DISCRETE MATHEMATICS

ELSEVIER

Discrete Mathematics 178 (1998) 93–107

Eulerian subgraphs containing given vertices and hamiltonian line graphs

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Received 5 April 1995; revised 16 July 1996

Abstract

Let G be a graph and let $D_1(G)$ be the set of vertices of degree 1 in G. Veldman (1994) proves the following conjecture from Benhocine et al. (1986) that if $G - D_1(G)$ is a 2-edge-connected simple graph with n vertices and if for every edge $xy \in E(G)$, $d(x)+d(y) > (2n)/5 - 2$, then for n large, $L(G)$, the line graph of G, is hamiltonian. We shall show the following improvement of this theorem: if $G - D_1(G)$ is a 2-edge-connected simple graph with n vertices and if for every edge $xy \in E(G)$, $\max\{d(x), d(y)\} \geqslant n/5 - 1$, then for n large, $L(G)$ is hamiltonian with the exception of a class of well characterized graphs. Our result implies Veldman's theorem.

1. Introduction

We use [2] for terminology and notation not defined here, and consider loopless finite graphs only. Let G be a graph. Then $\kappa(G)$, $\kappa'(G)$ and $\Delta(G)$ denote the connectivity, the edge-connectivity and the maximum degree of G, respectively. Let $O(G)$ denote the set of all vertices in G with odd degrees. An *eulerian* graph is a connected graph G with $O(G) = \emptyset$. The graph K_1 is an eulerian graph.

An eulerian subgraph H of a graph G is *dominating* if $G - V(H)$ is edgeless, and in this case we call H a *dominating eulerian subgraph* (DES). For an integer $i \geqslant 1$, define

$$D_i(G) = \{v \in V(G)\colon\ d(v) = i\}.$$

The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. There is a close relationship between dominating eulerian subgraphs in graphs and Hamilton cycles in $L(G)$.

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PII S0012-365X(96)00376-7

Theorem 1.1 (Harary and Nash-Williams [8]). *Let G be a graph with $|E(G)| \geqslant 3$. Then $L(G)$ is hamiltonian if and only if G has a DES.*

Various sufficient conditions for Hamilton cycles in $L(G)$ in terms of degrees of vertices in G have been found. See Catlin's survey [5] for references. The most recent result is obtained by Veldman. He proves the following which was conjectured in [1].

Theorem 1.2 (Veldman [9]). *If G is a simple graph with $\kappa'(G - D_1(G)) \geqslant 2$ and with n vertices, and if for every edge $xy \in E(G)$,*

$$d(x) + d(y) > \tfrac{2}{5}n - 2, \tag{1.1}$$

then for n large, $L(G)$ is hamiltonian.

Chen and Lai showed that if G is 3-edge-connected, then the bound in (1.1) can be further reduced with stronger conclusions [6, 7].

For every edge $xy \in E(G)$, if (1.1) holds, then we have

$$\max\{d(x), d(y)\} \geqslant \tfrac{1}{5}n - 1. \tag{1.2}$$

Therefore, it is natural to consider if (1.1) can be replaced by (1.2). In this note, we investigate this problem and find some exceptional cases.

In Section 2, we present the main results. In Section 3, Catlin's reduction method is introduced. The main result will be proved in Section 4, using Catlin's reduction method and a similar idea of Veldman [9]. The last section shall be devoted to an auxiliary result, which will be needed in the proof of the main result.

2. Main results

Let G be a graph and let $X \subseteq E(G)$. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If H is a subgraph of G, then we write G/H for $G/E(H)$. If H is a connected subgraph of G, and if v_H denotes the vertex in G/H to which H is contracted, then H is called the *preimage* of v_H. A vertex v in a contraction of G is *nontrivial* if v has a nontrivial preimage.

Let $K_{2,3}$, $K_{2,5}$, W_3', W_4', L_1, L_2 and L_3 be the graphs defined in Figs. 1–3, and let $\mathscr{F} = \{K_{2,3}, K_{2,5}, W_3', W_4', L_1, L_2, L_3\}$. Using the labels in Figs. 1–3, for each $L \in \mathscr{F}$, we define $B(L)$, the *bad set* of L, to be the vertex subset of $V(L)$ that are labeled with the b_i's. The notation $B(L)$ will be used throughout this paper.

Let G be a simple graph with n vertices. Let $p \geqslant 2$ be an integer, and define

$$J_p(G) = \left\{ v \in V(G): \ d(v) \geqslant \frac{n}{p} - 1 \right\}.$$

We are ready to state our main result of this paper.

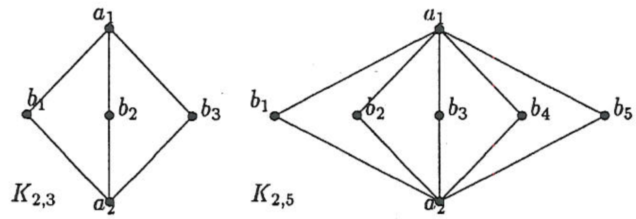

Fig. 1. The graphs $K_{2,3}$ and $K_{2,5}$.

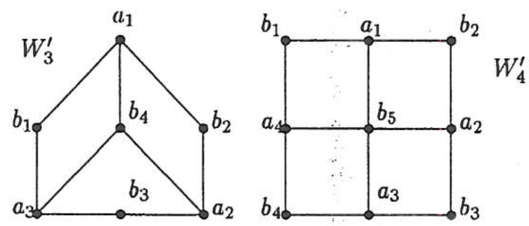

Fig. 2. The graphs W_3' and W_4'.

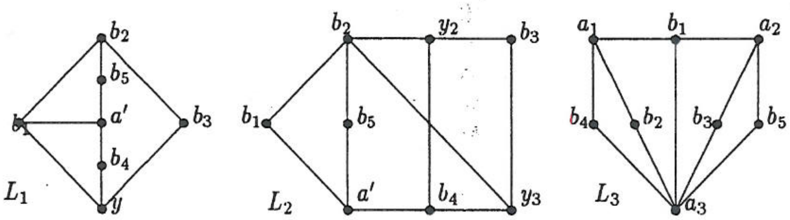

Fig. 3. The graphs L_1, L_2 and L_3.

Theorem 2.1. *Let G be a simple graph with $\kappa'(G - D_1(G)) \geqslant 2$ and with n vertices, and let $J = J_5(G)$. If (1.2) holds for every edge $xy \in E(G)$, then for n large, one of the following must hold:*

(i) *G has an eulerian subgraph L such that $J \subseteq V(L)$, or*

(ii) *G can be contracted to a member $L \in \mathscr{F}$ such that the preimage of every vertex in $B(L)$ intersects J.*

Note that if (1.2) holds for every edge of G, then $V(G) - J$ ($J = J_5(G)$) is an independent set of G. Hence, any eulerian subgraph H of G with $J \subseteq V(H)$ must be a DES of G. Therefore by Theorems 1.1 and 2.1, we have

Corollary 2.2. *Let G be a simple graph with $\kappa'(G - D_1(G)) \geqslant 2$ and with n vertices. If (1.2) holds for every edge $xy \in E(G)$, then for n large, either*

(i) *$L(G)$ is hamiltonian, or*

(ii) *G can be contracted to a member $L \in \mathscr{F}$ such that the preimage of every vertex in $B(L)$ intersects $J_5(G)$.*

Corollary 2.3. *Theorem* 1.2 *follows from Corollary* 2.2.

Proof. Let G be a graph satisfying the hypothesis of Theorem 1.2 with $n > 150$ vertices. By contradiction, assume that $L(G)$ is not hamiltonian. Then by Corollary 2.2, G can be contracted to a member $L \in \mathscr{F}$. By $n > 150$ and by Corollary 1.4(ii), each vertex in $B(L)$ is nontrivial.

Let H be the preimage of a nontrivial vertex in L. Since $E(H) \neq \emptyset$, by (1.1),

$$|V(H)| > \tfrac{1}{2}[(\tfrac{2}{5}n - 2) - \Delta(L)] + 1 \geqslant \tfrac{1}{5}n - 5. \tag{2.1}$$

Since $n > 150$, L has at most 5 nontrivial vertices.

Suppose that $L \neq K_{2,3}$ and L has 5 nontrivial vertices. Then by (2.1), the preimage of each nontrivial vertex must have at most $n - 4(n/5 - 5) = n/5 + 20$ vertices. Since $L \neq K_{2,3}$, there is an edge $e = xy \in E(G)$ such that $e \in E(L)$ and such that x is a trivial vertex in $V(L)$. Since $n > 150$

$$d(x) + d(y) \leqslant \Delta(L) + (\tfrac{1}{5}n + 20) = \tfrac{1}{5}n + 25 < \tfrac{2}{5}n - 2,$$

contrary to (1.1).

If $L = K_{2,3}$ and L has 5 nontrivial vertices, then since $n > 150$ and by $\Delta(K_{2,3}) = 3$, the preimage of each vertex in L must have an edge not adjacent to any edge in $E(L)$, and so by (1.1), the preimage of each vertex in L must have order at least

$$\tfrac{1}{2}(\tfrac{2}{5}n - 2) + 1 = \tfrac{1}{5}n.$$

Since G has n vertices, the preimage of each vertex in L must have exactly $n/5$ vertices. Let $e = xy \in E(G)$ such that $e \in E(L)$. Then

$$d(x) + d(y) \leqslant 2(\tfrac{1}{5}n - 1) = \tfrac{2}{5}n - 2,$$

contrary to (1.1).

Hence, we assume that L has at most 4 nontrivial vertices, and so $|B(L)| \leqslant 4$, which implies $L \in \{K_{2,3}, W_3'\}$. Note that L always has a trivial vertex of degree 3, and so there is an edge $e = xy \in E(G)$ such that $e \in E(L)$ and such that x is a trivial vertex in L. By (1.1) and since $n > 150$, each of the three vertices in L adjacent to x must be nontrivial, and the preimage of each of these 3 nontrivial vertices must have at least $(2n/5 - 2) - 3 = 2n/5 - 5$ vertices. It follows that G has at least $n > 6n/5 - 15$ vertices, contrary to $n > 150$.

This proves Corollary 2.3. □

3. The reduction method

Let G be a graph and let $F \subseteq V(G)$ be a vertex subset. An *eulerian subgraph* (ES) H of G is called an *F-eulerian subgraph* (F-ES) if $F \subseteq V(H)$. A graph G is *supereulerian* if it has a $V(G)$-ES (see [5] for supereulerian graphs). Catlin [4] invented a method to find a $V(G)$-ES for given G. A graph G is *collapsible* if for every subset $R \subseteq V(G)$ with $|R|$ even, G has a spanning connected subgraph H_R such that $O(H_R) = R$. In [4], Catlin showed that every graph G has a unique collection of maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The *reduction* of G is $G' = G/(\bigcup_{i=1}^{c} H_i)$, the graph obtained from G by contracting all nontrivial maximal collapsible subgraphs of G. A graph G is *reduced* if the reduction of G is G.

Theorem 3.1 (Catlin [4]). *Let G be a graph. Each of the following holds*:

(i) *Let L be a collapsible subgraph of G, v_L the vertex in G/L to which L is contracted, and $M \subseteq V(G) - V(L)$. Then G has an ES H such that $M \cup V(L) \subseteq V(H)$ if and only if G/L has an ES H' such that $M \cup \{v_L\} \subseteq V(H')$.*

(ii) *If G is reduced, then G is a simple graph with $\delta(G) \leqslant 3$ and with either $G \in \{K_1, K_2\}$, or $|E(G)| \leqslant 2|V(G)| - 4$.*

(iii) *Any subgraph of a reduced graph is reduced.*

Proposition 3.2. *Let G be a 2-edge-connected graph and let $F \subseteq V(G) - D_2(G)$. Suppose that $v \in D_2(G)$ and that $e \in E(G)$ such that v is incident with e in G. (Thus, we can regard $F \subseteq V(G/e) - D_2(G/e)$.) The following are equivalent.*

(i) *G has a F-ES.*

(ii) *G/e has an F-ES.*

Proof. That (i) $\Rightarrow$ (ii) is trivial since if H is an F-ES of G, then either H (if $e \notin E(H)$) or H/e (if $e \in E(H)$) is an F-ES of G/e. Conversely, suppose that G/e has an F-ES H', and assume that the two neighbors of v are u and w, and that the two ends of e are v and u.

If $v = w$, then $G/e = G - v$, and so an F-ES of G/e is also an F-ES of G. Thus we assume that $v \neq w$. Then wv can be regarded as an edge in $E(G/e)$ and that $E(H')$ can be regarded as an edge subset of G/e. Let

$$H = \begin{cases} G[E(H')] & \text{if } wv \notin E(H'), \\ G[E(H') \cup \{uv\}] & \text{if } wv \in E(H'). \end{cases}$$

It is easy to see that H is an F-ES of G. $\square$

Let G be a connected graph such that $G - D_1(G)$ is 2-edge-connected, and let $\tilde{G}$ be the graph obtained from $G - D_1(G)$ after eliminating all vertices in $D_2(G - D_1(G))$ by repeatedly contracting an edge that is incident with a vertex of degree 2 until no such vertex is left. Note that

$$\text{either } \tilde{G} = K_1, \text{ or both } \kappa'(\tilde{G}) \geqslant 2 \text{ and } \delta(\tilde{G}) \geqslant 3. \tag{3.1}$$

Also, if G is connected, but $G - D_1(G)$ is not 2-edge-connected, then we cannot guarantee $\kappa'(\tilde{G}) \geqslant 2$. An example is the graph obtained from two vertex disjoint K_4 by joining these two K_4's by a new edge.

Note also that if $X \subset E(G)$ denotes the set of edges that are incident with some vertex in $D_1(G)$, then $G - D_1(G) = G/X$, and so $\tilde{G}$ is in fact a contraction image of G. Therefore, if H is a connected subgraph of G which is contracted to a vertex v_H in $\tilde{G}$, then we say that H is the preimage of v_H, as before.

Let G be a connected graph and let $F \subseteq V(G) - (D_1(G) \cup D_2(G))$ be a vertex subset. Let $\tilde{F} \subseteq V(\tilde{G})$ be such that $v' \in \tilde{F}$ if and only if the preimage of v' in G contains at least one vertex in F.

Corollary 3.3. *Let G be a connected graph and let $F \subseteq V(G) - (D_1(G) \cup D_2(G))$. The following are equivalent.*

(i) *G has an F-ES.*

(ii) *$\tilde{G}$ has a $\tilde{F}$-ES.*

Proof. Since $F \cap D_1(G) = \emptyset$, G has an F-ES if and only if $G - D_1(G)$ has an F-ES. By Proposition 3.2 and by induction on $|D_2(G - D_1(G))|$, we can see that $G - D_1(G)$ has an F-ES if and only if $\tilde{G}$ has an $\tilde{F}$-ES. □

Let $\tilde{G}'$ denote the reduction of $\tilde{G}$. Let $\tilde{F}' \subseteq V(\tilde{G}')$ be the vertex subset such that $v \in \tilde{F}'$ if and only if the preimage of v in $\tilde{G}$ contains a vertex in $\tilde{F}$.

Proposition 3.4. *Let G be a connected graph and let $F \subseteq V(G) - (D_1(G) \cup D_2(G))$. Then G has an F-ES if and only if $\tilde{G}'$ has an $\tilde{F}'$-ES.*

Proof. This follows by Corollary 3.3 and Theorem 3.1(i) □

Proposition 3.5. *Let G be a 2-edge-connected graph and let $F \subseteq V(G)$ with $|F| \leqslant 5$. If $G - F$ is edgeless, and if G does not have an F-ES, then G is contractible to a member $L \in \mathscr{F}$ such that F intersects the preimage of every vertex in $B(L)$.*

The proof of Proposition 3.5 will be postponed to the last section.

4. Proof of Theorem 2.1

Throughout this section, we let G be a connected simple graph with $\kappa'(G - D_1(G)) \geqslant 2$ and with n vertices, and let $G'' = \tilde{G}'$, the reduction of $\tilde{G}$. Note that G'' is reduced by definition. Let $n' = |V(G'')|$.

We shall approach the problem slightly more generally. Let $p \geqslant 3$ be an integer, and consider the condition that for every edge $xy \in E(G)$,

$$\max\{d(x), d(y)\} \geqslant \frac{n}{p} - 1. \tag{4.1}$$

We shall assume that (4.1) holds and that n is sufficiently large (say $n \geqslant \max\{3p^3 + 10p^2 + 7p, 6p^2 + 15p + 1\}$).

Let $c = 3p + 7$ and let

$$W = \{v \in V(G''):\ d_{G''}(v) \leqslant c\} \quad \text{and} \quad W' = \{v \in W:\ v \text{ is nontrivial}\}.$$

We shall prove several lemmas to help us to establish the conclusion of our main result.

Lemma 4.1. *For any $v \in W'$, if H_v denotes the preimage of v in G, then*

$$|V(H_v)| \geqslant \frac{n}{p} - d_{G''}(v). \tag{4.2}$$

Proof. Let $\mathrm{Out}(H_v) = \{x \in V(H_v): N_G(x) \neq N_{H_v}(x)\}$ and $\mathrm{In}(H_v) = V(H_v) - \mathrm{Out}(H_v)$. If there is an edge $xy \in E(H_v)$ such that $d(x) \geqslant d(y)$ and such that $x \in \mathrm{In}(H_v)$, then by (4.1), $|V(H_v)| \geqslant d(x) + 1 \geqslant n/p$, and so Lemma 4.1 holds. Therefore, we assume that for any edge $xy \in E(H_v)$ with $d(x) \geqslant d(y)$, we always have $x \in \mathrm{Out}(H_v)$. Thus $|\mathrm{Out}(H_v)| \geqslant 1$, and so by (4.1),

$$|V(H_v)| = |\mathrm{In}(H_v)| + |\mathrm{Out}(H_v)| \geqslant 1 + d_G(x) - d_{G''}(v) \geqslant \frac{n}{p} - d_{G''}(v).$$

This proves Lemma 4.1. □

Corollary 4.2. $|W'| \leqslant p$.

Proof. By Lemma 4.1, we have $n \geqslant |W'|(n/p - c)$. This is equivalent to $|W'| \leqslant np/(n - pc)$. Since $|W'|$ is an integer, we have $|W'| \leqslant p$ when $n \geqslant 3p^3 + 10p^2 + 7p$. □

Lemma 4.3. $V(G'') = W$.

Proof. By contradiction, we assume that $V(G'') - W \neq \emptyset$. Note that every vertex in $V(G'') - W$ has degree at least $c + 1$ in G''. Since G'' is simple by Theorem 4.1(ii), this means

$$n' \geqslant c + 2. \tag{4.3}$$

Count the incidences to get $c|V(G'') - W| \leqslant 2|E(G'')| \leqslant 4n' - 8$, which means $|V(G'') - W| \leqslant (4n' - 8)/c$. It follows that

$$|W| = n' - |V(G'') - W| \geqslant \left(1 - \frac{4}{c}\right) n' + \frac{8}{c}. \tag{4.4}$$

By (3.1), G'' is 2-edge-connected and reduced, and every vertex in $W - W'$ has degree at least 3 in G''. Therefore, by Theorem 3.1(ii), by the fact that $W - W'$ is independent in G'' when $n \geqslant 6p^2 + 15p + 1$, by (4.4) and by Corollary 4.2,

$$2n' - 4 \geqslant |E(G')| \geqslant 3|W - W'| \geqslant \left(3 - \frac{12}{c}\right) n' + \frac{24}{c} - 3p.$$

It follows that

$$3p - 4 \geqslant \left(1 - \frac{12}{c}\right) n' + \frac{24}{c}. \tag{4.5}$$

By (4.3), $n' \geqslant c + 2$. Thus, (4.5) implies that

$$3p - 4 \geqslant \left(1 - \frac{12}{c}\right)(c+2) + \frac{24}{c} = c - 10,$$

and so $3p+6 \geqslant c = 3p+7$, a contradiction. Therefore, we must have $V(G'') = W$. □

Corollary 4.4. *Every vertex in $J_p(G)$ is contained in the preimage of some vertex in W'.*

Proof. Since $n \geqslant 3p^2 + 9p$, the degree of vertices in $J_p(G)$ will exceed c, and so Corollary 4.4 follows from Lemma 4.3. □

Proof of Theorem 2.1. Note that $J = J_5(G)$ in Theorem 2.1. Note also that by Corollary 4.4 and by Theorem 3.1(i), if G'' has a W'-ES, then G has a J-ES.

Applying the discussion above to the case when $p=5$, we conclude that $|W'| \leqslant p = 5$. Thus, by Proposition 3.5, either G'' has a W'-ES, thereby G has a J-ES and so Theorem 2.1(i) holds; or G'' can be contracted to a member $L \in \mathscr{F}$ such that W' intersects the preimage of every vertex in $B(L)$, and so Theorem 2.1(ii) must hold. □

5. Eulerian subgraphs that contain given vertices

We shall prove Proposition 3.5 in this section. Let G be a 2-edge-connected graph and let $F \subseteq V(G)$ with $|F| \leqslant 5$. By contradiction, we assume that

$$G \text{ is a counterexample to Proposition 3.5 with } |V(G)| \text{ minimized.} \tag{5.1}$$

Thus, we assume that

$$G - F \text{ is edgeless,} \tag{5.2}$$

and that

$$G \text{ does not have an } F\text{-ES.} \tag{5.3}$$

Lemma 5.1. *G is reduced, $\kappa(G) \geqslant 2$ and G is not contractible to a member $L \in \mathscr{F}$ such that F intersects the preimage of every vertex in $B(L)$.*

Proof. By (5.1), G cannot be contracted to a member $L \in \mathscr{F}$ such that F intersects the preimage of every vertex in $B(L)$.

Let G' denote the reduction of G, and let $F' \subseteq V(G')$ be a subset such that $v \in F'$ if and only if the preimage of v in G contains a vertex in F, and so $|F'| \leqslant |F| \leqslant 5$. By (5.2), $G' - F'$ is also edgeless. If G is not reduced, then $|V(G')| < |V(G)|$, and so by (5.1), G' can be contracted to a member $L \in \mathscr{F}$ such that F' intersects the preimage of every vertex in $B(L)$. Therefore, G is contractible to $L \in \mathscr{F}$ such that F intersects the preimage of every vertex in $B(L)$, contrary to (5.1). Hence G is reduced.

Assume then that G has a cut vertex v, and $G - v$ has components $H_1, \dots, H_c$. Let $G_i = G[V(H_i) \cup \{v\}]$, $(1 \leqslant i \leqslant c)$. Then $F \subseteq V(G) = \bigcup_{i=1}^{c} V(G_i)$. By (5.2), $F \cap V(G_i) \neq \emptyset$, $(1 \leqslant i \leqslant c)$. Let $F_i = (F \cap V(G_i)) \cup \{v\}$. Then $|F_i| \leqslant 5$. By (5.1), Proposition 3.5 holds for each G_i, and so either each G_i has an F_i-ES H_i, and so $H = \cup H_i$ is an F-ES of G, contrary to (5.3), or there is some i such that G_i is contractible to a member $L \in \mathscr{F}$ such that F intersects the preimage of every vertex in $B(L)$. So the same holds for G, contrary to (5.1). Hence $\kappa(G) \geqslant 2$. □

Lemma 5.2. *Let H be a subgraph of G such that $F \subseteq V(H)$. If P is a path in $G - E(H)$ with end vertices in $V(H)$ and all internal vertices in $V(G) - V(H)$, then one of the following must hold:*

(i) $|E(P)| = 1$, *and P has at least one end in F.*

(ii) $|E(P)| = 2$, *and both ends of P are in F.*

Proof. This follows from (5.2). □

Lemma 5.3. *If H is 2-connected, and if $b_1, b_2, b_3 \in V(H)$, then one of the following must hold:*

(i) *H has a cycle C with $b_1, b_2, b_3 \in V(C)$.*

(ii) *H has two vertices $a_1, a_2 \notin \{b_1, b_2, b_3\}$ and three internally disjoint (a_1, a_2)-paths P_1, P_2, P_3 such that $b_i \in V(P_i)$, $(1 \leqslant i \leqslant 3)$. (This subgraph $P_1 \cup P_2 \cup P_3$ of H is called a $K_{2,3}(b_1, b_2, b_3)$ of H.)*

Proof. Since $\kappa(H) \geqslant 2$, by Menger's Theorem, there is a cycle C containing b_1 and b_2. If $b_3 \in V(C)$, then (i) of Lemma 5.3 holds. Thus we assume that $b_3 \notin V(C)$. By Menger's Theorem again, there exists $a_1, a_2 \in V(C)$ such that G has a (a_1, b_3)-path Q and a (b_3, a_2)-path Q' with $V(Q) \cap V(Q') = \{b_3\}$. If $\{a_1, a_2\}$ does not separate b_1 and b_2 in C, then C has a (a_1, a_2)-path Q'' that contains both b_1 and b_2. It follows that $Q \cup Q' \cup Q''$ is a cycle containing b_1, b_2 and b_3, and so Lemma 5.3(i) holds. If each of the two (a_1, a_2)-paths P_1 and P_2 (say) contains exactly one of b_1 and b_2, then Lemma 5.3(ii) holds. □

Lemma 5.4. *Let M be a graph, and let $R \subset V(M)$ with $b \in R$. If M has a cycle C containing $R - \{b\}$, but no cycles containing R, and if M has a (a, b)-path Q and an*

(a',b)-path Q' with $V(Q)\cap V(Q)=\{b\}$, and with $a,a'\in V(C)$, then each component of $C-\{a,a'\}$ must contains a member of $R-\{b\}$.

Proof. If $C-\{a,a'\}$ has a component C' with $V(C')\cap R=\emptyset$, then $(C-V(C'))\cup Q\cup Q'$ is a cycle of M containing R, a contradiction. □

Lemma 5.5 (Catlin [4, 3]). *The graphs K_3 and $K_{3,3}-e$ ($K_{3,3}$ minus an edge) are collapsible.*

Therefore by Lemmas 5.1 and 5.5, and by Theorem 3.1(ii), we have

$$G \text{ is simple and does not have } K_3 \text{ or } K_{3,3}-e \text{ as a subgraph.} \tag{5.4}$$

We need one more notation. If $C=z\cdots u\cdots w\cdots v\cdots z$ denotes a cycle (together with the indicated orientation), then ${}_uC_v$ denotes the section $u\cdots w\cdots v$ of C. If $P=x_1x_2\cdots x_{n-1}x_n$ is a path, then $P^o=x_2\cdots x_{n-1}$ with the agreement that when $|E(P)|=1$, then P^o denotes the only edge in P.

Proof of Proposition 3.5. If $F=\{b_1,b_2,b_3\}$, then by Lemma 5.3, we assume that G has a $K_{2,3}(b_1,b_2,b_3)$ (denoted H). By (5.2) and by the fact that $F=\{b_1,b_2,b_3\}$, $H\cong K_{2,3}$. If $G-E(H)$ has a path Q, then by Lemma 5.2, $H\cup Q$ has either a K_3 or a $K_{3,3}-e$, contrary to (5.4). Hence we assume that $|F|\geqslant 4$.

Case 1: $F=\{b_1,b_2,b_3,b_4\}$. By Lemma 5.3, either G has a cycle C containing b_1,b_2,b_3, or G has a subgraph $G_1=K_{2,3}(b_1,b_2,b_3)$.

Case 1A: G has a cycle containing b_1,b_2,b_3. Since $\kappa(G)\geqslant 2$, there are $a,a'\in V(C)$ such that G has an (a,b_4)-path Q and an (a',b_4)-path Q' with $V(Q)\cap V(Q')=\{b_4\}$. By Lemma 5.4, each component of $C-\{a,a'\}$ contains a vertex in $\{b_1,b_2,b_3\}$. By (5.3), we assume $C=b_1\cdots b_2\cdots a\cdots b_3\cdots a'\cdots b_1$, where $b_2=a$ is possible but $a'\notin\{b_3,b_1\}$. (We shall use the convention here that if $b_2=a$, then b_2a denotes the single vertex b_2.) By (5.2), $ab_3,b_3a',a'b_1\in E(C)$, $Q=b_4a$ and $Q'=b_4a'$. Also by (5.2), either there is a vertex $a''\in V(C)$ such that $C=b_1a''b_2ab_3a'b_1$, or $C=b_1b_2ab_3a'b_1$. Let $H=C\cup Q\cup Q'$. If $G=H$, then

$$L=\begin{cases} G/\{b_1b_2\} & \text{if } C=b_1b_2ab_3a'b_1,\\ G/\{b_1a'',a''b_2\} & \text{if } C=b_1a''b_2ab_3a'b_1\end{cases}$$

is a desired member in $\mathscr{F}$. Thus, we suppose that there is a (x,y)-path P in $G-E(H)$ with $x,y\in V(H)$.

If such an (x,y)-path P exists only when $\{x,y\}\subseteq V({}_{b_1}C_{b_2})$, then G can be contracted to a $K_{2,3}$ whose edge set is $\{b_1a',b_2a,ab_4,a'b_4,ab_3,a'b_3\}$. Thus, we assume that $\{x,y\}\not\subseteq V({}_{b_1}C_{b_2})$. If $\{x,y\}\subset F$, then since we may assume that either $\{x,y\}=\{b_1,b_3\}$, whence $(H-\{b_1a',ab_3\})\cup P$ is an F-ES; or $\{x,y\}=\{b_1,b_4\}$, whence $(H-\{b_1a',b_4a\})\cup P$ is an F-ES, contrary to (5.3) in any case. Thus, by Lemma 5.2, we may assume that $x\in F$ and $y\notin F$.

If $x = b_1$, then by (5.4), y must be a, and so $(H-\{b_1a'\})+xy$ is an F-ES, contrary to (5.3). Similarly, $x \neq b_2$. By the symmetry of b_3 and b_4 in H, we may assume that $x = b_4$. By (5.4), we must have $C = b_1a''b_2ab_3a'b_1$ and $y = a''$. Note that any path in $G - E(H + b_4a'')$ will result in an F-ES of G, and so $G = H + b_4a'' \cong W_3'$ with $B(W_3') = F$. This completes the proof for Case 1A.

Case 1B: G has a subgraph $G_1 = K_{2,3}(b_1, b_2, b_3)$ and

$$G \text{ does not have a cycle containing any three members of } F. \tag{5.5}$$

By (5.5), $b_4 \notin V(G_1)$ and so by (5.2), G has two vertices a, a' such that

$$V(G_1) = \{a, a', b_1, b_2, b_3\} \quad \text{and} \quad E(G_1) = \{ab_1, ab_2, ab_3, a'b_1, a'b_2, a'b_3\}. \tag{5.6}$$

Since $\kappa(G) \geqslant 2$, G has two vertices $c, c' \in V(G_1)$ and a (c, b_4)-path Q, and a (c', b_4)-path Q' with $V(Q) \cap V(Q') = \{b_4\}$. By (5.5), we must have $\{c, c'\} = \{a, a'\}$, and so $G_1 \cup Q \cup Q'$ is an F-ES of G, contrary to (5.3).

Case 2: $F = \{b_1, b_2, b_3, b_4, b_5\}$. Since $\kappa(G) \geqslant 2$, G has a cycle C with $|V(C) \cap F|$ maximized. By (5.3), $2 \leqslant |V(C) \cap F| \leqslant 4$.

Case 2A: $|V(C) \cap F| = 4$. We assume that $C = b_1 \cdots b_2 \cdots b_3 \cdots b_4 \cdots b_1$. Since $b_5 \notin V(C)$ and since $\kappa(G) \geqslant 2$, G has a (a, b_5)-path Q and a (a', b_5)-path Q' with $V(Q) \cap V(Q') = \{b_5\}$, and with $a, a' \in V(C)$. By Lemma 5.4, each component of $C - \{a, a'\}$ has a vertex in $F - \{b_5\}$.

Case 2A1: $a, a' \in F - \{b_5\}$. By Lemma 5.4, we may assume that $a = b_2$ and $a' = b_4$. Let $H_1 = C \cup Q \cup Q'$. If $G = H_1$, then G can be contracted to $K_{2,3} \in \mathscr{F}$. So we assume that $G - E(H_1)$ has an (x, y)-path P with $x, y \in V(H_1)$. If $x, y \in F$, then without loss of generality, either $\{x, y\} = \{b_1, b_2\}$, whence $(H_1 - ({}_{b_4}C^o_{b_1})) \cup P$ is an F-ES; or $\{x, y\} = \{b_1, b_5\}$, whence $(H_1 - ({}_{b_1}C^o_{b_2} \cup Q'^o)) \cup P$ is an F-ES; or $\{x, y\} = \{b_2, b_4\}$, whence $H_1 \cup P$ is an F-ES, contrary to (5.3) in any case. Thus, by Lemma 5.2, we may assume that $x \in F$ and $y \notin F$ with $E(P) = \{xy\}$. Without loss of generality, we assume that $y \in V(Q)$, and so by (5.2), $Q = b_5yb_2$. By (5.4), $x \in \{b_1, b_3, b_4\}$. If $x = b_1$ (or b_3), then $(H_1 - ({}_{b_4}C^o_{b_1} \cup \{yb_2\})) + b_1y$ (or $(H_1 - ({}_{b_3}C^o_{b_4} \cup \{yb_2\})) + b_3y$) is a cycle of G containing F; if $x = b_4$, then $(H_1 - \{y_{b_2}\}) + b_4y$ is an F-ES, contrary to (5.3) in any case. This proves Case 2A1.

Case 2A2: $a \in F - \{b_5\}$ and $a' \notin F$. We may assume that $a = b_2$. Then by (5.3), a' cannot be in ${}_{b_1}C_{b_3}$, and so we may assume that, by (5.2), $C = b_1 \cdots b_2 \cdots b_3 \cdots b_4a'b_1$, and $Q' = b_5a'$. Let $H_1 = C \cup Q \cup Q'$. If $G = H_1$, then G is a subdivision of a $K_{2,3}$ which can be contracted to a desired member in $\mathscr{F}$. Therefore we assume that $G - E(H_1)$ has an (x, y)-path P with $x, y \in V(H_1)$. If such an (x, y)-path P exists only when $\{x, y\} \subseteq V({}_{b_3}C_{b_4})$, then G can be contracted to a $K_{2,3}$. Hence, we assume that $\{x, y\} \nsubseteq V({}_{b_3}C_{b_4})$.

If $\{x, y\} \subseteq F$, then either $\{x, y\} = \{b_1, b_3\}$, whence $(H_1 - ({}_{b_2}C^o_{b_3} \cup_{a'} C^o_{b_1})) \cup P$ is an F-ES; or we have a case similar to Case 2A1. Hence, we may assume that $x \in F$ and $y \notin F$ with $E(P) = \{xy\}$. If $y = a'$, then by (5.4), either $x = b_3$, whence $(H_1 -_{b_2} C^o_{b_3}) + b_3a'$ is an F-ES of G; or $x = b_2$ whence $H_1 + b_2a'$ is an F-ES of G, contrary to (5.3). Hence, $y \neq a'$, and so for some i, $(1 \leqslant i \leqslant 3)$, $y \in V({}_{b_i}C_{b_{i+1}})$. By (5.2), ${}_{b_i}C_{b_{i+1}} = b_i y b_{i+1}$.

Suppose that $i = 1$. Then by (5.4), $x \in \{b_3, b_4, b_5\}$. If $x \in \{b_4, b_5\}$, then $(H_1 - \{yb_2, xa'\}) + xy$ is an F-ES, and so $x = b_3$. Suppose that $i = 2$. Then $x \in \{b_1, b_4, b_5\}$. If $x \in \{b_1, b_5\}$, then $(H_1 - \{xa', yb_2\}) + xy$ is an F-ES of G, and so $x = b_4$.

Thus, we may summarize these discussions to conclude that any edge in $E(G) - E(H_1)$ must be $e_1 = y_1 b_3$, where $y_1 \in V({}_{b_1}C^o_{b_2})$; or $e_2 = y_2 b_4$, where $y_2 \in V({}_{b_2}C^o_{b_3})$, or $e_3 = y_3 b_1$, or $e_4 = y_3 b_2$, or $e_5 = y_3 b_5$, where $y_3 \in V({}_{b_3}C^o_{b_4})$. Table 1 shows that if the indicated $e_i, e_j \in E(G) - E(H_1)$, then the subgraph Γ defined in Table 1 is an F-ES of G.

Thus, by Table 1, $|E(G) - E(H_1)| \leqslant 2$, where equality holds if and only if $E(G) - E(H_1) = \{e_1, e_4\}$ or $E(G) - E(H_1) = \{e_2, e_4\}$. Table 2 shows that in all the possible cases of $E(G) - E(H_1)$, G can be contracted to a graph W which can be easily shown to be isomorphic or can be contracted to the element of $\mathscr{F}$ in the third column.

This completes the proof of Case 2A2.

Case 2A3: $a, a' \notin F$. Let c be the maximum number such that G has (a_k, b_5)-paths Q_k, $1 \leqslant k \leqslant c$, with $a_1, \ldots, a_c \in V(C) - F$ and with $V(Q_i) \cap V(Q_j) = \{b_5\}$, whenever $i \neq j$. By (5.2),

$$|E(Q_k)| = 1 \quad \text{and so } Q_k = b_5 a_k \quad \text{for each } 1 \leqslant k \leqslant c. \tag{5.7}$$

Since $\kappa(G) \geqslant 2$ and by (5.3), $2 \leqslant c \leqslant 4$. Let $H_1' = C \cup (\bigcup_{k=1}^{c} Q_k)$.

Suppose that $c = 4$. Then by (5.7), we may assume that $C = b_1 a_1 b_2 a_2 b_3 a_3 b_4 a_4 b_1$. Note that if $G = H_1'$, then $G \cong W_4' \in \mathscr{F}$. Thus, $G - E(H_1')$ has an (x, y)-path P with $x, y \in V(H_1)$. Then by Lemma 5.2, $|\{x, y\} \cap F| \geqslant 1$. Table 3 defines a subgraph Γ which is an F-ES of G, contrary to (5.3).

The missing cases in Table 3 can be either obtained by symmetry, or eliminated by (5.4).

Suppose that $c = 3$. Then we may assume that $C = b_1 a_1 b_2 a_2 b_3 a_3 b_4 \cdots b_1$, with possibly a vertex a_4 inserted between b_4 and b_1. If $G = H_1'$, then $G/E({}_{b_4}C_{b_1}) \cong W_3' \in \mathscr{F}$. Therefore, we assume that $G - E(H_1')$ has an (x, y)-path P with $x, y \in V(H_1)$ and $|\{x, y\} \cap F| \geqslant 1$. If such an (x, y)-path exists only when $\{x, y\} \subseteq V({}_{b_4}C_{b_1})$, then G can

Table 1

e_i,	$e_j \in E(G) - E(H_1)$	Subgraph Γ
e_1	e_2	$(H_1 - \{a'b_4, y_1b_2, y_2b_3\}) + \{e_1, e_2\}$
e_1	e_3	$(H_1 - \{a'b_1, y_1b_2, y_3b_3\}) + \{e_1, e_3\}$
e_1	e_5	$(H_1 - \{a'b_5, y_1b_2, y_3b_3\}) + \{e_1, e_5\}$
e_2	e_3	$(H_1 - \{a'b_1, y_2b_2, y_3b_4\}) + \{e_2, e_3\}$
e_2	e_5	$(H_1 - \{a'b_5, y_2b_2, y_3b_4\}) + \{e_2, e_5\}$
e_3	e_4	$(H_1 - \{a'b_1\}) + \{e_3, e_4\}$
e_3	e_5	$((H_1 - \{a'b_1\}) - Q^o) + \{e_3, e_5\}$
e_4	e_5	$(H_1 - \{a'b_5\}) + \{e_4, e_5\}$

Table 2

$E(G)-E(H_1)$	Contraction W	Element of $\mathcal{F}$
$\{e_1\}$	$G/(E(_{b_2}C_{b_3}) \cup \{e_1, b_2y_1\})$	$K_{2,3}$
$\{e_2\}$	$G/(E(_{b_3}C_{b_4}) \cup \{e_2, b_3y_2\})$	$K_{2,3}$
$\{e_3\}$	G	L_1
$\{e_4\}$	$G/(E(_{b_2}C_{b_3}) \cup \{e_4, b_3y_3\})$	$K_{2,3}$
$\{e_5\}$	G	W_3'
$\{e_1, e_4\}$	$G/(E(_{b_2}C_{b_3}) \cup \{e_1, e_4, b_2y_2, y_2b_3\})$	$K_{2,3}$
$\{e_2, e_4\}$	G	L_2
$\emptyset$	$G/\{e_4\}$	$K_{2,3}$

Table 3

x	y	Subgraph Γ
b_1	b_2	$(H_1' - \{b_1a_1, b_2a_2, b_5a_3, b_5a_4\}) \cup P$
b_1	b_3	$(H_1' - \{b_1a_4, b_3a_2, b_5a_1, b_5a_3\}) \cup P$
b_1	b_5	$(H_1' - \{b_1a_1, b_5a_2, b_5a_3, b_5a_4\}) \cup P$
b_1	a_2	$(H_1' - \{b_1a_1, b_5a_3, b_5a_4\}) \cup P$

Table 4

x	y	Subgraph Γ
b_1	b_2	$(H_1' - \{b_1a_1, b_2a_2, b_5a_3\}) \cup P$
b_1	b_3	$(H_1' - \{b_1a_1, b_3a_3, b_5a_2\}) \cup P$
b_1	b_5	$(H_1' - \{b_1a_1, b_5a_2, b_5a_3\}) \cup P$
b_1	a_2	$(H_1' - \{b_1a_1, b_5a_3\}) \cup P$
b_1	a_3	$(H_1' - \{b_1a_1, b_5a_2\}) \cup P$
b_2	b_3	$(H_1' - \{b_5a_1, b_3a_3, b_2a_2\}) \cup P$
b_2	b_5	$(H_1' - \{b_5a_1, b_2a_2\}) \cup P$
b_2	a_3	$(H_1' - \{b_5a_1, b_2a_2\}) \cup P$

be contracted to a desired $W_3' \in \mathcal{F}$. Hence, we assume that $\{x, y\} \not\subseteq V(_{b_4}C_{b_1})$. Table 4 defines a subgraph Γ which is an F-ES of G.

The missing cases in Table 4 can be either obtained by symmetry, or eliminated by (5.4).

Suppose that $c = 2$. By Lemma 5.4, we may assume that either $C = b_1a_1b_2a_2b_3 \cdots b_4 \cdots b_1$ or $C = b_1 \cdots b_2a_2b_3 \cdots b_4a_1b_1$.

First we assume that $C = b_1a_1b_2a_2b_3 \cdots b_4 \cdots b_1$. If $G - E(H_1')$ has a (b_2, b_5)-path P, then $(H_1' - \{b_2a_2, a_1b_5\}) \cup P$ is an F-ES of G, and so we exclude this possibility. If $G = H_1'$, or if there is no path in $G - E(H_1')$ joining $\{a_1, b_2, a_2, b_5\}$ to $V(G) - \{a_1, b_2, a_2, b_5\}$, then $G/(E(_{b_3}C_{b_1}) \cong K_{2,3}$ is a desired contraction. Therefore, we assume that there is an (x, y)-path in $G - E(H_1')$ with $x \in \{a_1, b_2, a_2, b_5\}$ and $y \in V(G) - \{a_1, b_2, a_2, b_5\}$. Note that $H_1' - b_2$ is a cycle containing $F - \{b_2\}$. So we can assume $x \notin \{b_2, b_5\}$, otherwise we are in Case 2A1 or 2A2 or we have $c \geqslant 3$. Therefore, by (5.4) and Lemma 5.2, $E(P) = \{xy\}$ and $xy \in \{a_1b_4, a_2b_4, a_1b_3, a_2b_1\}$. If $\{x,y\} = \{a_1,b_3\}$ (resp. $\{a_2,b_1\}$), then $(H_1' - \{a_2b_3\}) \cup P$ (resp. $(H_1' - \{a_1b_1\}) \cup P$) is an F-ES. If both $a_1b_4, a_2b_4 \in E(G)$, then $H_1' + \{a_1b_4, a_2b_4\}$ is an F-ES. Thus, we may assume that $P = a_1b_4$.

If for some $a \in V({}_{b_4}C_{b_1})$, $G - E(H_1')$ has an (a,b_3)-path P', then $((H_1' - \{a_2b_3\}) - {}_{b_4}C_a^o) \cup P \cup P'$ is an F-ES of G, and so we assume no such paths exist. Therefore if ${}_{b_3}C_{b_4} = b_3b_4$, then $G/E({}_{b_4}C_{a_1}) \cong K_{2,3}$, and if ${}_{b_3}C_{b_4} = b_3a_3b_4$, then $G/E({}_{a_3}C_{a_1}) \cong K_{2,3}$. In either case, we have a desired contraction image in $\mathscr{F}$.

Then we assume that $C = b_1 \cdots b_2a_2b_3 \cdots b_4a_1b_1$. Since $H_1'/(E({}_{b_1}C_{b_2}) \cup E({}_{b_3}C_{b_4})) \cong K_{2,3} \in \mathscr{F}$, we assume that there is an (x,y)-path P in $G - E(H_1')$ with $x, y \in V(H_1')$ and with both $\{x,y\} \not\subset V({}_{b_1}C_{b_2})$ and $\{x,y\} \not\subset V({}_{b_3}C_{b_4})$. We can assume $b_5 \notin \{x,y\}$, otherwise we have a case already examined. Note that if such (x,y)-path exists only when $\{x,y\} \subseteq V({}_{b_4}C_{b_1})$ (or $\{x,y\} \subseteq V({}_{b_1}C_{b_4}^o)$), then G is contractible to a desired $K_{2,3} \in \mathscr{F}$. Thus, we assume that $\{x,y\} \nsubseteq V({}_{b_4}C_{b_1})$ and $\{x,y\} \nsubseteq V({}_{b_1}C_{b_4}^o)$. Note also that if there are two (b_1,x_i)-paths P_i, $(1 \leqslant i \leqslant 2)$, where $x_i \in V({}_{b_3}C_{b_4}^o) - \{b_4\}$, then G is contractible to a desired $K_{2,3} \in \mathscr{F}$, and that if there is only one (b_1,x_1)-path with $x_1 \in V({}_{b_3}C_{b_4}^o) - \{b_4\}$, then G is contractible to a desired $L_1 \in \mathscr{F}$. Therefore, without loss of generality, we assume that either $\{x,y\} = \{b_1,b_3\}$, whence $(H_1' - \{a_1b_1, a_2b_3\}) \cup P$ is an F-ES of G; or $\{x,y\} = \{b_1,a_2\}$, whence $(H_1' - \{a_1b_1\}) \cup P$ is an F-ES of G, contrary to (5.3) in either case. This concludes Case 2A.

Case 2B: $|V(C) \cap F| = 3$, and

$$G \text{ has no cycle that contains 4 vertices in } F. \tag{5.7}$$

We may assume that $C = b_1 \cdots b_2 \cdots b_3 \cdots b_1$. Since $\kappa(G) \geqslant 2$, G has a (b_4,a_1)-path Q_1 and a (b_4,a_3)-path Q_2 such that $a_1, a_3 \in V(C)$ with $V(Q_1) \cap V(Q_2) = \{b_4\}$. Similarly, G has a (b_5,a)-path Q_3 and a (b_5,a')-path Q_4 such that $a, a' \in V(C) - F$ with $V(Q_3) \cap V(Q_4) = \{b_5\}$. By (5.7), $b_4 \notin V(Q_1 \cup Q_2)$ and $b_5 \notin V(Q_1 \cup Q_2)$. By (5.2), $|E(Q_i)| = 1$, $(1 \leqslant i \leqslant 4)$. Therefore, $V(Q_1 \cup Q_2) \cap V(Q_3 \cup Q_4) = \emptyset$.

By (5.7), $\{a_1,a_3\} \nsubseteq F$. Hence, we may assume that $C = b_1a_1b_2 \cdots b_3a_3b_1$, where $a_3 = b_3$ is possible (in this case b_3a_3 denotes the single vertex b_3).

If $\{a,a'\} = \{a_1,a_3\}$, then $C \cup (\bigcup_{i=1}^4 Q_i)$ is an F-ES of G, contrary to (5.3). Hence $\{a,a'\} \neq \{a_1,a_3\}$. By (5.2), there is at most one vertex not in F that lies between b_2 and b_3 in C, and so by $\{a,a'\} \neq \{a_1,a_3\}$, we may assume that $a' = a_3$, and $a = a_2$ (say) and that $C = b_1a_1b_2a_2b_3a_3b_1$. Let $H_2 = C \cup (\bigcup_{i=1}^4 Q_i)$. By (5.2), we have

$$V(H_2) = V(C) \cup \{b_4, b_5\} \quad \text{and} \quad E(H_2) = E(C) \cup \{b_4a_1, b_4a_3, b_5a_3, b_5a_2\}.$$

If $G = H_2$, then $G/\{a_2b_3, b_3a_3, a_3b_5, b_5a_2\} \cong K_{2,3}$ is a desired member in $\mathscr{F}$. Therefore, $G - E(H_2)$ has an (x,y)-path P.

A straightforward analysis shows that the only possibilities for $\{x,y\}$ are $\{x,y\} = \{b_1,a_2\}, \{b_2,a_3\}, \{b_3,a_1\}, \{b_4,a_2\}$ or $\{b_5,a_1\}$. All other choices for $\{x,y\}$ contradict (5.2), (5.4), Lemma 5.2, or they give a cycle that contains 4 vertices of F, hence contradicting (5.7). If $\{x,y\} \in \{\{b_1,a_2\}, \{b_5,a_1\}, \{b_3,a_1\}, \{b_4,a_2\}\}$, then it is easy to find an F-ES. So the only case remaining is $\{x, y\} = \{b_2, a_3\}$. But then the graph can be contracted to L_3. This proves Case 2B.

Case 2C: $|V(C) \cap F| = 2$, and

$$G \text{ has no cycle that contains 3 vertices in } F. \tag{5.8}$$

By Lemma 5.3, G has a subgraph $G_1 = K_{2,3}(b_1, b_2, b_3)$. By (5.8), $b_4, b_5 \notin V(G_1)$, and so by (5.2), G has two vertices a, a' such that (5.6) holds.

Since $\kappa(G) \geqslant 2$, G has two vertices $c, c' \in V(G_1)$ and a (c, b_4)-path Q, and a (c', b_4)-path Q' with $V(Q) \cap V(Q') = \{b_4\}$. By (5.6), and by (5.4), we must have $\{c, c'\} = \{a, a'\}$, and so $b_4a, b_4a' \in E(G)$. Similarly, $b_5a, b_5a' \in E(G)$. Let H_3 be the subgraph of G with

$$V(H_3) = V(G_1) \cup \{b_4, b_5\} \quad \text{and} \quad E(H_3) = E(G_1) \cup \{b_4a, b_4a', b_5a, b_5a'\}.$$

Then $H_3 \cong K_{2,5}$. Thus, we assume that $G - E(H_3)$ has an (x, y)-path P. By Lemma 5.2, $|\{x, y\} \cap F| = 1$. It is straightforward to check that we always can form a cycle containing 3 vertices of F, contradicting (5.8).

This completes the proof of Proposition 3.5. □

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