Cycle covers of planar graphs

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Abstract

Bondy conjectured in [1] that every 2-connected simple graph with n vertices admits a cycle cover with at most $(2n-1)/3$ cycles. In this paper we shall show that every 2-connected planar graph with $n \geq 6$ vertices admits a cycle cover with at most $(2n-2)/3$ cycles. This bound is best possible.

1. Introduction

We follow the notation of Bondy and Murty [2], unless otherwise noted. In particular, $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and the edge-connectivity of a graph G, respectively. An edge e of a graph G is called a multiple edge if $G-e$ has an edge f that has the same ends as e in G. We allow multiple edges but forbid loops. When $v, v' \in V(G)$, vv' would denote any one edge in G with ends v and v'. For $X \subseteq E(G)$, the contraction G/X is the graph obtained from G by identifying the ends of each edge of X and then deleting the resulting loops. We shall use G/e for $G/\{e\}$ and when H is a subgraph of G, we write G/H for $G/E(H)$. For $v \in V(G)$, $N(v)$, the neighborhood of v in G, denotes the set of vertices adjacent to v in G. If H is a subgraph of G and $P = v_1v_2\cdots v_k$ is a path of G, then we shall write $H + v_1v_2\cdots v_k$ for the subgraph $G[E(H) \cup E(P)]$.

A cycle cover (CC) is a collection C of cycles in G such that every edge in G lies in at least one cycle in C. It is clear that G has a cycle cover if and only if $\kappa'(G) \geq 2$. For a 2-edge-connected graph, let

$$cc(G) = \min\{|C| \ : \ C \text{ is a CC of } G\}. \tag{1}$$

In [B], Bondy posed the following conjecture:

Conjecture SCC: (Bondy [1]) If G is a 2-connected simple graph G with n vertices, then

$$cc(G) \leq \frac{2n-1}{3}. \tag{2}$$

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We shall work on a multigraph version of this conjecture. For a graph G, define a relation on $E(G)$ such that e is related to e' if and only if $e = e'$ or e is parallel to e' in G. It is easy to check that this is an equivalence relation. Let $[e]$ denote the equivalence class containing e, and $[G]$ the collection of all equivalence classes. Define

$$\mu(G) = \sum_{[e]\in[G]} (|[e]| - 1). \tag{3}$$

Thus G is simple if and only if $\mu(G) = 0$, and so a multigraph version of Conjecture SCC can be stated as follows: If G is a 2-edge-connected graph with order n, then

$$cc(G) \leq \frac{2n-1}{3} + \frac{\mu(G)}{2}. \tag{4}$$

Call a multigraph G a plane triangulation if G can be embedded in the plane such that every face of the embedding has degree 2 or 3. In [3], we showed the following:

Theorem 1.1 If G is a planar triangulation with $n \geq 6$ vertices, then

$$cc(G) \leq \frac{2n-3}{3} + \frac{\mu(G)}{2}. \tag{5}$$

In this note, we shall show that Conjecture SCC holds for planar graphs:

Theorem 1.2 If G is a planar graph with $n \geq 6$ vertices and with $\kappa(G) \geq 2$, then

$$cc(G) \leq \frac{2n-2}{3} + \frac{\mu(G)}{2}. \tag{6}$$

This result is best possible, in the sense that there exists a collection of planar graphs in which the bound in (6) is attained (see [4]).

2. Lemmas. We shall argue with a minimum counterexample, and so we need to take care of graphs with small orders.

Lemma 2.1 Let H be a 2-connected simple planar graph with $4 \leq |V(H)| \leq 5$. If $\delta(H) = 3$, then $H \in \{K_4, J_1, J_2\}$, (see Figure 1 for definition of J_1 and J_2).

Proof: This is trivial if $|V(H)| = 4$ and so we assume that $|V(H)| = 5$. By $\delta(H) = 3$, we have $2|E(H)| \geq 15$ and so $|E(H)| \geq 8$. Since H is a simple plane graph with 5 vertices, we have $|E(H)| \leq 9$, with equality if and only if H is a triangulation. Hence either G is a triangulation ($H \cong J_2$) or H is a triangulation minus an edge with $\delta(H) = 3$ ($H \cong J_1$). □

Lemma 2.2 Let H be a 2-edge-connected simple planar graph. Each of the following holds:

(i) If $|V(H)| = 4$ and if $e \in E(H)$ is given, then H has two cycles that covers H such that e can be covered at twice.

(ii) If $|V(H)| = 5$ and if $\delta(H) = 2$, or if $H = J_1$, then $cc(H) \leq 2$.

(iii) Let $H \in \{J_1, J_2\}$. If $e \in E(H)$ is given, then H has 3 cycles covering H such that e is covered at least twice.

(iv) Given $e \in E(J_2)$, J_2 has 3 cycles covering J_2 so that e is covered 3 times.

(v) Given $e_1, e_2 \in E(J_2)$, J_2 has 3 cycles covering J_2 so that e_1 and e_2 are covered at least twice.

Proof: We shall use the notation in Figure 1 in the proof. (i) of Lemma 2.2 is obvious. If $\delta(H) = 2$ in (ii), then one can contract an edge incident with a vertex of degree 2 to get (ii) from (i). Note that J_1 can be covered by $z_1z_2z_3z_5z_4z_1$ and $z_1z_4z_2z_5z_3z_1$. This shows that $cc(J_1) = 2$ and the case $H = J_1$ in (iii), since the edge e can be covered by the third cycle. The case $H = J_2$ in (iii) follows from (iv). By symmetry of J_2, we may assume that edge $e \in \{z_1z_4, z_2z_3, z_3z_5\}$. Table 1 below exhibits 3 desired cycles for (iv).

the edge e	C_1	C_2	C_3
z_1z_4	$z_1z_4z_5z_3z_1$	$z_1z_4z_2z_3z_1$	$z_1z_4z_3z_5z_2z_1$
z_2z_3	$z_1z_2z_3z_4z_1$	$z_1z_4z_5z_2z_3z_1$	$z_4z_5z_3z_2z_4$
z_3z_5	$z_1z_4z_5z_3z_1$	$z_1z_4z_3z_5z_2z_1$	$z_4z_5z_3z_2z_4$

Table 1

Let $e_1, e_2 \in E(J_2)$ be given. There is a cycle C_1 containing both e_1 and e_2, together with a third edge e_3. If e_3 is either incident with a vertex of degree 3 and a vertex of degree 4 (say $e_3 = z_4z_5$), or e_3 is incident with two vertices of degree 4 (say $e_3 - z_3z_4$), then in any case, $J_2 - e_3$ can be covered by two cycles and so (v) of Lemma 2.2 follows. □

Lemma 2.3 If $H \in \{L_8, L_8', L_8'', L_8'''\}$ (see Figure 4 for definitions), then $cc(H) \leq 4$.

Proof: We shall use the notation in Figure 4. The cycles $x_1x_2x_3x_5x_6x_4x_8x_7x_1$, $x_2x_5x_4x_7x_2$, $x_1x_7x_8x_2x_4x_1$ and $x_2x_4x_3x_5x_6x_2$ cover L_8; $x_1x_2x_3x_6x_4x_7x_8x_1$, $x_1x_7x_2x_5x_3x_4x_1$, $x_2x_4x_5x_2$ and $x_2x_5x_6x_4x_7x_8x_2$ cover L_8'; $x_1x_2x_3x_6x_4x_8x_7x_1$, $x_1x_7x_2x_5x_4x_1$, $x_2x_4x_3x_5x_2$ and $x_2x_9x_7x_4x_6x_5$ cover L_8''; and $x_1x_8x_7x_2x_5x_6x_3x_4x_1$, $x_1x_2x_3x_5x_6x_4x_8x_7x_1$, $x_2x_5x_4x_7x_2$ and $x_2x_5x_4x_2$ cover L_8'''. □

Lemma 2.4 If $H \in \{J_3, J_4\}$ (see Figure 3 for definitions), then $cc(H) \leq 3$.

Proof: We shall use the notation in Figure 3. The cycles $z_1z_2z_5u_iz_3z_1$, $z_1z_4z_5u_iz_2z_3z_1$ and $z_2z_5u_iz_3z_4z_2$ cover J_3; and the cycles $z_1z_4z_2z_5u_iz_3z_1$, $z_1z_2u_iz_3z_4z_1$ and $z_3z_4z_5z_3$ cover J_4. □

3. Proof of Theorem 1.2 From now on we assume that

$$G \text{ is a counterexample to Theorem 1.2} \tag{7}$$

such that

$$|V(G)| \text{ is minimized,} \tag{8}$$

and subject to (8),

$$\mu(G) \text{ is minimized.} \tag{9}$$

(iv) Given $e \in E(J_2)$, J_2 has 3 cycles covering J_2 so that e is covered 3 times.

(v) Given $e_1, e_2 \in E(J_2)$, J_2 has 3 cycles covering J_2 so that e_1 and e_2 are covered at least twice.

Proof: We shall use the notation in Figure 1 in the proof. (i) of Lemma 2.2 is obvious. If $\delta(H) = 2$ in (ii), then one can contract an edge incident with a vertex of degree 2 to get (ii) from (i). Note that J_1 can be covered by $z_1z_2z_3z_5z_4z_1$ and $z_1z_4z_2z_5z_3z_1$. This shows that $cc(J_1) = 2$ and the case $H = J_1$ in (iii), since the edge e can be covered by the third cycle. The case $H = J_2$ in (iii) follows from (iv). By symmetry of J_2, we may assume that edge $e \in \{z_1z_4, z_2z_3, z_3z_5\}$. Table 1 below exhibits 3 desired cycles for (iv).

the edge e	C_1	C_2	C_3
z_1z_4	$z_1z_4z_5z_3z_1$	$z_1z_4z_2z_3z_1$	$z_1z_4z_3z_5z_2z_1$
z_2z_3	$z_1z_2z_3z_4z_1$	$z_1z_4z_5z_2z_3z_1$	$z_4z_5z_3z_2z_4$
z_3z_5	$z_1z_4z_5z_3z_1$	$z_1z_4z_3z_5z_2z_1$	$z_4z_5z_3z_2z_4$

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Let $e_1, e_2 \in E(J_2)$ be given. There is a cycle C_1 containing both e_1 and e_2, together with a third edge e_3. If e_3 is either incident with a vertex of degree 3 and a vertex of degree 4 (say $e_3 = z_4z_5$), or e_3 is incident with two vertices of degree 4 (say $e_3 - z_3z_4$), then in any case, $J_2 - e_3$ can be covered by two cycles and so (v) of Lemma 2.2 follows. □

Lemma 2.3 If $H \in \{L_8, L_8', L_8'', L_8'''\}$ (see Figure 4 for definitions), then $cc(H) \leq 4$.

Proof: We shall use the notation in Figure 4. The cycles $x_1x_2x_3x_5x_6x_4x_8x_7x_1$, $x_2x_5x_4x_7x_2$, $x_1x_7x_8x_2x_4x_1$ and $x_2x_4x_3x_5x_6x_2$ cover L_8; $x_1x_2x_3x_6x_4x_7x_8x_1$, $x_1x_7x_2x_5x_3x_4x_1$, $x_2x_4x_5x_2$ and $x_2x_5x_6x_4x_7x_8x_2$ cover L_8'; $x_1x_2x_3x_6x_4x_8x_7x_1$, $x_1x_7x_2x_5x_4x_1$, $x_2x_4x_3x_5x_2$ and $x_2x_9x_7x_4x_6x_5x_2$ cover L_8''; and $x_1x_8x_7x_2x_5x_6x_3x_4x_1$, $x_1x_2x_3x_5x_6x_4x_8x_7x_1$, $x_2x_5x_4x_7x_2$ and $x_2x_5x_4x_2$ cover L_8'''. □

Lemma 2.4 If $H \in \{J_3, J_4\}$ (see Figure 3 for definitions), then $cc(H) \leq 3$.

Proof: We shall use the notation in Figure 3. The cycles $z_1z_2z_5u_iz_3z_1$, $z_1z_4z_5u_iz_2z_3z_1$ and $z_2z_5u_iz_3z_4z_2$ cover J_3; and the cycles $z_1z_4z_2z_5u_iz_3z_1$, $z_1z_2u_iz_3z_4z_1$ and $z_3z_4z_5z_3$ cover J_4. □

3. Proof of Theorem 1.2 From now on we assume that

$$G \text{ is a counterexample to Theorem 1.2} \tag{7}$$

such that

$$|V(G)| \text{ is minimized,} \tag{8}$$

and subject to (8),

$$\mu(G) \text{ is minimized.} \tag{9}$$

Lemma 3.1 $\mu(G) \leq 1$.

Proof: Suppose that $\mu(G) \geq 2$. Then either there is an edge e with $|[e]| \geq 3$ or there are two edges e_1, e_2 with $|[e_1]| \geq 2$, $|[e_2]| \geq 2$ and with $[e_1] \neq [e_2]$. In either case we can pick two edges e, e' (say), in such a way that $\mu(G - \{e, e'\}) = \mu(G) - 2$. By $\kappa(G) \geq 2$, there is a cycle C containing e and e'. By (9) and by the fact that any CC $\mathcal{C}$ of $G - \{e, e'\}$ together with the cycle C containing e and e' will form a CC of G, we have

$$cc(G) \leq cc(G - \{e, e'\}) + 1 \leq \frac{2n-2}{3} + \frac{\mu(G) - 2}{2} + 1,$$

contrary to (7). □

Lemma 3.2 G does not have any vertex v of degree 3 that is incident with some multiple edges.

Proof: We argue by contradiction. Suppose that G has a vertex of degree 3 that is incident with some multiple edges. Let e_1, e_2, e_3 be the edges incident with v. By $\kappa(G) \geq 2$ and by Lemma 3.1, we may assume that $[e_1] = \{e_1, e_2\}$ and $[e_3] = \{e_3\}$. Let the two vertices adjacent to v be u, u' such that u is incident with e_3 and u' with e_1 and e_2. Define

$$G' = \begin{cases} G - v + uu' & \text{if } uu' \notin E(G) \\ G - v & \text{if } uu' \in E(G) \end{cases} \tag{10}$$

It is then easy to see that

$$cc(G) \leq cc(G') + 1. \tag{11}$$

In fact, let $\mathcal{C}'$ be a CC of G' and let $C' \in \mathcal{C}'$ be a cycle that contains the edge uu'. Since every cycle in $\mathcal{C}' - \{C'\}$ can be extended to a cycle in G by possibly replacing uu' by e_1, e_3, we shall use $\mathcal{C}$ to denote the collection of cycles in G corresponding to the cycles in $\mathcal{C}' - \{C'\}$. Let $C_1 = C' - uu' + \{e_1, e_3\}$. If $uu' \notin E(G)$, then let $C_2 = G[\{e_1, e_2\}]$, and if $uu' \in E(G)$, then let $C_2 = G[\{uu', e_2, e_3\}]$. Thus $\mathcal{C} \cup \{C_1, C_2\}$ would form a CC for G in either case and so (11) holds.

Since $\kappa(G) \geq 2$ and by (10), $\kappa(G') \geq 2$ also. Note that $\mu(G') = \mu(G) - 1$. If $|V(G')| \geq 6$, then by the minimality of G, we have

$$cc(G) \leq cc(G') + 1 \leq \frac{2|V(G')| - 2}{3} + \frac{\mu(G')}{2} + 1 \leq \frac{2|V(G)| - 2}{3} + \frac{\mu(G)}{2},$$

contrary to (7). Thus we assume that $|V(G')| = 5$ and so $|V(G)| = 6$.

Since the edges in $[e_1]$ are deleted, by Lemma 3.1, G' is simple with $\delta(G') \geq 2$. If $\delta(G') = 2$, or if $G' = K_4$, then by (i) or (ii) of Lemma 2.2, $cc(G') = 2$ and so by (11), $cc(G) \leq 3$, contrary to (7).

Hence by Lemma 2.1 we assume that $G' \in \{J_1, J_2\}$. If $uu' \notin E(G)$, then by (iii) of Lemma 2.2, G' can have a CC $\mathcal{C}$ with 3 cycles such that the edge uu' can be covered by 2 cycles, whence $cc(G) = 3$, contrary to (7). If $G' \cong J_1$ and $uu' \in E(G)$, then since $cc(J_1) = 2$, it follows by (11) that $cc(G) \leq 3$ also, contrary to (7). Thus we must have $G' \cong J_2$ and $uu' \in E(G)$. By the symmetry of J_2, either both u, u' are of degree 4 in J_2

(say $\{u, u'\} = \{z_2, z_3\}$), or one of $\{u, u'\}$ has degree 4 and the other has degree 3 in J_2 (say $\{u, u'\} = \{z_1, z_2\}$). Table 2 below uses the notation in Figure 1 and shows that $cc(G) \leq 3$ in any case.

u	u'	C_1	C_2	C_3
z_1	z_2	$z_1vz_2z_3z_5z_4z_1$	$z_1vz_2z_4z_3z_1$	$z_1z_2z_5z_4z_1$
z_2	z_1	$z_1vz_2z_3z_5z_4z_1$	$z_1vz_2z_4z_3z_1$	$z_1z_2z_5z_4z_1$
z_2	z_3	$z_1z_2vz_3z_5z_4z_1$	$z_1z_3vz_2z_4z_1$	$z_4z_5z_2z_3z_4$

Table 2

Thus G satisfies (6) in any case, contrary to (7). □

Lemma 3.3 If $w \in V(G)$ with $N(w) = \{w_1, w_2, w_3\}$ and if $w_1w_3 \notin E(G)$, then $G[N(w)]$ is disconnected.

Proof: By contradiction, we may assume that $w_1w_2, w_2w_3 \in E(G)$. By Lemma 3.1 and without loss of generality, we may assume that $|[w_1w_2]| = |[ww_3]| = 1$. Let $G' = G/ww_3$ and let $\mathcal{C}$ be a CC of G'. Denote $e_1 = w_2w_3$, $e_2 = ww_2$ and $e_3 = w_1w_2$, and let $C_i \in \mathcal{C}$ be cycles containing e_i, $(1 \leq i \leq 3)$. Note that e_1 and e_2 become edges with the same ends in G', and so we may assume that $C_1 \neq C_3$.

If $C_1 = C_2$ is a 2-cycle, then let $C_3' = C_3 - e_3 + w_1ww_2$ and $F'' = w_1ww_3w_2w_1$, and extend any cycle $L \in \mathcal{C} - \{C_1, C_3\}$ to a cycle L' in G by adding ww_3, if necessary. Thus $\{L' | L \in \mathcal{C} - \{C_1\}\} \cup \{F''\}$ is a CC of G. Thus we may assume that neither C_1 nor C_2 is a 2-cycle. In Table 3, C_1' and C_2' are defined according to the different situations of $C_1 - e_1$ and $C_2 - e_2$ in G.

$C_1 - e_1$ in G	$C_2 - e_2$ in G	The new cycle C_1'	The new cycle C_2'
(w_2, w_3)-path	(w_2, w_3)-path	C_1	$C_2 - e_2 + w_2ww_3$
(w_2, w_3)-path	(w_2, w)-path	$C_1 - e_1 + w_2ww_3$	$C_2 - e_2 + ww_3w_2$
(w_2, w)-path	(w_2, w_3)-path	$C_1 - e_1 + ww_3w_2$	$C_2 - e_2 + w_2ww_3$
(w_2, w)-path	(w_2, w)-path	$C_1 - e_1 + ww_3w_2$	C_2

Table 3

Extend any cycle $C \in \mathcal{C} - \{C_1, C_2\}$ to a cycle C' in G. Thus $\{C' | C \in \mathcal{C}\}$ gives

$$cc(G) \leq cc(G'). \tag{12}$$

Since $N(w) = \{w_1, w_2, w_3\}$ and by $\kappa(G) \geq 2$, $\{w, w_3\}$ is not a vertex cut of G and so $\kappa(G') \geq 2$. By (8) and (12), if $|V(G')| \geq 6$, then G satisfies (6), contrary to (7). Since $|V(G)| \geq 6$, we must have $|V(G')| = 5$ and so G' is spanned by J_1 or J_2. By (iii) of Lemma 2.2 and by (12), G satisfies (6) also, contrary to (7). □

If S is a vertex cut of a connected graph H and if the components of $H - S$ have vertex sets $V_1, V_2, \cdots, V_p$, then $H[V_i \cup S]$ $(1 \leq i \leq p)$ is called an S-component of H.

Lemma 3.4 G does not have a vertex cut $\{u_1, u_2\}$ with $u_1u_2 \in E(G)$ such that G has a $\{u_1, u_2\}$-component of at most 3 vertices.

Proof: Suppose such an edge u_1u_2 exists. Let L' be a $\{u_1, u_2\}$-components of G with $V(L) = \{u_1, u_2, \cdots, u_k\}$ and with $k \leq 5$. Let $L = L'$ if $|[u_1u_2]| = 1$ and L be the underlying simple graph of L' if $|[u_1u_2]| = 2$. Throughout the proof, let $e \notin E(G)$ be an edge parallel to u_1u_2.

Suppose first that $k = 3$. By Lemma 3.2, $d(u_3) = 2$. Let $G_1 = (G - u_3) + e$. Then $cc(G) \leq cc(G_1)$ since the edge e can be replaced by the path $u_1u_3u_2$ in any cycle containing e. By the minimality of G, we may assume that $|V(G_1)| = 5$. But then $|V(G)| = 6$, and so by (ii) or (iii) of Lemma 2.2, $cc(G) \leq cc(G_1) \leq 3$, contrary to (7). Thus $k \neq 3$.

Since we did not use the fact that $u_1u_2 \in E(G)$, we have in fact shown:

Corollary 3.5 $\delta(G) \geq 3$. □

We continue the proof of Lemma 3.4 and assume that $k = 4$. Let

$$G_2 = \begin{cases} G - \{u_3, u_4\} & \text{if } \mu(L) = 0 \\ G - \{u_3, u_4\} + e & \text{if } \mu(L) = 1. \end{cases}$$

Since $u_1u_2 \in E(G)$, $\kappa(G_2) \geq 2$. By $\delta(G) \geq 3$ and by Lemma 3.2, L must be spanned by a K_4. We claim that $cc(G) \leq cc(G_2) + 1$. When $\mu(L) = 0$, let $\mathcal{C}$ be a CC of G_2 and let $C \in \mathcal{C}$ be a cycle in G_2 containing u_1u_2. Since $C - u_1u_2 + u_1u_3u_4u_2$ and $u_1u_2u_3u_4u_1$ are two cycles in G, it follows that $cc(G) \leq cc(G_2) + 1$. The case when $\mu(L) = 1$ is similar. By (8), if $|V(G_2)| \geq 6$, then

$$cc(G) \leq cc(G_2) + 1 \leq \frac{2(n-2) - 2}{3} + \frac{\mu(G)}{2} + 1 < \frac{2n - 2}{3} + \frac{\mu(G)}{2},$$

contrary to (7). Thus $|V(G_2)| \leq 5$. If $|V(G_2)| = 4$, then $|V(G)| = 6$ and by (i) of Lemma 2.2, $cc(G) \leq cc(G_2) + 1 \leq 3$, contrary to (7). If $|V(G_2)| = 5$, then $|V(G)| = 7$ and by (ii) or (iii) of Lemma 2.2, $cc(G) \leq cc(G_2) + 1 \leq 4$, contrary to (7) also. This excludes $k = 4$.

Assume that $k = 5$. Let $G_3 = G - \{u_3, u_4, u_5\}$. Let $\mathcal{C}$ be a CC of G_3 and let $\{C_1, C_2, C_3\}$ be a CC of L so that u_1u_2 is covered by C_1 and C_2. Let $C \in \mathcal{C}$ be a cycle containing u_1u_2. Then $(\mathcal{C} - \{C\}) \cup \{C_2, C_3, G[E(C) \cup E(C_1) - \{e\}]\}$ is a CC of G and so $cc(G) \leq cc(G_3) + cc(L) - 1$. By the minimality of G, we may assume $|V(G_3)| \leq 5$. But if $|V(G_3)| < 5$, then we are back to the cases of $3 \leq k \leq 4$. Therefore we assume that $|V(G_3)| = |V(L)| = 5$ and so $|V(G)| = 8$. If $cc(G_3) = 2$ or $cc(L) = 2$, then by (ii) or (iii) of Lemma 2.2 and by $cc(G) \leq cc(L) + cc(G_3) - 1$, we have $cc(G) \leq 4$, contrary to (7). If $\mu(G) = 1$, then by the same reasons, $cc(G) \leq 5$, contrary to (7) again. Thus we have $\mu(G) = 0$ and $G_3 \cong L \cong J_2$. By the symmetry of J_2, $G \in \{L_8, L_8', L_8'', L_8'''\}$, and so by Lemma 2.3, G satisfies (6), contrary to (7). □

Fix a planar embedding of G. For a cycle C in G, $IntC$, called the interior of C, denotes the vertices of G lying inside C, excluding $V(C)$. The exterior of C is defined similarly. A cycle C of a plane graph G is trivial if $IntC = \emptyset$. A vertex $v \in V(G)$ is cyclic if $G[N(v)]$ is spanned by an m-cycle C_m with $m = deg(v)$. This cycle C_m is called the rim cycle of v. It is clear that if every vertex of G is cyclic, then G has a triangulation planar embedding.

For any vertex v in a simple plane graph G, the vertices in $N(v)$ can be viewed as an ordered string $< v_1, v_2, \cdots, v_m >$ such that vv_i and vv_{i+1} are incident with the same face of G, $i = 1, 2, \cdots, m$, $(mod\ m)$. We call this string an ordered neighborhood of v in G. (The definition for multigraphs is similar.) Note that for each $v \in V(G)$, one can have two different ordered neighborhoods: the clockwise one and the anticlockwise one.

If $v \in V(G)$ is not a cyclic vertex, then for any plane embedding of G, there are two vertices $u, u' \in N(v)$ such that vu and vu' are incident with the same face but $uu' \notin E(G)$. We called u, u' a bad pair in $N(v)$.

Lemma 3.6 Let G be a connected planar graph satisfying (7), (8) and (9). Then G has no cyclic vertices.

Proof: We shall show by contradiction that if $v \in V(G)$ is a cyclic vertex, then every vertex in $N(v)$ is also a cyclic vertex. Then by the connectedness of G, every vertex of G is cyclic and so G is a triangulation. A contradiction follows from Theorem 1.1.

Fix an embedding of G such that the interior of the rim cycle of v consists of v only. Let $N(v) =< v_1, v_2, ..., v_m >$ be an ordered neighborhood of v in G. By contradiction, we assume that v_1 is not cyclic. Let $N(v_1) = \{v_1', v_1'', ...\}$ be the neighborhood of v_1 in G. Since v_1 is not cyclic, we may assume that $v_1', v_1'' \in N(v_1)$ such that

$$v_1' \text{ and } v_1'' \text{ are a bad pair in } N(v_1). \tag{13}$$

Claim 1: $\{v_1', v_1''\} \neq \{v_2, v_m\}$.

In fact, if $\{v_1', v_1''\} = \{v_2, v_m\}$, then since $v_1'v_1'' \notin E(G)$ and since the interior of the rim cycle of v consists of v only, we have $|N(v_1)| = 3$, contrary to Lemma 3.3.

Therefore, we assume from now on that one can choose v, v_1 and v_1' such that $v_1' \notin \{v_2, v_m\}$.

Claim 2: One can choose v, v_1 and v_1' so that $v_1' \notin N(v)$.

Suppose that $v_1'v \in E(G)$. Since $v_1' \notin \{v_2, v_m\}$, we may assume that v_2 is in the interior of $C_v = vv_1v_1'v$ and v_1'' is in the exterior of C_v. For each cyclic vertex v with a noncyclic neighbor v_1, we choose v_1 so that

$$|IntC_v| \text{ is minimized,} \tag{14}$$

subject to the condition that C_v separates v_2 and v_1''.

If v_2 is a cyclic vertex, then since v_1'' lies in $ExtC_v$, $v_1'' \notin N(v_2)$. Thus one can replace v, v_1, v_1' by v_2, v_1, v_1'', respectively, and so Claim 2 holds.

If v_2 is not a cyclic vertex, then there are two vertices v_2', v_2'' that are a bad pair in $N(v_2)$. If $v_2' \notin N(v)$, then v, v_1, v_1' can be replaced by v, v_2, v_2'. Suppose that $v_2' \in N(v)$. Let $C_v' = vv_2v_2'v$. Then we may assume that $IntC_v' = \emptyset$. For otherwise C_v' separates a vertex in $N(v_2)$ and v_2'', violating (14).

If $v_2'' = v_1$, then $d_G(v_2) = 3$ and $v_1v_2' \notin E(G)$, violating Lemma 3.3. Thus v_2'' must be a vertex in the interior of the cycle $vv_2'v_2v_1v_1'v$ and so $v_2'' \notin N(v)$. Hence we can replace v, v_1, v_1' by v, v_2, v_2'' and so Claim 2 holds.

Claim 3: One can choose v, v_1 and v_1' so that $v_1' \notin N(v)$ and $v_1'v_m \notin E(G)$ or $v_1'v_2 \notin E(G)$.

By Claim 2, we can have v, v_1 and v_1' so that $v_1' \notin N(v)$. If $v_1'v_2, v_1'v_m \in E(G)$, then by (13), $v_1'' \notin \{v_2, v_m\}$ and without loss of generality, we may assume that v_1'' is in the interior of $vv_1v_1'v_2v$. Thus $v_1'' \notin N(v)$ and $v_1''v_m \notin E(G)$, and so one can replace v, v_1, v_1' by v, v_1, v_1'' to establish Claim 3.

By Claim 3, we can choose a cyclic vertex v, and a noncyclic vertex $v_1 \in N(v)$ with ordered neighborhoods

$$N(v) = < v_1, \cdots, v_m > \text{ and } N(v_1) = < u_1, \cdots, u_s >,$$

such that $u_1 = v$, $u_2 = v_2$ and

$$u_ju_{j+1} \in E(G),\ (1 \le j \le i-1),\ u_i \notin N(v), u_iu_{i+1}, u_iv_m \notin E(G). \tag{15}$$

In other words, u_i, u_{i+1} are a bad pair. Let P denote the (u_2, u_{i-1})-path $u_2u_3 \cdots u_{i-1}$. (Note that P may consist of a single vertex $u_2 = v_2$ if $u_{i-1} = v_2$).

Claim 4 $\{v_1, u_i\}$ is not a vertex cut of G.

Proof of Claim 4: Suppose that $\{v_1, u_i\}$ is a vertex cut of G. Let L_1 be the $\{v_1, u_i\}$-component of G containing v. Note that v_2, v_m, u_{i-1} are also in $V(L_1)$. Let L_2 be the union of other $\{v_1, u_i\}$-components of G. For convenience, we assume that L_1 always contains only one edge in $[v_1u_i]$, even when $|[v_1u_i]| = 2$. Thus

$$\mu(L_1) + \mu(L_2) = \mu(G). \tag{16}$$

Let $L_1' = (L_1 - u_iu_{i-1})/\{v_1u_i\}$ and denote by v' the vertex in L_1' to which v_1u_{i-1} is contracted. Since $v', v, u_{i-1}, u_{i-2} \in V(L_1')$ and since $u_iv_1 \in E(L_2)$, both $\kappa(L_1') \ge 2$ and $\kappa(L_2) \ge 2$. We claim that

$$cc(G) \le cc(L_1') + cc(L_2). \tag{17}$$

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two CC's of L_1' and L_2, respectively. Let $C_1 \in \mathcal{C}_1$ be a cycle containing $v'u_{i-1}$ and let $C_2 \in \mathcal{C}_2$ be a cycle containing v_1u_i. Let $C_1' = C_1 - v'u_{i-1} + v_1u_iu_{i-1}$ (if $C_1 - v'u_{i-1}$ is a (v_1, u_{i-1})-path) or $C_1' = C_1 - v'u_{i-1} + u_{i-1}v_1u_i$ (if $C_1 - v'u_{i-1}$ is a (u_i, u_{i-1})-path), and let $C_2' = C_2 - v_1u_i + v_1u_{i-1}u_i$. Then $(\mathcal{C}_1 - \{C_1\}) \cup (\mathcal{C}_2 - \{C_2\}) \cup \{C_1', C_2'\}$ is a CC of G, and so (17) holds.

By Lemma 3.4, $|V(L_1)| \geq 6$ and $|V(L_2)| \geq 6$. If $|V(L_1')| \geq 6$, or if $|V(L_1')| = 5$ and $cc(L_1') = 2$, then by (17) and by (8), G satisfies (6), contrary to (7). Hence we may assume that L_1' is spanned by J_2. If $\mu(L_1) = 1$, then by (16), $\mu(L_2)$ is simple, and so by (17) and by Lemma 2.2,

$$cc(G) \leq cc(L_2) + cc(L_1') \leq \frac{2(n-4)-2}{3} + 3 < \frac{2n-2}{3} + \frac{1}{2},$$

contrary to (7). Thus $\mu(L_1) = 0$. Then by (v) of Lemma 2.2, L_1' can be covered by 3 cycles so that $v'u_{i-1}$ is covered 3 times. It follows that L_1 can have 3 cycles C_1', C_1'' and C_1''' (say) so that v_1u_i is covered by C_1' and C_1''. Let C_2 be a CC of L_2 with $C_2 \in \mathcal{C}_2$ so that $v_1u_i \in E(C_2)$. Then $(\mathcal{C}_2 - \{C_2\}) \cup \{C_1', C_1''', G[E(C_1'') \cup E(C_2) - \{v_1u_i\}]\}$ is a CC of G with at most $(2(n-4)-2)/3 + \mu(G)/2 + 2 < (2n-2)/3 + \mu(G)/2$ cycles, contrary to (7). This proves Claim 4.

<u>Case 1</u> Both $\|[vv_1]\| = \|[v_1u_i]\| = 1$.

Define $G_1 = (G - \{v_1v_2, u_{i-1}u_i, v_1v_m\})/\{v_1v, v_1u_i\}$ (see Figure 2). We shall show

$$cc(G) \leq cc(G_1) + 1. \tag{18}$$

Let v' to denote the vertex in G_1 to which v_1v and v_1u_i are contracted. By Claim 4, v' is not a cut vertex of G and so $\kappa(G_1) \geq 2$. Let $\mathcal{C}$ be a CC of G_1 and let $C_1, C_2, C_3 \in \mathcal{C}$ such that $v'v_m \in C_1$, $v'v_2 \in C_2$ and $v'u_{i-1} \in C_3$, and such that $C_l = C_j$ whenever C_l contains 2 edges in $\{v'v_m, v'v_2, v'u_{i-1}\}$ that are assumed to be in C_l and C_j, respectively. Denote

$$C_1'' = G[E(C_1) - \{v'v_m\}],\ C_2'' = G[E(C_2) - \{v'v_2\}],\ C_3'' = G[E(C_3) - \{v'u_{i-1}\}]. \tag{19}$$

Note that any cycle L in $\mathcal{C} - \{C_1, C_2, C_3\}$ can be extended to a cycle L' in G by adding edges in $\{v_1v_2, u_{i-1}u_i, v_1v_m, v_1v, v_1u_i\}$, if necessary. In each of the subcases below, we shall exhibit a CC $\{L' : L \in \mathcal{C}\} \cup \{F\}$ of G and so (18) follows.

<u>Case 1A</u> $|E(C_j) \cap \{v'v_m, v'v_2, v'u_{i-1}\}| = 1$, $(1 \leq j \leq 3)$.

Then C_1'' is a (v_m, w_1)-path, C_2'' is a (v_2, w_2)-path and C_3'' is a (u_{i-1}, w_3)-path, for some $w_j \in \{v, v_1, u_i\}$, $(1 \leq j \leq 3)$.

If $w_1 = v$ in G, then set $C_1' = C_1'' + v_mv_1v$ and extend C_2'', C_3'' to cycles C_2' and C_3' in G, respectively, so that $v_1v_2 \in E(C_2)$ and $v_1u_i \in E(C_3)$. Note that since C_3'' is a (u_{i-1}, w_3)-path, either v_1u_{i-1} or $u_{i-1}u_i$ is in $E(C_3')$. Let F be a cycle in G that contains P, v_mv, vv_2 and either v_1u_{i-1} or $u_{i-1}u_i$, depending on whether $u_{i-1}u_i \in E(C_3')$ or $v_1u_{i-1} \in E(C_3')$, respectively.

If $w_1 = v_1$ or $w_1 = u_i$, then let $C_1' = C_1'' + v_mvv_1$ or $C_1' = C_1'' + v_mvv_1u_i$, respectively. Extend C_2'', C_3'' to cycles C_2' and C_3' in G, respectively, so that $v_1v_2 \in E(C_2)$ and $v_1u_i \in E(C_3)$. Note again that either v_1u_{i-1} or $u_{i-1}u_i$ is in $E(C_3')$. Let F be a cycle in $G[\{v\} \cup N(v)]$ that contains P, v_mv, vv_2 and either v_1u_{i-1} or $u_{i-1}u_i$, depending on whether $u_{i-1}u_i \in E(C_3')$ or $v_1u_{i-1} \in E(C_3')$, respectively. (See Table 5 at the end for details).

Case 1B: $C_1 = C_2$. Then $C_1 \neq C_3$ and C_1'' is a (v_m, v_2)-path in G.

Let $C_1' = C_1'' + v_m v v_1 v_2$ and extend C_3'' to a cycle C_3' in G so that $v_1 u_i \in E(C_3')$. Note that either $v_1 u_{i-1}$ or $u_{i-1} u_i$ is in $E(C_3')$. Let

$$F = \begin{cases} P + v_2 v v_m v_1 u_{i-1} & \text{if } u_i u_{i-1} \in E(C_3') \\ P + v_2 v v_m v_1 u_i u_{i-1} & \text{if } u_i u_{i-1} \notin E(C_3'). \end{cases}$$

Case 1C: $C_1 = C_3$. Then $C_1 \neq C_2$ and C_1'' is a (v_m, u_{i-1})-path in G.

Let $C_1' = C_1'' + v_m v v_1 u_i u_{i-1}$ and extend C_2'' to a cycle C_2' in G so that $v_1 v_2 \in E(C_2')$. Let $F = P + v_2 v v_m v_1 u_{i-1}$.

Case 1D: $C_2 = C_3$. Then $C_2 \neq C_1$ and C_2'' is a (v_2, u_{i-1})-path in G.

Let $C_2' = C_2'' + v_2 v_1 u_i u_{i-1}$ and extend C_1'' to a cycle C_1' in G so that $v_1 v \in E(C_1')$. Let $F = P + v_2 v_m v_1 u_{i-1}$.

By (18) and by (8), we may assume that $|V(G_1)| \leq 5$. Since $|V(G)| \geq 6$, we have $|V(G_1)| \geq 4$. By Corollary 3.5, $\delta(G_1) \geq 3$. If $cc(G_1) \leq 2$, then by (18), $cc(G) \leq 3$, contrary to (7). Hence we may assume that G_1 is spanned by J_2. By (iii) of Lemma 2.2, $cc(G_1) \leq 3$ and so by (18), $cc(G) \leq 4$. Since $n = |V(G_1)| + 2 = 7$, G satisfies (6), contrary to (7).

Case 2 $|[vv_1]| = 2$.

Let $G_2 = (G - \{v_1 v_m, v_1 v_2\})/\{vv_1\}$. Let v' denote the vertex in G_2 to which v and v_1 are contracted, and let $\mathcal{C}$ be a CC of G_2. We shall show that

$$cc(G) \leq cc(G_2) + 1. \tag{20}$$

Let $C_1, C_2 \in \mathcal{C}$ be two cycles such that $v' v_m \in E(C_1)$ and $v' v_2 \in E(C_2)$. If $C_1 = C_2$, then let $C' = C_1 - \{v' v_m, v' v_2\} + v_m v v_1 v_2$ and $F = v v_2 v_3 \cdots v_m v_1 v$. Note that both edges in $[vv_1]$ are covered by C' and F. Thus $(\mathcal{C} - \{C_1\}) \cup \{C', F\}$ is a CC of G, and so (20) holds.

If $C_1 \neq C_2$, then $C_i'' = C_i - \{v' v_m, v' v_2\}$ is a (v_m, w_i)-path in G for some $w_i \in \{v, v_1\}$, $(1 \leq i \leq 2)$. Table 4 defines C_1' and C_2' and F so that $(\mathcal{C} - \{C_1'', C_2''\}) \cup \{C_1', C_2'\}$ is a CC of G.

w_1	w_2	C_1'	C_2'	F
v	v	$C_1'' + v_m v_1 v$	$C_2'' + v_2 v$	$v v_1 v_2 \cdots v_m v$
v	v_1	$C_1'' + v_m v_1 v$	$C_2'' + v_2 v v_1$	$v v_1 v_2 \cdots v_m v$
v_1	v	$C_1'' + v_m v v_1$	$C_2'' + v_2 v_1 v$	$v v_2 \cdots v_m v_1 v$
v_1	v_1	$C_1'' + v_m v v_1$	$C_2'' + v_2 v_1$	$v v_2 \cdots v_m v_1 v$

Table 4

Note that both edges in $[vv_1]$ are covered by two of C_1', C_2' and F, and so (20) holds.

Since the rim cycle of v consists of v only, we have $\kappa(G_2) \geq 2$. By $|[vv_1]| = 2$ and by Lemma 3.1, G_2 is simple. If $n \geq 7$, then by (8) and (20),

$$cc(G) \leq cc(G_2) + 1 \leq \frac{2(n-1)-2}{3} + 1 < \frac{2n-2}{3} + \frac{\mu(G)}{2},$$

contrary to (7). Hence we assume that $n = 6$ and so $|V(G')| = 5$. By (15), G_2 is not a triangulation and so $G_2 \neq J_2$, which implies by Lemma 2.2 that $cc(G_2) = 2$. Thus by (20), $cc(G) \leq 3$, contrary to (7).

Case 3 $|[v_1u_i]| = 2$.

Let $G_3 = (G - u_iu_{i-1})/\{v_1u_i\}$ and let v' denote the vertex in G_3 to which v_1u_i is contracted. We shall show that

$$cc(G) \leq cc(G_3) + 1 \tag{21}$$

Let $\mathcal{C}$ be a CC of G_3 and let $C \in \mathcal{C}$ be a cycle with $v'u_{i-1} \in E(C)$. Note that $C'' = C - v'u_{i-1}$ is a (u_{i-1}, w)-path in G for $w \in \{v_1, u_i\}$. Let $F = v_1u_iu_{i-1}v_1$. If $w = v_1$, then set $C' = C'' + u_{i-1}u_iv_1$, and if $w = u_i$, then set $C' = C'' + u_{i-1}v_1u_i$. Thus $(\mathcal{C} - \{C\}) \cup \{C', F\}$ is a CC of G and so (21) must hold.

By Claim 4, $\kappa(G_3) \geq 2$. By (21) and by (7), we may again assume that $G_3 \cong J_2$. By $|[v_1u_i]| = 2$ and by Lemma 3.2, $d_G(u_i) \geq 4$ and so $d_{G_3}(v') \geq 4$ also. It follows that $v_2 = u_{i-1}$ in G and so $v' \in \{z_2, z_3, z_4\}$ in J_2 (using the notation of Figure 1). By the symmetry of J_2, we may assume $v' = z_3$. It follows that either $G \cong J_3$ with $|[u_iz_3]| = 2$ ($v = z_1, v_1 = z_3, v_2 = z_2, v_m = z_4$), or $G \cong J_4$ with $|[u_iz_3]| = 2$ ($v = z_4, v_1 = z_3, v_2 = z_5, v_m = z_1$). By Lemma 2.4, G satisfies (6), contrary to (7). □

Lemma 3.7 G does not have a nontrivial 4-cycle C with $|IntC| = 1$.

Proof: By contradiction, assume that $C = v_1v_2v_3v_4v_1$ is a 4-cycle in G with $IntC = \{v\}$. Thus $N(v) \subseteq V(C)$. By Lemma 3.6, $N(v) \neq V(C)$. If $|N(v)| = 2$, then by Corollary 3.5, v must have degree 3, contrary to Lemma 3.2. If $|N(v)| = 3$, then by Lemma 3.6, v is not a cyclic vertex and so Lemma 3.3 must be violated. □

Lemma 3.8 If Γ_3 is a nontrivial 3-cycle of G such that

$$|Int\Gamma_3| \text{ is minimized.} \tag{22}$$

then each holds:

(i) $|Int\Gamma_3| > 1$;

(ii) for any $v \in Int\Gamma_3$ and for any consecutive bad pair $v_1, v_2 \in N(v)$ (vv_1 and vv_2 are incident with the same face), neither vv_1 nor vv_2 lies in a 3-cycle of G.

Proof: If $|Int\Gamma_3| = 1$, then either Lemma 3.6 or Lemma 3.3 would be violated, and so (i) of Lemma 3.8 follows.

We shall show (ii) by contradiction. Suppose that vv_1v_3v is a 3-cycle. Let $e \notin E(G)$ be an edge parallel to vv_3. Since $v \in Int\Gamma_3$ and by (22), vv_1v_3v must be a trivial 3-cycle. By

Lemma 3.1, $|[vv_1]| \leq 2$ in G. Let

$$G_4 = \begin{cases} G/vv_1 & \text{if } |[vv_1]| = 1 \\ (G+e)/vv_1 & \text{if } |[vv_2]| = 2 \end{cases}$$

We shall show $cc(G) \leq cc(G_4)$. Let v' denote the vertex in G_4 to which vv_1 is contracted and let C' be a CC of G_4. We consider two cases.

Case 1: $|[vv_1]| = 2$ in G. Then $|[v'v_3]| = 3$ in G_4 and we may assume that $[v'v_3] = \{e_1, e_2, e_3\}$. Let $C_i' \in C'$ be a cycle containing e_i, $(1 \leq i \leq 3)$. One can adjust the cycles in $C' - \{C_1', C_2', C_3'\}$ to cycles in G, and denote this family of cycles in G by C.

Suppose that $C_1' = C_2' = G_4[\{e_1, e_2\}] \in C'$. Note that $C_3' - e_3$ is a (v_3, w)-path, with $w \in \{v, v_1\}$. If $w = v$, then set $C_3 = C_3' - e_3 + v_3v_1v$, and if $w = v_1$, then set $C_3 = C_3' - e_3 + v_3vv_1$. In any case, $C \cup \{C_3\}$ is a CC of G and so $cc(G) \leq cc(G_4)$.

Hence we may assume that the C_i's are distinct. Note that $C_i' - e_i$ is a (v_3, w_i)-path with $w_i \in \{v, v_1\}$, $(1 \leq i \leq 3)$. If $w_i \neq w_j$ for some i and j, say $w_1 = v$ and $w_2 = v_1$, then set $C_1 = C_1' - e_1 + v_3vv_1$, $C_2 = C_2' - e_2 + v_3v_1v$, and C_3 be a cycle in G obtained by extending $C_3' - e_3$. If $w_i = v$, $(1 \leq i \leq 3)$, then set $C_1 = C_1' - e_1 + v_3v$, $C_2 = C_2' - e_2 + v_3v_1v$ and $C_3 = C_3' - e_3 + v_3vv_1$. If $w_i = v_1$, $(1 \leq i \leq 3)$, then set $C_1 = C_1' - e_1 + v_3v_1$, $C_2 = C_2' - e_2 + v_3vv_1$ and $C_3 = C_3' - e_3 + v_3vv_1$. In any case, $C \cup \{C_1, C_2, C_3\}$ is a CC of G and so $cc(G) \leq cc(G_4)$.

Since G is plane, vv_1 is incident with two faces, one being the 3-cycle vv_1v_3v and the other being $v_1vv_2 \cdots v_1$, which is a cycle of length at least 4. Therefore $\mu(G_4) = \mu(G) + 1$, and so if $n \geq 7$, then by (8),

$$cc(G) \leq cc(G_4) \leq \frac{2(n-1)-2}{3} + \frac{\mu(G)+1}{2},$$

contrary to (7). Thus we may assume that $n = 6$ and $|V(G_4)| = 5$. Then by (ii) or (iii) of Lemma 2.2, $cc(G) \leq cc(G_4) \leq 3$, contrary to (7).

Case 2: $|[vv_1]| = 1$. The proof is similar to and simpler than that of Case 1, and so it is omitted. □

Since the minimality of Γ_3 is used in the proof of Lemma 3.8 only to guarantee that vv_1v_3v is a trivial 3-cycle (and so $\mu(G_4) \leq \mu(G) + 1$), we have also proved:

Corollary 3.9: If G has no nontrivial 3-cycles, then for any $v \in V(G)$ and for any consecutive bad pair $v_1, v_2 \in N(v)$, neither vv_1 nor vv_2 lies in a 3-cycle of G.

Lemma 3.10 If Γ_3 is a nontrivial 3-cycle of G such that (22) holds, then $G[V(\Gamma_3) \cup Int\Gamma_3]$ does not contain a nontrivial 4-cycle.

Proof: By contradiction, we choose a nontrivial 4-cycle C with

$$|IntC| \text{ is minimized.} \qquad (23)$$

Let $C = u_1u_2u_3u_4u_1$ and let $v \in IntC$. Let $N(v) =< v_1, v_2, \cdots, v_m >$ be an ordered neighborhood of v. By Lemma 3.6, we may assume that $v_1v_2 \notin E(G)$, and so at most one of v_1 and v_2 is in $V(\Gamma_3)$. Note that

$$|\{v_1, v_2\} \cap V(C)| \leq 1. \tag{24}$$

For if $v_1, v_2 \in V(C)$, then since $v_1v_2 \notin E(G)$, we may assume that $v_1 = u_1$ and $v_2 = u_3$. By Lemma 3.7, $|IntC| > 1$ and so one of the 4-cycles $vu_1u_2u_3v$ and $vu_1u_4u_3v$ is nontrivial, contrary to (23). Similarly, by (23) and by Lemma 3.7, we have,

$$|N(v_1) \cap N(v_2) - \{v\}| \leq 1. \tag{25}$$

Define

$$G_5 = \begin{cases} G/\{v_1v, v_2v\} & \text{if } v_1vv_2 \text{ does not lie in a 4-cycle of } G \\ (G - v_1w)/\{v_1v, vv_2\} & \text{if for some } w \in V(G),\ wv_1vv_2w \text{ is a 4-cycle} \end{cases}$$

and we shall show that

$$cc(G) \leq cc(G_5) + 1. \tag{26}$$

By Lemma 3.8, neither vv_1 nor vv_2 lies in a 3-cycle. This, together with (25), implies that no new multiple edges will be created in getting G_5, and so $\mu(G_5) \leq \mu(G)$.

Case 1: v does not lie in a 4-cycle of G, and so $G_5 = G/\{v_1v, vv_2\}$.

Let C' be a CC of G_5. Note that every cycle $L' \in C'$ can be extended to a cycle L in G, by using edges in $\{vv_1, vv_2\}$, if necessary. By $\kappa(G) \geq 2$, there is a cycle F in G containing v_1vv_2. Hence $\{L|L' \in C'\} \cup \{F\}$ is a CC of G, and so (26) holds.

Case 2: $F' = vv_1wv_2v$ is a 4-cycle of G, and so $G_5 = (G - v_1w)/v_1v, vv_2\}$.

Note that $N(v_1) \cap N(v_2) = \{v, w\}$. Using the notation in Case 1, we conclude that $\{L|L' \in C'\} \cup \{F'\}$ is also a CC of G, and so (26) holds.

By (8) and (26), we may assume that $|V(G_5)| = 5$. By Lemma 2.2, $cc(G_5) \leq 3$, and so by (26), $cc(G) \leq 4$. Since $n = |V(G_5)| + 2 = 7$, G satisfies (6), contrary to (7). □

Using Corollary 3.9 in place of (ii) of Lemma 3.8 in the proof of Lemma 3.10, we have,

Corollary 3.11 If G has no nontrivial 3-cycles, then G has no nontrivial 4-cycles. □

The argument in the proof for Lemma 3.10 also proves the following:

Corollary 3.12 Let C be a nontrivial cycle in G such that $IntC$ has no nontrivial 4-cycles. Let $v \in IntC$ and define G_5 as in Lemma 3.10. Then $cc(G) \leq cc(G_5) + 1$.

Proof of Theorem 1.2, continued If G has a nontrivial 3-cycle, then G has a nontrivial 3-cycle Γ_3 such that (22) holds. Let $v \in Int\Gamma_3$. By Lemma 3.6, there are $v_1, v_2 \in N(v)$ forming a consecutive bad pair in $N(v)$. By Lemma 3.10, (25) must hold. Define G_5 as in

Lemma 3.10. By Lemma 3.10 and Corollary 3.12, $cc(G) \leq cc(G_5) + 1$, and so by (8) and (26), $|V(G_5)| = 5$, which implies that G is not a counterexample, contrary to (7).

If G does not have a nontrivial 3-cycle, then by Corollaries 3.11 and 3.12, one can define G_5 and argue as above to obtain a contradiction to (7).

Since contradiction arises in any case, the proof of Theorem 1.2 is now completed. □

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Table 5

w_1	w_2	w_3	C_1'	C_2'	C_3'	F
v	v	v	$C_1'' + v_m v_1 v$	$C_2'' + v v_1 v_2$	$C_3'' + v v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
v	v	v_1	$C_1'' + v_m v_1 v$	$C_2'' + v v_1 v_2$	$C_3'' + v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
v	v	u_i	$C_1'' + v_m v_1 v$	$C_2'' + v v_1 v_2$	$C_3'' + u_i v_1 u_{i-1}$	$P + v_2 v v_m v_1 u_i u_{i-1}$
v	v_1	v	$C_1'' + v_m v_1 v$	$C_2'' + v_1 v_2$	$C_3'' + v v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
v	v_1	v_1	$C_1'' + v_m v_1 v$	$C_2'' + v_1 v_2$	$C_3'' + v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
v	v_1	u_i	$C_1'' + v_m v_1 v$	$C_2'' + v_1 v_2$	$C_3'' + u_i v_1 u_{i-1}$	$P + v_2 v v_m v_1 u_i u_{i-1}$
v	u_i	v	$C_1'' + v_m v_1 v$	$C_2'' + u_i v_1 v_2$	$C_3'' + v v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
v	v_1	v_1	$C_1'' + v_m v_1 v$	$C_2'' + u_i v_1 v_2$	$C_3'' + v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
v	v_1	u_i	$C_1'' + v_m v_1 v$	$C_2'' + u_i v_1 v_2$	$C_3'' + u_i v_1 u_{i-1}$	$P + v_2 v v_m v_1 u_i u_{i-1}$
v_1	v	v	$C_1'' + v_m v v_1$	$C_2'' + v v_1 v_2$	$C_3'' + v v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
v_1	v	v_1	$C_1'' + v_m v v_1$	$C_2'' + v v_1 v_2$	$C_3'' + v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
v_1	v	u_i	$C_1'' + v_m v v_1$	$C_2'' + v v_1 v_2$	$C_3'' + u_i v_1 u_{i-1}$	$P + v_2 v v_m v_1 u_i u_{i-1}$
v_1	v_1	v	$C_1'' + v_m v v_1$	$C_2'' + v_1 v_2$	$C_3'' + v v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
v_1	v_1	v_1	$C_1'' + v_m v v_1$	$C_2'' + v_1 v_2$	$C_3'' + v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
v_1	v_1	u_i	$C_1'' + v_m v v_1$	$C_2'' + v_1 v_2$	$C_3'' + u_i v_1 u_{i-1}$	$P + v_2 v v_m v_1 u_i u_{i-1}$
v_1	u_i	v	$C_1'' + v_m v v_1$	$C_2'' + u_i v_1 v_2$	$C_3'' + v v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
v_1	u_i	v_1	$C_1'' + v_m v v_1$	$C_2'' + u_i v_1 v_2$	$C_3'' + v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
v_1	u_i	u_i	$C_1'' + v_m v v_1$	$C_2'' + u_i v_1 v_2$	$C_3'' + u_i v_1 u_{i-1}$	$P + v_2 v v_m v_1 u_i u_{i-1}$
u_i	v	v	$C_1'' + v_m v v_1 u_i$	$C_2'' + v v_1 v_2$	$C_3'' + v v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
u_i	v	v_1	$C_1'' + v_m v v_1 u_i$	$C_2'' + v v_1 v_2$	$C_3'' + v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
u_i	v	u_i	$C_1'' + v_m v v_1 u_i$	$C_2'' + v v_1 v_2$	$C_3'' + u_i v_1 u_{i-1}$	$P + v_2 v v_m v_1 u_i u_{i-1}$
u_i	v_1	v	$C_1'' + v_m v v_1 u_i$	$C_2'' + v_1 v_2$	$C_3'' + v v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
u_i	v_1	v_1	$C_1'' + v_m v v_1 u_i$	$C_2'' + v_1 v_2$	$C_3'' + v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
u_i	v_1	u_i	$C_1'' + v_m v v_1 u_i$	$C_2'' + v_1 v_2$	$C_3'' + u_i v_1 u_{i-1}$	$P + v_2 v v_m v_1 u_i u_{i-1}$
u_i	u_i	v	$C_1'' + v_m v v_1 u_i$	$C_2'' + u_i v_1 v_2$	$C_3'' + v v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
u_i	u_i	v_1	$C_1'' + v_m v v_1 u_i$	$C_2'' + u_i v_1 v_2$	$C_3'' + v_1 u_i u_{i-1}$	$P + v_2 v v_m v_1 u_{i-1}$
u_i	u_i	u_i	$C_1'' + v_m v v_1 u_i$	$C_2'' + u_i v_1 v_2$	$C_3'' + u_i v_1 u_{i-1}$	$P + v_2 v v_m v_1 u_i u_{i-1}$

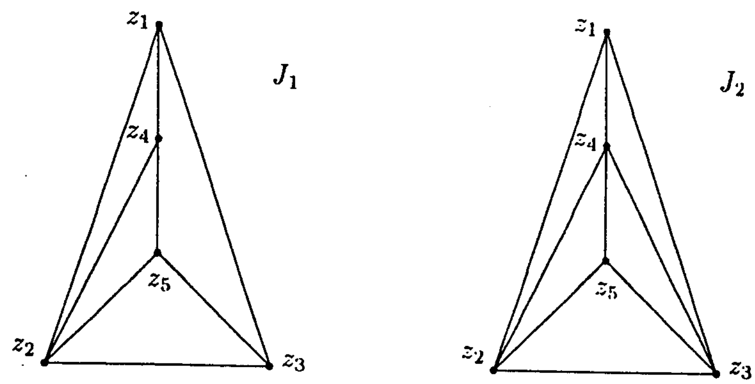

Figure 1: J_1 and J_2

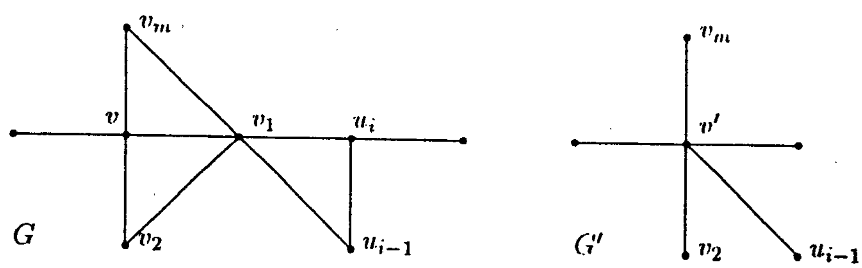

Figure 2: G and G_1 in Lemma 6

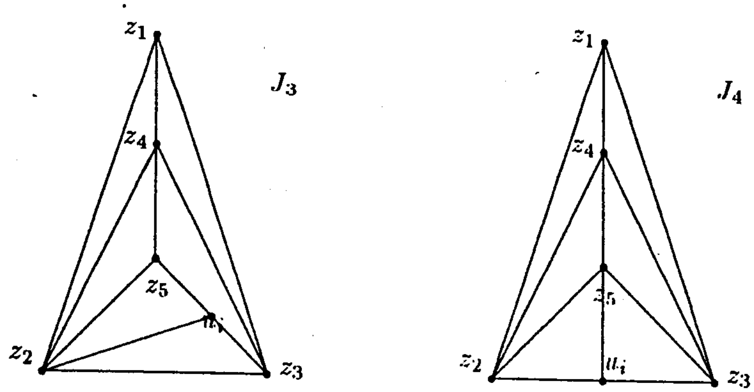

Figure 3: J_3 and J_4

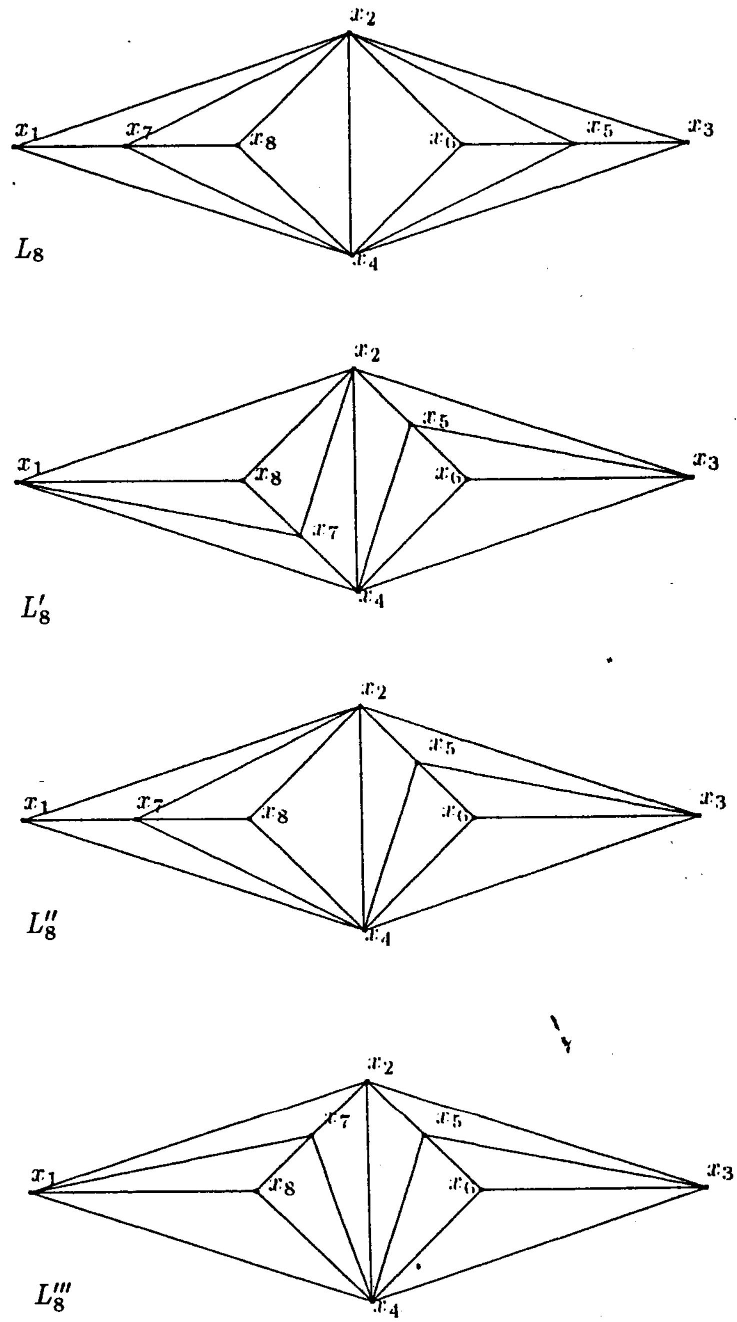

Figure 4 : Graphs L_8, L'_8, L''_8 and L'''_8