. Zhang, Integer flows and cycle covers of graphs, Marcel Dekker, New York, 1997.