Group Coloring and Group Connectivity of Graphs

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Abstract

Let G be a graph with a fixed orientation and let A be an Abelian group. Let $F(G,A)$ denote the set of all functions f: $E(G) \to A$. The graph G is **A-colorable** if for any function $f \in F(G,A)$, there is a function c: $V(G) \to A$ such that for every directed $e = uv \in E(G)$, $c(u) - c(v) \neq f(e)$. It is shown that whether G is A-colorable is independent of the choice of the orientation. The **group chromatic number** of a graph G is defined to be the minimum m for which G is A-colorable for any Abelian group A of order $\geq m$ under a given orientation D, and is denoted by $\chi_1(G)$. In this note, we summarize the results in a recent completed paper and provide a proof for the duality between group coloring and group connectivity.

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1. Introduction

Graphs in this note are finite and loopless, unless otherwise stated. We use Bondy and Murty [2] as a reference for undefined terms. Thus $\chi(G)$ denotes the chromatic number of a graph G and $\Delta(G)$ denotes the maximum degree of graph G. We use $H \subseteq G$ to denote the fact that H is a subgraph of G.

Let $G = (V, E)$ be a graph and A a non-trivial Abelian group, and let $F(G, A)$ denote the set of all functions f: $E(G) \to A$. Denote by D an orientation of $E(G)$. An oriented edge uv of G (assumed to be directed from u to v) is called an arc uv. The graph G under the orientation D is sometimes denoted by $D(G)$.

For $f \in F(G, A)$, an **A-coloring** of G under the orientation D is a function c: $V(G) \to A$ such that for every arc $e = uv \in E(G)$ $c(u) - c(v) \neq f(e)$; G is **A-colorable** under the orientation D if and only if for every $f \in F(G, A)$ there exists an A-coloring. The **group chromatic number** of a graph G is defined to be the minimum m for which G is A-colorable for any group A of order at least m under the orientation D, and is denoted by $\chi_1(G)$.

We first indicate that the group colorability is independent of the choice of the orientation of the graph.

Proposition 1.1 Let D be an orientation of $E(G)$ and E_0 be a subset of $E(G)$. Let D' be the orientation of $E(G)$ obtained from D by reversing the direction of every arc in E_0. Assume that A is a non-trivial Abelian group. If G is A-colorable under the orientation D, then G is also A-colorable under the orientation D'.

Proof Let $f' \in F(G, A)$. We consider the ordered pair (D, f), where f is defined as follows:

$$f(e) = \begin{cases} f'(e), & if\ e \notin E_0 \\ -f'(e), & if\ e \in E_0. \end{cases} \quad (1)$$

Since G is A-colorable under the orientation D, there exists a function c: $V(G) \to A$ such that for every arc $e = xy \in E[D(G)]$, $c(x) - c(y) \neq f(e)$. If $e \notin E_0$, then $e \in E[D'(G)]$ and $c(x) - c(y) \neq f(e) = f'(e)$; if $e \in E_0$, then $yx \in E[D'(G)]$ and $c(x) - c(y) \neq f(e)$, namely, $c(y) - c(x) \neq -f(e) = f'(e)$. Hence, G is A-colorable under the orientation D'. □

In this note, we summarize the results in a recent completed paper [9] (the proofs of these results can be found in [9]). We also provide a proof for the duality between group coloring and group connectivity, which will be introduced later in this note.

2. Some Results on Group Colorability

In [8], Jaeger *et al* proved the following result.

Theorem 2.1 (Jeager, Linial, Payan, and Tarsi [8], Proposition 4.2). If G is a simple planar graph, then $\chi_1(G) \leq 6$.

This has been improved by Lai and Zhang.

Theorem 2.2 If G is a simple planar graph, then $\chi_1(G) \leq 5$.

It is well known that $\chi(G) = 2$ if and only if G is a bipartite graph. But this does not hold for group colorability, as shown by Theorem 2.3 bellow.

Theorem 2.3 Let G be a graph. Then $\mu_1(G) = 2$ if and only if G is a forest.

Having Theorem 2.3 in mind, it would be natural to see that the analogue to Brooks' Theorem would be somewhat different.

Theorem 2.4 Let G be a connected simple graph. Then

$$\chi_1(G) \leq \Delta(G) + 1$$

with equality if and only if either $\Delta(G) = 2$ and G is a cycle; or $\Delta(G) \geq 3$ and G is a complete graph.

A natural question is how the chromatic number $\chi(G)$ and the group chromatic number are related. Right from the definition, one can conclude that $\chi(G) \leq \chi_1(G)$. The next result indicates the gap between the two can be arbitrarily large.

Theorem 2.5 For any positive integers m and k, there exists a graph G such that $\chi(G) = m$ and $\chi_1(G) = m + k$.

The following extends the well known Nordhaus and Gaddum Theorem.

Theorem 2.6 Let G be a simple graph of order n and let G^c denote the complement of G. Then

$$2\sqrt{n} \leq \chi(G) + \chi(G^c) \leq \chi_1(G) + \chi_1(G) \leq n + 1$$

and

$$n \leq \chi(G)\chi(G^c) \leq \chi_1(G)\chi_1(G) \leq \left(\frac{n+1}{2}\right)^2.$$

All the bounds in these inequalities can be reached.

3. The Duality Between Group Colorability and Group Connectivity The purpose of this section is to show the duality between group colorability and group connectivity. We shall refer to [13] for matroid terminology.

F. Jaeger *et al* in [8] proposed the definition of group colorability of graphs as the matroid dual of the group connectivity of M, where M is a cographic matroid. Clearly, an A-colorable graph is $|A|$-colorable (take $f = 0$) and A-colorability is the dual of local A-connectivity, in the same way that k-colorability is the dual of admitting a k-nowhere-zero flow.

Let $D = D(G)$ be an orientation of an undirected graph G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let **tail**$(e) = u$ and **head**$(e) = v$. For a vertex $v \in V(G)$, let

$$E_D^-(v) = \{e \in E(D) \ : \ v = \text{ tail}(e)\}, \text{ and}$$

$$E_D^+(v) = \{e \in E(D) \ : \ v = \text{ head}(e)\}.$$

The subscript D may be omitted when $D(G)$ is understood from the context. Let $E_G(v)$ denote the subset of edges incident with v in G.

Let A denote an (additive) abelian group with identity 0, and recall that $F(G, A)$ denotes the set of all functions from $E(G)$ to A. Given a function $f \in F(G, A)$, let $\partial f : V(G) \mapsto A$ be given by

$$\partial f(v) = \sum_{e \in D^+(v)} f(e) - \sum_{e \in D^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. Unless otherwise stated, we shall adopt the following convention: if $X \subseteq E(G)$ and if $f : X \mapsto A$ is a function, then we regard f as a function $f : E(G) \mapsto A$ such that $f(e) = 0$ for all $e \in E(G) - X$.

Fix an orientation D of G. Let A be a nontrivial abelian group with identity 0, and let A^* denote the set of nonzero elements in A. Recall $F(G, A) = \{f : E(G) \mapsto A\}$ and define $F^*(G, A) = \{f : E(G) \mapsto A^*\}$.

Let $Z(G, A)$ denote the collection of all functions $b : V(G) \mapsto A$ satisfying $\sum_{v \in V(G)} b(v) = 0$. For a given $b \in Z(G, A)$, a function $f \in F^*(G, A)$ is an (A, b)-**NZF** if $\partial f = b$. A graph G is A-**connected** if G has an orientation D such that for every function $b \in Z(G, A)$, there is an (A, b)-NZF $f \in F^*(G, A)$. For an abelian group A, let $\langle A \rangle$ denote the family of graphs that are A-connected. As noted in [8], that $G \in \langle A \rangle$ is independent of the orientation D of G.

An A-**flow** in G is a function $f \in F(G, A)$ where ∂f is identically zero. We shall denote by $F_0(G, A)$ the set of all A-flows. An A-**nowhere-zero-flow** (abbreviated as an A-NZF) in G is a function $f \in F^*(G, A)$ such that $\partial f = 0$. The nowhere-zero-flow problems were introduced by Tutte [11], and recently surveyed by Jaeger in [7].

Theorem 3.1 (Tutte [12]) Let A be an abelian group with $|A| = k$. Then a graph G has an A-NZF if and only if G has a k-NZF.

Let $\mathbf{Z}$ denote the set of integers. For an integer $k \geq 2$, a **nowhere-zero** k-**flow** (abbreviated as a k-NZF) of G is an orientation D of G together with a map $f \in F(G, \mathbf{Z})$ such that for each $e \in E(G)$, $0 < |f(e)| < k$, and for each vertex $v \in V(G)$, $\partial f(v) = 0$. As noted in [8], the existence of a nowhere-zero k-flow of a graph G is independent of the choice of the orientation D.

Theorem 3.2 (Tutte [12]) Let G be a plane graph. Then G^*, the geometric dual of G, has a k-NZF if and only if $\chi(G) \leq k$.

Jaeger *et al* established some equivalence definitions for the notion of group connectivity.

Theorem 3.3 (Jeager, Linial, Payan and Tarsi, [8]) Let G be a connected graph and let A be a nontrivial Abelian group. The following are equivalent.

(i) $G \in \langle A \rangle$.

(ii) Given any $\bar{f} \in F(G, A)$, there exists an $f \in F_0(G, A)$, the set of all A-flows, such that $f(e) \neq \bar{f}(e)$, for every $e \in E(G)$.

Let M be a connected regular matroid and A be an abelian group, and let $F(M, A)$ denote the set of all functions $f : E(M) \mapsto A$. Define the **circuit matrix** D_M of M to be the incident matrix of circuits against elements. Thus d_{ij}, the (i, j)-entry of D_M, is one if the circuit labeled i contains the element labeled j, and is zero otherwise. Denote by $D_M(C)$ the row of D_M corresponding to the circuit C of M. Define the **cocircuit matrix** $D^*_M = D_{M^*}$. When M is understood from the context, we shall write D and D^* for D_M and D^*_M, respectively.

An orientation of M is a matrix $w(D)$ obtained from D by multiplying each entry of D by a factor in $\{1, -1\}$ such that $w(D)(w(D^*))^T = 0$. Let $F_0(M, A)$ denote the subset of $F(M, A)$ such that $f \in F_0(M, A)$ if and only if the matrix product $fw(D^*)^T = 0$, where $w(D^*)^T$ is the transpose of $w(D^*)$, and where each function $f \in F(M, A)$ is viewed as a row vector indexed by the elements in $E(M)$ in the same way that the columns of D

are indexed. By the definition of $F_0(M,A)$, and since a cocircuit of M^* is a circuit of M, we have the following lemma.

Lemma 3.5 $f \in F_0(M^*, A)$ if and only if, for any circuit C of M, the dot product $fw(D_M(C))^T = 0$.

Theorem 3.3 is then applied to define group connectivity of a regular matroid. Let M be a connected regular matroid and A a nontrivial Abelian group. Then M is **A-connected** if and only if for any $\bar{f} \in F(M,A)$, there exists an $f \in F_0(M,A)$ such that $f(e) \neq \bar{f}(e)$, for every $e \in E(M)$.

Let G be a graph, $M = M(G)$ be the cycle matroid of G and $M^* = M^*(G)$ be the dual matroid of M. The following was observed in [8] without a proof. We shall give a proof for the sake of completeness. An **arborescence rooted at** v, T, is an orientation of a tree such that the indegree of every vertex other than v is exactly one, while the indegree of v is zero. Therefore, if G is a connected graph with a distinguished vertex v, then G can have an orientation D such that $D(G)$ has a spanning arborescence rooted at v.

Theorem 3.6 Let G be a 2-connected graph and A be an abelian group. Then G is A-colorable if and only if $M^*(G)$ is A-connected.

Proof Let $V(G) = \{v_1, v_2, \cdots, v_n\}$. We shall fix an orientation $D = D(G)$ of G so that $D(G)$ has a spanning arborescense T rooted at v_1. Since $E(M^*) = E(M) = E(G)$, $F(M^*, A) = F(M,A) = F(G,A)$.

First we assume that G is A-colorable. Given $f' \in F(M^*, A) = F(G,A)$, there is an (A, f')-coloring $c : V(G) \mapsto A$ such that $c(x) - c(y) \neq f'(x,y)$, for each directed edge (x,y) in $D(G)$.

Define $f \in F(T, A)$ by

$$f(v_i, v_j) = c(v_i) - c(v_j), \text{ whenever } (v_i, v_j) \in E(T). \tag{2}$$

Extend f to a function in $F(G,A)$ as follows: for each edge $e = (x,y) \in E(G) - E(T)$, the undirected underlying graph of $T + (x,y)$ has a unique circuit C_e, the fundamental circuit of e with respect to T. The orientation D of G gives rise to an orientation $w(D(C_e))$ of C_e. Define $f(e)$ to be the unique value which makes the dot product $f(w(D(C_e)))^T = 0$. Now for each fundamental circuit C with respect to T, $f(w(D(C)))^T = 0$, and so the matrix product $f(w(D))^T = 0$, since any circuit, when viewed as a vector in the cycle space, is a linear combination of the fundamental circuits. Therefore, by Lemma 3.5, $f \in F_0(M^*, A)$.

By (2) and by the way f is defined on $E(G) - E(T)$, for each directed edge (x,y) in $D(G)$, $f(x,y) = c(x) - x(y) \neq f'(x,y)$, and so M^* is A-connected.

Conversely, assume that M^* is A-connected. Let $f' \in F(G,A) = F(M^*,A)$, there is a function $f \in F_0(M^*,A)$ such that $f(e) \neq f'(e)$ for each directed edge e in $D(G)$. We define a map $c : V(T) \mapsto A$ inductively as shown by the algorithm below.

Algorithm: Definition of c:

(Step 1) Set $c(v_1) := 0$ and $S := \{v_1\}$.

(Step 2) While $\bar{S} = V(G) - S \neq \emptyset$, Do if $[S,\bar{S}]_T = \{(v_{i_1}, v_{j_1}), (v_{i_2}, v_{j_2}), \cdots (v_{i_k}, v_{j_k})\}$, then $c(v_{j_t}) := c(v_{i_t}) - f(v_{i_t}, v_{j_t})$, where $1 \leq t \leq k$; and $S := S \cup \{v_{j_1}, v_{j_2}, \cdots v_{j_k}\}$.

Since $V(G) = V(T)$, c defined above is also a map from $V(G)$ to A. For each directed edge $e = (x,y)$ in $D(G)$, if $e \in E(T)$, then $f(x,y) = c(x) - c(y) \neq f'(x,y)$. Assume that $e \in E(G) - E(T)$. Then the undirected underlying graph of $T + e$ has a unique circuit C_e. Since $f \in F_0(G,A)$, the dot product $f(D(C_e))^T = 0$, and so we have both $f(x,y) = c(x) - c(y)$ and $f(x,y) \neq f'(x,y)$. Thus G is A-colorable. □

Corollary 3.7 Let G be a connected plane graph, G^* the geometric dual of G, and A an abelian group. Then the following are equivalent.

(i) G is A-connected.

(ii) G^* is A-colorable.

Proof This follows from the relationship between the cycle matroids $M(G^*) \cong (M(G))^*$.

Corollary 3.8 Let G be a connected plane triangulation with $\delta(G) \geq 3$. Then G is A-connected, for any abelian group of order $|A| \geq 3$ if and only if $G \ncong K_4$.

Proof By Corollary 3.7, G is A-connected if and only if G^* is A-colorable.

Let G^* denote the dual of G. Then G^* is 3-regular and simple. By Theorem 2.4, $\chi_1(G^*) \leq 4$ and $\chi_1(G^*) = 4$ if and only if $G^* \cong K_4$, namely $G \cong K_4$. Hence, if $G \ncong K_4$, then $\chi_1(G^*) = 3$ and so G^* is A-colorable for any abelian group of order $|A| \geq 3$. □

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