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Supereulerian graphs and excluded induced minors

Hong-Jian Lai[1]

West Virginia University, Morgantown, WV 26506, USA

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Abstract

We show that if a graph G with $\kappa'(G) \geqslant 2$ does not have an induced subgraph contractible to $K_{2,3}$ or to one of the subdivided wheels, then G has a spanning eulerian subgraph. As a corollary, such a graph has a nowhere-zero 4-flow.

1. Introduction

We follow the notations of Bondy and Murty [3], unless otherwise stated. A cycle of length n is called an n-cycle. A graph G is *even* if every vertex of G has even degree. If G is both connected and even, then G is *eulerian*. A graph G is *supereulerian* if it has a spanning eulerian subgraph. Thus by definition, K_1 is supereulerian. Regarding K_1 as having arbitrary high edge-connectivity, we observe that a supereulerian graph is 2-edge-connected. Eulerian and supereulerian graphs have been discussed by many authors. See [13, 7] for surveys in the literature.

An edge of G is said to be *subdivided* when it is deleted and replaced by a path of length 2 connecting its ends, the internal vertex of this path being a new vertex. A *subdivision* of a graph G, denoted by TG, is a graph that can be obtained from G by a sequence of edge subdivisions.

A *wheel* W_n is the graph obtained from the n-cycle $C_n = v_1 v_2 \cdots v_n v_1$, where $n \geqslant 2$, by adding an extra vertex v and new edges $\{vv_i\colon 1 \leqslant i \leqslant n\}$. The edges $\{vv_i\colon 1 \leqslant i \leqslant n\}$ are called *spokes* of W_n and the cycle C_n is called the *rim cycle* of W_n. Define the *subdivided wheel* W_n^* to be the graph obtained from W_n by replacing each edge $v_i v_{i+1}$, $(1 \leqslant i \leqslant n)$, (mod n)) by a path of length 2, $v_i v_i' v_{i+1}$ (say), where $\{v_1', \ldots, v_n'\} \cap V(W_n) = \emptyset$. We shall call the vertex $v \in V(W_n^*)$ the *center* of W_n^*, and the vertices v_i *spoke-vertices* of W_n^*, and the edges in $W_n^* - v$ the *rim edges* of W_n^*. Note that $W_2^* \cong K_{2,3}$. Let

$$\mathscr{W} = \{W_n^*\colon n \geqslant 2\}.$$

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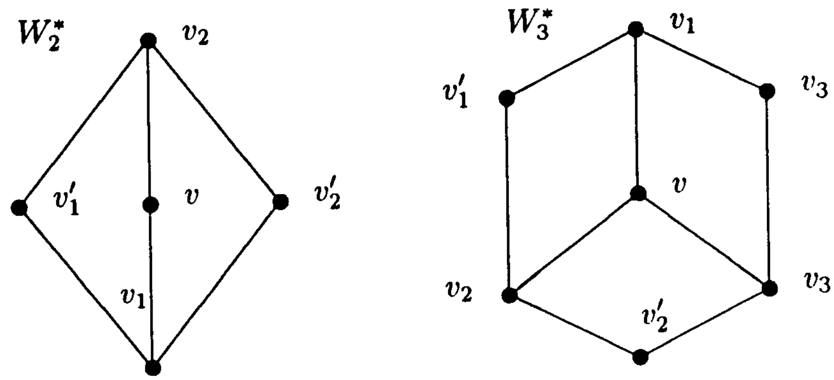

Fig. 1. The graphs W_2^* and W_3^*.

It should be noted that the subdivided wheels are not arbitrary subdivisions of wheels. Fig. 1. shows the graphs of W_2^* and W_3^*.

For a subset $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the ends of each edge in X and then deleting the resulting loops. Note that the edges in $E(G/X)$ can be regarded as edges in $E(G)$. If H is a subgraph, then we use G/H for $G/E(H)$. Note that by this definition, if H is a connected subgraph of G, then

$$G/H = G/G[V(H)]. \tag{1}$$

A graph H is a *minor* of G if H is isomorphic to the contraction image of a subgraph of G. We call H an *induced minor* of G if H is isomorphic to the contraction image of an induced subgraph of G.

In this note, we shall show the following result, whose proof appears in Section 4.

Theorem 1.1. *Let G be a 2-edge-connected graph. The following are equivalent:*

(i) *Every 2-edge-connected induced subgraph of G is supereulerian.*

(ii) *G has no induced minor isomorphic to a member in* $\mathscr{W}$.

As G is an induced subgraph of itself, Corollary 1.2 is an immediate consequence of Theorem 1.1.

Corollary 1.2. *If G is 2-edge-connected and does not have an induced minor isomorphic to a member in* $\mathscr{W}$, *then G is supereulerian.*

Corollary 1.2 has an application to a special kind of cycle double cover of graphs. A *cycle double cover* of a graph G is a collection of even subgraphs $H_1, H_2, \ldots, H_m$ of G such that every edge of G occurs in exactly two of the H_i's. When $m = 3$, we say that G has a cycle double cover with 3 even subgraphs. Bermond et al. [2] showed that a graph G admits a cycle double cover with 3 even subgraphs if and only if G has a nowhere-zero 4-flow. (For the terms and for a survey on flows, see [11].)

There have been many excluded minor results (see [8–12, 16, 18], among others). A prominent conjecture of Tutte [14] states that if a 2-edge-connected graph G does not have a Petersen graph minor, then G has a cycle double cover with 3 even subgraphs.

Theorem 1.3 (Walton and Welsh [17]). *Every 2-edge-connected graph without a subgraph contractible to $K_{3,3}$ has a cycle double cover with 3 even subgraphs.*

It is known (see [6] for a proof) that every supereulerian graph has a cycle double cover with 3 even subgraph, and so Corollary 1.4 below follows immediately from Corollary 1.2.

Corollary 1.4. *If G is 2-edge-connected and does not have an induced minor isomorphic to a member in $\mathscr{W}$, then G has a cycle double cover with 3 even subgraphs.*

Corollary 1.4 cannot be derived from Theorem 1.3. Consider the complete graph K_n. When $n \geqslant 6$, K_n does have $K_{3,3}$ as its subgraph, and so one cannot conclude that K_n has a cycle double cover with 3 even subgraphs from Theorem 1.3. On the other hand, every induced subgraph of K_n is a complete subgraph and so K_n does not have any induced minor isomorphic to a member in $\mathscr{W}$. Therefore, Corollary 1.4 assures that K_n has a cycle double cover with 3 even subgraphs.

As K_4 satisfies the hypothesis of Corollary 1.4, one can see that a graph satisfying the hypothesis of Corollary 1.4 may not have a nowhere-zero 3-flow, and so the conclusion of Corollary 1.4 cannot be improved in that direction.

We also indicate that the proof of Theorem 1.3 depends on the 4-color-theorem [1], whereas the proof of Theorem 1.1 does not depend on the 4-color-theorem.

2. Induced minors

It is known that the minor of a minor is also a minor. This in fact is also true for induced minors.

Proposition 2.1. *If H is an induced minor of G and if L is an induced minor of H, then L is an induced minor of G.*

Proof. Let H' be an induced subgraph of G and let $H_1, \ldots, H_k$ be vertex-disjoint connected subgraphs of H' such that

$$H' / \left(\bigcup_{i=1}^{k} E(H_i) \right) \cong H.$$

By (1), we assume that all the H_i's are induced subgraphs in G as well as in H'. Let $v_1, \ldots, v_k$ be the vertices in $V(H)$ to which $H_1, \ldots, H_k$ are contracted, respectively. Let

L' be an induced subgraph of H and let $L_1, \ldots, L_m$ be vertex-disjoint connected subgraphs of L' such that

$$L' / \left(\bigcup_{j=1}^{m} E(L_j) \right) \cong L.$$

By (1) again, all L_j's are assumed to be induced subgraphs in H as well as in L'. By the definition of contradiction, edges in $E(H)$ and in $E(L)$ can be regarded as edges in $E(G)$. For $1 \leqslant j \leqslant m$, we may assume that

$$v_{j_1}, \ldots, v_{j_{n(j)}} \in V(L_j). \tag{2}$$

For $1 \leqslant j \leqslant m$, let $R_j = G[E(L_j) \cup (\bigcup_{t=1}^{n(j)} E(H_{j_t}))]$, and let

$$R = G\left[E(L) \cup \left(\bigcup_{j=1}^{m} E(R_j) \right) \right]. \tag{3}$$

Note that the R_j's and R are subgraphs of G. We shall show that R is an induced subgraph of G and $R/(\bigcup_{j=1}^{m} E(R_j)) \cong L$.

Suppose, by contradiction, that $e = uv \in E(G) - E(R)$ with $u, v \in V(R)$. By (3), each of u and v must be either incident in G with edges in $E(L)$, or lying in $\bigcup_{j=1}^{m} V(R_j)$. If both u and v are in some $V(H_i)$ or in some $V(R_j)$, then since all H_i's and all L_j's are induced subgraphs, $uv \in E(R)$ by (3), a contradiction. Thus $e = uv$ must be an edge in $E(L)$, and so must be in $E(R)$, by (3), contrary to the assumption that $e \notin E(R)$.

To see that $R/(\bigcup_{j=1}^{m} E(R_j)) \cong L$, we observe that by (3) and by (2), $R/(\bigcup_{i=1}^{k} E(H_i)) \cong L'$ and so by (3),

$$\begin{aligned} R \Big/ \left(\bigcup_{j=1}^{m} E(R_j) \right) &\cong \left(R \Big/ \left(\bigcup_{i=1}^{k} E(H_i) \right) \right) \Big/ \left(\bigcup_{j=1}^{m} E(L_j) \right) \\ &\cong L' \Big/ \left(\bigcup_{j=1}^{m} E(L_j) \right) \cong L. \end{aligned} \tag{4}$$

This proves Proposition 2.1. □

3. Reduced graphs

We follow the notation of Catlin [4]. For any $R \subseteq V(G)$ with $|R|$ even, a subgraph T of G is called an *R-subgraph* if $G - E(T)$ is connected and R is the set of vertices of odd degree in T. A graph G is *collapsible* if for every subset $R \subseteq V(G)$ with $|R|$ even, G has an R-subgraph. In [4], Catlin has shown that every graph has a unique collection of maximal collapsible subgraphs. The graph obtained from G by contracting each maximal collapsible subgraph to a single vertex is called the *reduction* of G. A graph is *reduced* if it is the reduction of some graph.

Theorem 3.1 (Catlin [4]).*Let G be a graph. Each of the following holds.*
(i) *G is supereulerian if and only if G′, the reduction of G, is supereulerian.*
(ii) *If G is reduced, then G does not contain cycles of length at most* 3.
(iii) *G is reduced if and only if G has no nontrivial collapsible subgraphs.*
(iv) *If G is supereulerian, and if* $X \subseteq E(G)$, *then the contraction* G/X *is also supereulerian.*

Let $F = W_n^* \in \mathscr{W}$ and assume that G contains a TF as subgraph. Define the *spoke-vertices of* TW_n^* to be the spoke-vertices of W_n^*, and the *rim cycle of* TW_n^* to be the cycle in TF corresponding to the rim cycle of W_n^*. Then TF is said to be *adjustable* in G if G has an edge $e \in E(G) - E(TF)$ that joins two spoke-vertices of TF.

Proposition 3.2. *Let G be a reduced graph that is spanned by a TF for some* $F \in \mathscr{W}$. *Then one of the following must hold:*
(i) *G has an induced minor that is isomorphic to a member in* $\mathscr{W}$.
(ii) *TF is adjustable in G.*

Proof. Let G be a counterexample with minimum number of vertices. By (iii) of Theorem 3.1, any subgraph H of G is reduced. If G has a proper induced subgraph H that is spanned by a nonadjustable TF' for some $F' \in \mathscr{W}$, then by the minimality of G, H has an induced minor isomorphic to a member in $\mathscr{W}$. It then follows by Proposition 2.1 that G has an induced minor isomorphic to a member in $\mathscr{W}$, a contradiction. Hence we assume that if H is a proper induced subgraph of G, then

$$H \text{ is not spanned by a nonadjustable } TF' \text{ for some } F' \in \mathscr{W}. \tag{5}$$

Case 1: G is spanned by a nonadjustable $TK_{2,3}$.

Let $P_1 = vu_1u_2 \cdots u_m u$, $P_2 = vv_1v_2 \cdots v_k u$ and $P_3 = vw_1w_2 \cdots w_p u$ denote the three internally vertex-disjoint paths in G that form a $TK_{2,3}$ which spans G, where $m, k, p \geqslant 1$. If G is the union of these three paths, then G itself is a $TK_{2,3}$ and so G can be contracted to a $K_{2,3} \in \mathscr{W}$. Hence we may assume that

$$E(G) - \bigcup_{i=1}^{3} E(P_i) \neq \emptyset. \tag{6}$$

For convenience, we denote in the proof of Case 1 that

$$v = v_0 = u_0 = w_0 \quad \text{and} \quad u = v_{m+1} = u_{k+1} = w_{p+1}.$$

Claim 1. *Each of the following holds:*
(a) *For* $0 \leqslant i < i+1 < j \leqslant m+1$, $u_iu_j \notin E(G)$.
(b) *For* $0 \leqslant i < i+1 < j \leqslant k+1$, $v_iv_j \notin E(G)$.
(c) *For* $0 \leqslant i < i+1 < j \leqslant p+1$, $w_iw_j \notin E(G)$.

Proof. We only need to show (a) of Claim 1. By contradiction we assume that $u_iu_j \in E(G)$. Since this $TK_{2,3}$ is nonadjustable, $uv \notin E(G)$. Thus $G - \{u_l: i < l < j\}$ is also spanned by a nonadjustable $TK_{2,3}$, contrary to (5). This proves Claim 1. □

By (6) and by Claim 1, we assume that for some i and j with $0 < i < m + 1$ and with $0 < j < k + 1$, $u_iv_i \in E(G)$.

Claim 2. *If $u_iv_j \in E(G)$, then one of the following holds:*
(a) $k > j > 1$, $i = 1$ *and* $m = 1$;
(b) $m > i > 1$, $j = 1$ *and* $k = 1$.

Since G has no 2-cycles nor 3-cycles, (by (ii) of Theorem 3.1), it is impossible to have both $i = 1$ and $j = 1$, and so we may assume that $j > 1$.

Proof. Suppose first that $j < k$. (Thus $1 < j < k$.) If $i > 1$, then $G - \{v_t: j + 1 \leqslant t \leqslant k\}$ will be spanned by a $T_1 \cong TK_{2,3}$ consisting of three internally vertex-disjoint paths: $vu_1u_2 \cdots u_i$, $vv_1 \cdots v_ju_i$ and $vw_1 \cdots w_nuu_mu_{m-1} \cdots u_i$. By Claim 1, $u_iv \notin E(G)$ and so T_1 is nonadjustable in G, contrary to (5). Similarly, a contradiction obtains if $i < m$, by considering the graph $G - \{v_t: 1 \leqslant t \leqslant j - 1\}$. Thus if $j < k$, then $m = i = 1$.

Suppose then that $j = k$. By (ii) of Theorem 3.1, G has no cycles of length less than 4, and so $1 \leqslant i < m$. Thus by $k = j > 1$, $G - \{v_t: 1 \leqslant t < j\}$ is a proper induced subgraph of G spanned by a $T_2 \cong TK_{2,3}$ consisting of three internally disjoint paths: $u_iu_{i+1} \cdots u_mu$, u_iv_ku, $u_iu_{i-1} \cdots u_1vw_1 \cdots w_pu$. By Claim 1, $u_iu \notin E(G)$, and so T_2 is nonadjustable, contrary to (5). This completes the proof of Claim 2. □

By Claim 2, we may assume that (a) of Claim 2 holds, and so $m = 1$. If there is no edge in G joining a w_i to a vertex in $V(P_2) - \{v, u\}$, then by Claims 1 and 2, and by the fact that G has no 3-cycles (ii) of Theorem 3.1), G itself is a TW_n^* with center u_1, for some $n \geqslant 3$.

Hence there are edges in G joining some w_i to a v_j. Applying Claim 2 to v_jw_i, (or repeating the argument of Claim 2) we must have $i = p = 1$ (or $j = k = 1$). As G does not have 3-cycles, ((ii) of Theorem 3.1), there exist integers i, i' with $1 < i < i' < k$ such that $v_iu_1, v_{i'}w_1 \in E(G)$. Thus $G[\{v_l: 0 \leqslant l \leqslant i'\} \cup \{u_1, w_1\}]$ is a proper induced subgraph of G spanned by a $T_3 = TK_{2,3}$ consisting of three internally disjoint paths: $vu_1v_i, vv_1 \cdots v_{i-1}v_i$, and $vw_1v_{i'}v_{i'-1} \cdots v_i$. By Claim 1, $vv_i \notin E(G)$ and so T_3 is not adjustable, contrary to (5). This proves Case 1.

Case 2: G is spanned by a TW_n^*, for some $n \geqslant 3$.

Let v be the center of this TW_n^* and let $C = w_0w_1 \cdots w_mw_0$ be the sequence of vertices of the rim cycle of the TW_n^*. If w_i is a spoke-vertex of this TW_n^*, then we use $P(w_i)$ to denote the (v, w_i)-path in the TW_n^*. The spoke-vertices will then divide the cycle C into several sections.

Claim 3. *For every spoke vertex w_i, $|E(P(w_i))| = 1$.*

Proof. Suppose, to the contrary, that $|E(P(w_1))| > 1$. Let $X = V(P(w_i)) - \{w_i, v\}$. Since $|E(P(w_1))| > 1$, $X \neq \emptyset$. Since G is spanned by a TW_n^* with $n \geqslant 3$, $G - X$ is an induced subgraph of G spanned by a TW_{n-1}^*. Since the TW_n^* is nonadjustable in G, this TW_{n-1}^* is nonadjustable in G also, contrary to (5). This proves Claim 3. □

Proof of Proposition 3.2 (*continued*). By Claim 3, G is spanned by a $TK_{2,3}$ and so we are back to Case 1. This proves Proposition 3.2. □

4. Proof of Theorem 1.1

Assume that (i) of Theorem 1.1 holds. Since every member in $\mathscr{W}$ in nonsupereulerian, it follows by (iv) of Theorem 3.1 that G cannot have an induced minor in $\mathscr{W}$. This proves (i) $\Rightarrow$ (ii) of Theorem 1.1.

Now we assume the truth of (ii) of Theorem 1.1 to establish (i) of Theorem 1.1 by contradiction. Suppose that G is a counterexample to (ii) $\Rightarrow$ (i) of Theorem 1.1 with

$$|V(G)| \text{ is minimized.} \tag{7}$$

Since G is a counterexample, G has no induced minor in $\mathscr{W}$ but G has an induced 2-edge-connected subgraph that is not supereulerian. By (i) of Theorem 3.1 and by the minimality of G, we may assume that

$$G \text{ is reduced, not supereulerian and } \kappa(G) \geqslant 2. \tag{8}$$

By (8), G is reduced and so by (ii) of Theorem 3.1,

$$G \text{ has no 3-cycles.} \tag{9}$$

If G has an induced subgraph spanned by a nonadjustable TF for some $F \in \mathscr{W}$, then by Propositions 3.2 and 2.1, G has an induced minor in $\mathscr{W}$, contrary to (ii) of Theorem 1.1. Hence we assume that for any induced subgraph L of G,

$$L \text{ is not spanned by a } TF \text{ for some } F \in \mathscr{W}. \tag{10}$$

By (8), $\kappa(G) \geqslant 2$. Thus G has an eulerian subgraph, and so an induced supereulerian subgraph. Let H be an induced supereulerian subgraph of G such that $|V(H)|$ is maximized. Since G is not supereulerian, $|V(G)| > |V(H)|$.

Lemma 4.2. *If $|V(G) - V(H)| \geqslant 2$, then for any $v \in V(G) - V(H)$, $G - v$ has a cut-edge $e_v \in E(G) - E(H)$ such that $G - \{v, e_v\}$ has H as one of the components.*

Proof. If $\kappa'(G - v) \geqslant 2$, then since $G - v$ is a proper induced subgraph of G, by the minimality of $G, G - v$ is supereulerian. This contradicts the maximality of H, since $|V(G) - V(H)| \geqslant 2$. Then by (8), $\kappa(G) \geqslant 2$ and so $G - v$ is still connected. Therefore $G - v$ must have a cut-edge e_v so that the order of the component of $G - \{v, e_v\}$

containing H is minimized. Then this component must be 2-edge-connected. By the maximality of H, this component must be H itself. □

Lemma 4.3. *$V(G) - V(H)$ consists of a single vertex v (say) and $N(v) \subseteq V(H)$ where $N(v)$ is the set of neighbors of v in G.*

Proof. It suffices to show that $|V(G) - V(H)| = 1$. Suppose that $|V(G) - V(H)| \geqslant 2$. Let $w \in V(G) - V(H)$. By Lemma 4.2, there is an edge $e_w \in E(G) - E(H)$ such that $G - \{w, e_w\}$ has H as one of its components. Let $x \in V(G) - V(H)$ be the vertex incident with e_w. Since e_w is a cut-edge of $G - w$,

$$|N(x) \cap V(H)| = 1. \tag{11}$$

By Lemma 4.2, $G - x$ has a cut-edge e_x such that H is a component of $G - \{x, e_x\}$. Let $y \in V(G) - V(H)$ be the vertex incident with e_x. Since e_x is a cut-edge of $G - x$,

$$|N(y) \cap V(H)| = 1. \tag{12}$$

It follows by (11) and (12) that $\{e_w, e_x\}$ is an edge-cut of G. Note that $\{e_x, e_w\}$ is an edge-cut of G lying in $E(G) - E(H)$ such that one of the components of $G - \{e_w, e_x\}$ contains H. Suppose that, among all such edge-cuts of size 2 of G that are in $E(G) - E(H)$, $\{e', e''\}$ is so chosen that the component H' of $G - \{e', e''\}$ which does not contain H, has as few vertices as possible. The minimality of $|V(H')|$ forces that either $|V(H')| = 1$, or both $|V(H')| \geqslant 2$ and $\kappa'(H') \geqslant 2$.

Assume first that $|V(H')| \geqslant 2$ and $\kappa'(H') \geqslant 2$. Let $G_1 = G/H'$. Then G_1 is an induced minor of G. Since G does not have induced minor in $\mathscr{W}$ and by Proposition 2.1, G_1 has no induced minor in $\mathscr{W}$ either. That $\kappa'(G_1) \geqslant 2$ follows from $\kappa'(G) \geqslant 2$. Thus by the minimality of G, G_1 is supereulerian. Note that $E(G_1) \subset E(G)$. If Γ' is a spanning eulerian subgraph of G_1, then since $\{e', e''\}$ is an edge-cut, $\Gamma = G[E(\Gamma')]$ must be a trial with $V(G) - V(H') \subset V(\Gamma)$ such that the two ends of Γ are in $V(H')$. (These two ends may be identical.) Since H' is connected, Γ can be extended to a eulerian subgraph that contains at least one vertex in $V(H')$, contrary to the maximality of H.

Hence we assume that $|V(H')| = 1$. Without loss of generality, we assume that v is the only vertex in $V(H')$ and e is an edge incident with v. Let $G_2 = G/\{e\}$. Again G_2 does not have any induced minor in $\mathscr{W}$, by (ii) of Theorem 1.1 and by Proposition 2.1. Thus by the minimality of G, G_2 has a spanning eulerian subgraph Γ_2'. Since the degree of v is 2, $\Gamma_2 = G[E(\Gamma_2')]$ is connected and is either a spanning eulerian subgraph of G, contrary to the assumption that G is not supereulerian, or is an eulerian subgraph of G that contains all vertices of $V(G) - \{v\}$, contrary to the maximality of H, since $|V(\Gamma_2)| = |V(G)| - 1 > |V(H)|$.

These contractions establish Lemma 4.3. □

Theorem 4.4 (Veblen [15]). *A connected graph is eulerian if and only if it is an edge-disjoint union of cycles.*

By Lemma 4.3, $V(G) - V(H) = \{v\}$. Let Γ denote a spanning eulerian subgraph of H. Then by Theorem 4.4, Γ is an edge-disjoint union of cycles $C_1, C_2, \ldots, C_q$ (say).

Lemma 4.5. *For any i ($1 \leqslant i \leqslant q$), $|V(C_i) \cap N(v)| \leqslant 1$.*

Proof. Suppose $|V(C_i) \cap N(v)| = s \geqslant 2$. By (9), G has no 3-cycles, and so $G[V(C_i) \cup \{v\}]$ is spanned by a nonadjustable W_s^*, contrary to (10). □

Define a new graph $C(\Gamma)$ whose vertices are $\{C_1, C_2, \ldots, C_q\}$, where two vertices C_i and C_j in $V(\Gamma)$ are adjacent if and only if $V(C_i) \cap V(C_j) \neq \emptyset$. As Γ is eulerian, $C(\Gamma)$ is connected.

By (8), we have $\delta(G) \geqslant 2$, and so the only vertex v in $V(G) - V(H)$ must have degree at least 2. Let the set of neighbors of v in G be

$$N(v) = \{v_1, v_2, \ldots, v_m\}.$$

Recall that $C(\Gamma)$ is connected. Without loss of generality, we may assume that $P = C_1 C_2 \ldots C_l$ is a shortest path in $C(\Gamma)$ such that

(P1) $v_1 \in V(C_1)$ and $v_2 \in V(C_l)$, and that

(P2) no internal vertex C_i in this path satisfies $V(C_i) \cap N(v) \neq \emptyset$, for all $1 < i < l$.

Lemma 4.6. *$v_1 \notin V(C_2)$ and $v_2 \notin V(C_{l-1})$.*

Proof. This follows from the assumption that P is a shortest path satisfying (P1) and (P2). □

Lemma 4.7. *For each i, $1 \leqslant i \leqslant l-1$, there is a vertex $w_i \in V(C_i) \cap V(C_{i+1})$ such that $G[\bigcup_{j=i+1}^{l} V(C_j)]$ has a (w_i, v_2)-path Q_i and such that $V(Q_i) \cap [\bigcup_{j=1}^{i} V(C_j)] = \{w_i\}$.*

Proof. Assume first that $i = l-1$. By Lemma 4.6, $v_2 \notin V(C_{l-1})$. Choose $w_{l-1} \in V(C_{l-1}) \cap V(C_l)$ such that w_{l-1} and v_2 are as close as possible in C_l, and so one of the two (w_{l-1}, v_2)-paths is the Q_{l-1} we want in C_l.

Inductively, we assume that there is a vertex $w_k \in V(C_k) \cap V(C_{k+1})$ such that $G[\bigcup_{j=k+1}^{l} V(C_j)]$ has a (w_k, v_2)-path Q_k and such that $V(Q_k) \cap V(C_k) = \{w_k\}$, and that $k > 1$. Since P is shortest, $w_k \notin V(C_{k-1})$ for otherwise C_k can be deleted from P. Choose $w_{k-1} \in V(C_{k-1}) \cap V(C_k)$ such that w_{k-1} and w_k are as close as possible in C_k, and so one of the two (w_{k-1}, w_k)-paths together with Q_k is the Q_{k-1} we want in $G[\bigcup_{j=k}^{l} V(C_j)]$.

Lemma 4.7 is now proved by induction. □

Lemma 4.8. *Let $w_1, w_2, \ldots, w_{l-1}$ be defined in Lemma 4.7 and write $w_0 = v_1$ and $w_l = v_2$. Then for each i, $0 \leqslant i \leqslant l-1$, $w_i w_{i+1} \in E(G)$.*

Proof. Assume first that $v_1 w_1 = w_0 w_1 \notin E(G)$. Then C_1 has two internally vertex-disjoint (v_1, w_1)-paths P_1 and P_2 (say), each having length at least 2. By Lemma 4.7, Q_1 is a (w_1, v_2)-path which is also internally vertex-disjoint from P_1 and P_2. Therefore, $G[V(P_1) \cup V(P_2) \cup V(Q_1) \cup \{v\}]$ is spanned by a $TK_{2,3}$. As $w_1 v_1 \notin E(G)$, this $TK_{2,3}$ is not adjustable, contrary to (10).

Inductively, we assume that $w_i w_{i+1} \in E(G)$ for all $0 \leqslant i \leqslant k-1$ and that $k \leqslant l-1$. Let $Q = v_1 w_2 w_3 \cdots w_k$ denote this (v_1, w_k)-path in $G[\bigcup_{j=1}^{k} V(C_i)]$. Suppose that $w_k w_{k+1} \notin E(G)$. Then C_{k+1} has two internally vertex-disjoint (w_k, w_{k+1})-paths P_1' and P_2' (say), each having length at least 2. By Lemma 4.7, Q_{k+1} is a (w_{k+1}, v_2)-path which is also internally vertex-disjoint from P_1' and P_2'. Therefore, $G[V(P_1') \cup V(P_2') \cup V(Q) \cup V(Q_{k+1}) \cup \{v\}]$ is spanned by a $TK_{2,3}$. As $w_k w_{k+1} \notin E(G)$, this $TK_{2,3}$ is not adjustable, contrary to (10).

Lemma 4.8 is now proved by induction. □

Proof of Theorem 1.1 (*continued*). For two sets X and Y, the *symmetric difference* of X and Y is

$$X \oplus Y = (X \cup Y) - (X \cap Y).$$

By Lemma 4.8, $\Gamma' = v v_1 w_2 \cdots w_{l-1} v_2 v$ is a cycle. Since each C_i contains at most one edge (namely $w_{i-1} w_i$) in Γ', $\Gamma - E(\Gamma')$ is still connected. Therefore $G[E(\Gamma) \oplus E(\Gamma')]$ is a spanning eulerian subgraph of G, contrary to (8).

This contradiction completes the proof of Theorem 1.1. □

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References

[1] K. Appel and W. Haken, Every planar map is four colorable, Bull. Amer. Math. Soc. 82 (1976) 711–712.
[2] J.-C. Bermond, B. Jackson and F. Jaeger, Shortest coverings of graphs with cycles, J. Combin. Theory (B) 35 (1983) 297–308.
[3] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Elsevier, New York, 1976).
[4] P.A. Catlin, A reduction method to find spanning eulerian subgraphs, J. Graph Theory 12 (1988) 29–45.
[5] P.A. Catlin, Double cycle covers and the Petersen graph, J. Graph Theory 13 (1989) 465–483.
[6] P.A. Catlin, Double cycle and the Petersen graph, II. Congr. Numer. 76 (1990) 173–181.
[7] P.A. Catlin, Supereulerian graphs: A survey, J. Graph Theory 16 (1992) 177–196.
[8] H. Fleischner, Eulerian Graphs and Related Topics, Part 1, Vol. 1, Ann. of Discrete Math. 45 (North-Holland, New York, 1990).
[9] H. Fleischner, Eulerian Graphs and Related Topics, Part 1, Vol. 2, Ann. of Discrete Math. 50 (North-Holland, New York, 1991).

[10] F. Harary and W.T. Tutte, A dual form of Kuratowski's theorem, Canad. Math. Bull. 8 (1965) 17–20.
[11] F. Jaeger, Nowhere-zero flow problems, in: L.W. Beineke and R.J. Wilson, eds., Selected Topics in Graph Theory, Vol. 3 (Academic Press, New York 1988) 71–95.
[12] K. Kuratowski, Sur le probleme des courbes gauches en topologie, Fund. Math. 15 (1930) 271–283.
[13] L. Lesniak and O.R. Oellermann, An eulerian exposition, J. Graph Theory 10 (1986) 277–297.
[14] W.T. Tutte, On the algebraic theory of graph colorings, J. Combin. Theory 1 (1966) 15–50.
[15] O. Veblen, An application of modular equations in analusis situs, Ann. Math. 14 (1912, 1913) 86–94.
[16] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, Math. Ann. 114 (1937) 570–590.
[17] P.N. Walton and D.J.A. Welsh, On the chromatic number of binary matroids, Mathematika 27 (1980) 1–9.
[18] D.J.A. Welsh. Matroid Theory (Academic Press, London, 1976).