The Size of Graphs Without Nowhere-Zero 4-Flows

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ABSTRACT

Let G be a 2-edge-connected simple graph with order n. We show that if $|V(G)| \le 17$, then either G has a nowhere-zero 4-flow, or G is contractible to the Petersen graph. We also show that for n large, if

$$|E(G)| \ge \binom{n-17}{2} + 34,$$

then either G has a nonwhere-zero 4–flow, or G can be contracted to the Petersen graph. © 1995 John Wiley & Sons, Inc.

1. INTRODUCTION

Graphs in this article are finite and loopless. Multiple edges are allowed. For undefined terms, see [1]. Let G be a graph. Then $\kappa'(G)$ denotes the edge-connectivity of G. Let $X \subseteq E(G)$ be an edge subset. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and by deleting the resulting loops. If a connected subgraph H is contracted to a vertex v_H in G/H, then H is called the contraction *preimage* of v_H, and when H is nontrivial (i.e., $E(H) \neq \emptyset$), v_H is called a *nontrivial* vertex in the contraction.

The set of all odd degree vertices of G is denoted by $O(G)$. A connected graph G is *eulerian* if $O(G) = \emptyset$, and G is *supereulerian* if G has a spanning eulerian subgraph.

Let $k \ge 3$ be an integer. A *nowhere-zero* k-*flow* of G is an assignment of edge directions and integer weights in the range of $\{-k+1, \ldots, -1, 1, \ldots, k-1\}$ to the edges of G such that at each vertex of G, the amount of flow in is the same as the amount of flow out. Following [8], we denote the set of graphs that admit a nowhere-zero k-flow by F_k.

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Tutte [8] has conjectured that if a 2-edge-connected graph does not have a subgraph contractible to the Petersen graph, then G is in F_4.

What is the maximum size a graph without nowhere-zero-4-flows may have? It has been noted that supereulerian graphs are all in F_4 [5]. Proving a conjecture of Cai [2], Catlin and Chen showed the following:

Theorem 1.1 (Catlin and Chen [7]). Let G be a 3-edge-connected simple graph with n vertices. If

$$|E(G)| \geq \binom{n-9}{2} + 16, \tag{1}$$

then G is supereulerian.

Corollary 1.2. Let G be a 2-edge-connected simple graph with n vertices. If (1) holds, then $G \in F_4$.

The extremal graphs G used to show the sharpness of both Theorem 1.1 and Corollary 1.2 are graphs with a complete subgraph $H \cong K_{n-9}$, such that G/H is the Petersen graph. We note that the Petersen graph and the two Blanuša snarks (see the survey of Watkins and Wilson [9] for snarks) are the three smallest 2-edge-connected graphs not in F_4, where the Petersen graph has 10 vertices and each of the Blanuša snarks has 18 vertices. Motivated by these, we in this article prove the following main result whose proof is in the last section of this paper.

Theorem 1.3. Let G be a 2-edge-connected simple graph with $n \geq 19$ vertices. If

$$|E(G)| \geq \binom{n-17}{2} + 34, \tag{2}$$

then either $G \in F_4$ or G can be contracted to the Petersen graph.

The bound in (2) is asymptotically best possible, in the following sense. Let B denote a Blanuša snark of order 18. Let $B(n)$ denote the graph obtained from B by replacing exactly one vertex of B by a K_{n-17}. Note that (see [9]) B is neither in F_4 nor contractible to the Petersen graph, and so $B(n)$ is neither in F_4 (by Corollary 2.2 in Section 2) nor contractible to the Petersen graph. We also note that

$$\lim_{n\to\infty} \frac{\binom{n-17}{2} + 34}{|E(B(n))|} = 1.$$

2. REDUCTION

Following Catlin [6], a graph G is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph H_R of G such that $O(H_R) = R$. Note that K_1 is collapsible, and that collapsible graphs are all supereulerian. (See Catlin's survey [3]). In [6] it is shown that every graph G has a unique collection of maximal collapsible subgraphs $H_1, \ldots, H_c$ (say). The *reduction* of G is obtained by contracting all nontrivial collapsible subgraphs of G. We call a graph *reduced* if it is the reduction of some graph.

Theorem 2.1 (Catlin [4]). Let H be a subgraph of G. If H is collapsible or if H is a 4-cycle, then

$$G \in F_4 \iff G/H \in F_4 . \tag{3}$$

Corollary 2.2. Let G' be the reduction of G. Then $G' \in F_4$ if and only if $G \in F_4$.

Theorem 2.3 (Catlin [6]). Let G be a 2-edge-connected nontrivial reduced graph. Then G is simple and

$$|E(G)| \leq 2|V(G)| - 4 . \tag{4}$$

Let $v \in V(G)$ denote a vertex of degree $d \geq 4$ in G. Let $N(v) = \{v_1, \ldots, v_d\}$ denote the set of vertices adjacent to v in G. For fixed i, j with $1 \leq i < j \leq d$, let G_{ij} denote the graph $G - \{vv_i, vv_j\} + v_iv_j$, and let $e_{ij} = v_iv_j$ denote the new edge in G_{ij}.

Proposition 2.4. If for some i, j, $G_{ij} \in F_4$, then $G \in F_4$.

Proposition 2.5. If G has a vertex u of degree 2 in G and if u is incident with an edge e in G, then

$$G/\{e\} \in F_4 \iff G \in F_4 .$$

Proposition 2.6. Let G be a 2-edge-connected graph and let $X \subset E(G)$ be an edge cut of G such that $G - X$ has two components H_1 and H_2. If $|X| \leq 3$ and if both $G/E(H_1)$ and $G/E(H_2)$ are in F_4, then $G \in F_4$.

Propositions 2.4, 2.5, and 2.6 are well-known results and so their proofs are omitted.

3. GRAPHS OF SMALL ORDERS

Proposition 3.1 is a well-known result for cubic graphs, which can be found in the survey paper [9].

For a fixed vertex $v \in V(G)$ with $N(v) = \{v_1, v_2, \ldots, v_d\}$ and with $d \geq 4$, recall that $G_{ij} = G - \{vv_i, vv_j\} + e_{ij}$ where $e_{ij} = v_iv_j$, and where $1 \leq i, j \leq d$.

Claim 4. For any $1 \leq i < j \leq d$, G_{ij} is contractible to the Petersen graph.

By Claim 2, $\kappa'(G_{ij}) \geq 2$ and so by (6), either $G_{ij} \in F_4$, or G_{ij} is contractible to the Petersen graph. Since $G_{ij} \in F_4$ implies that $G \in F_4$ by Proposition 2.4, the conclusion of Claim 4 must hold.

Let $G_1 = G_{12}$. By Claim 4, G_1 must be contractible to a graph G_1' that is isomorphic to the Petersen graph. Label $V(G_1')$ with $u_1, u_2, \ldots, u_{10}$ as in Figure 1, and let U_i $(1 \leq i \leq 10)$ denote the subgraph of G_1 whose contraction image is u_i. Note that

$$v \text{ and } e_{12} \text{ do not belong to the same } U_i, \text{ for some } i. \tag{7}$$

For otherwise the Petersen graph G_1' is a contraction of G, contrary to the assumption that G is a counterexample. Hence we may assume that U_1 contains v, and so either $e_{12} \in E(G_1')$, or $e_{12} \in E(U_i)$, for some $i > 1$.

Claim 5. Let $W \subset V(G)$ be a subset. If for some G_{ij}, $W \subset V(G_{ij})$ and G_{ij} does not have nonperipheral edge cuts of size 3 separating vertices in W, and if more than one vertex in W are contracted into a vertex w (say) in G_{ij}', the Petersen contraction of G_{ij}, then all vertices in W must be contracted to this same vertex w in G_{ij}'.

In fact, if vertices in W are contracted to distinct vertices in G_{ij}', then since G_{ij}' is cubic, the 3 edges incident with w in G_{ij} would be a nonperipheral edge cut of G_{ij}, contrary to the assumption that no such cuts exist. This proves Claim 5.

Case 1. $e_{12} \in E(G_1')$.

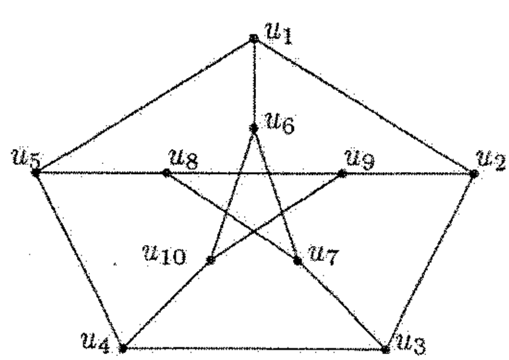

FIGURE 1. The labeled Petersen graph.

By Claim 2, for all i, $(2 \le i \le 10)$, $|V(U_i)| = 1$ and so

$$u_i \text{ is a vertex of } G \qquad (2 \le i \le 10). \tag{8}$$

Also by Claim 2, e_{12} is not incident with u_1 in G_1', for otherwise we may assume that $e_{12} = u_1u_2$ in G_1' with $v_1 \in V(U_1)$ and $v_2 = u_2$, and so $\{u_1u_5, u_1u_6, vv_2\}$ would form a nonperipheral edge cut of size 3 of G, contrary to Claim 2. Thus either exactly one end of e_{12} is adjacent to u_1, or both ends of e_{12} are not adjacent to u_1.

Subcase 1A. Suppose that one end of e_{12} is adjacent to u_1. Without loss of generality, we may assume that $u_2 = v_1$ and $u_3 = v_2$. Now consider $G_2 = G_{23}$ (Figure 2). By Claim 4, G_2 must be contractible to a Petersen graph, which we denote G_2'. By (8), $u_2, \ldots, u_{10}$ are vertices in G_2. If there is a nonperipheral edge cut X of size 3 in G_2 separating the vertices in $\{u_2, \ldots, u_{10}\}$, then either X consists of u_2u_9, v_1v, and a cut edge in U_1 separating the ends of u_1u_6, u_1u_5 from that of v_3 in U_1, or X consists of u_5u_8, u_4u_5, and a cut edge in U_1 separating the one end of u_1u_5 in U_1 from the ends in U_1 of other edges in $G - E(U_1)$. It follows that G would have a nonperipheral edge cut of size 3 in either case, contrary to Claim 2. Thus there is no nonperipheral edge cut of size 3 in G_2 that separates $\{u_2, \ldots, u_{10}\}$. It follows that if more than one vertex in $\{u_2, \ldots, u_{10}\}$ are contracted into a single vertex in G_2', then by Claim 5, all vertices in $\{u_2, \ldots, u_{10}\}$ must be contracted into the same vertex. Since $|V(G)| \le 17$, the preimage of any vertex in a Petersen graph contraction of G must have at most 8 vertices. Therefore G_2' cannot have a vertex whose contraction preimage contains all vertices in $\{u_2, \ldots, u_{10}\}$, and so $u_2, \ldots, u_{10}$ are vertices of G_2'. It follows that in G_2', the only nontrivial vertex must be adjacent to u_5, u_6, u_2, and u_3, and so G_2' cannot be the Petersen graph, contrary to Claim 4.

Subcase 1B. Suppose that both ends of e_{12} are not adjacent to u_1 in G_1'. Without loss of generality, we may assume that $u_3 = v_1$, $u_4 = v_2$. Again we denote $G_2 = G_{23}$ (Figure 3) and G_2' the Petersen contraction of G_2. We first

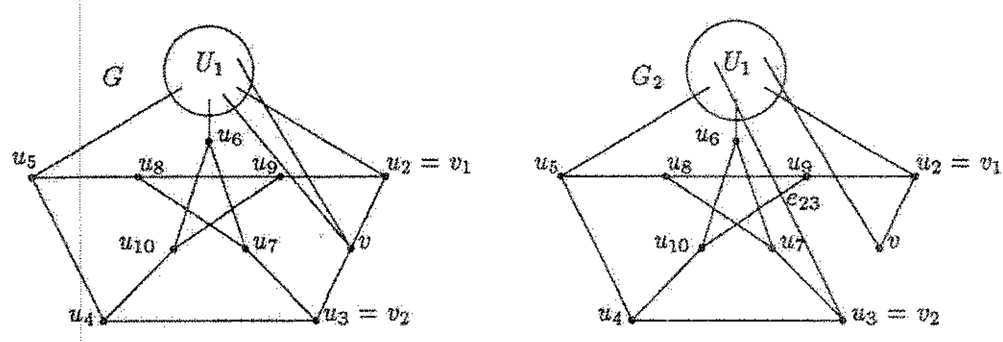

FIGURE 2. The graph G_2 in Subcase 1A.

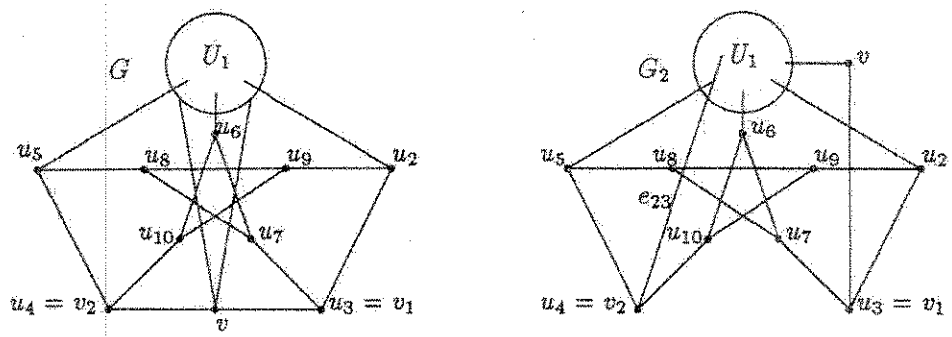

FIGURE 3. The graph G_2 in Subcase 1B.

note that no nonperipheral edge cut of size 3 in G_2 can separate the vertices in $\{u_6, u_7, u_8, u_9, u_{10}\}$. We claim that $u_6, \ldots, u_{10}$ are all vertices in G_2'.

Suppose, to the contrary, that at least two vertices of $u_6, \ldots, u_{10}$ are contained in the preimage W (say) of a nontrivial vertex w in G_2'. Then by Claim 5, all $u_6, \ldots, u_{10}$ are contained in W. If u_2, u_3, u_4, u_5 are all in this same preimage, then G_2' does not have enough vertices to make a Petersen graph since $|V(G)| \leq 17$. Hence we may assume that u_4 or u_5 is a vertex of G_2'. Since W contains $u_6, \ldots, u_{10}$, the edges u_5u_8 or u_4u_{10} would be in cycle of length at most 3 in G_2', contrary to the assumption that G_2' is the Petersen graph. This contradiction establishes the claim and so $u_6, \ldots, u_{10}$ are all vertices in G_2'.

If u_4 and u_5 are not vertices of G_2', then they are contained in the preimage of some vertex of G_2'. Since u_8, u_9, u_{10} are vertices of G_2'. G_2' must have a 4-cycle using edges u_8u_9, u_9u_{10}, $u_{10}u_4$, and u_5u_8, contrary to the assumption that G_2' is the Petersen graph. Thus u_4 and u_5 must be vertices in G_2'. Similarly, u_2 and u_3 must be in $V(G_2')$ also. However, since $u_3u_4 \notin E(G_2)$, the distance from u_3 to u_4 in G_2' is at least 3, and so G_2' cannot be the Petersen graph, as the diameter of the Petersen graph is 2, a contradiction.

Case 2. $e_{12} \notin E(G_1')$.

Thus there is an i $(2 \leq i \leq 10)$, such that $e_{12} \in E(U_i)$. By (7) and by the fact that the diameter of the Petersen graph is 2, either u_i is distance 1 from u_1, or u_i is distance 2 from u_1.

Subcase 2A. Suppose first that u_i is of distance 1 from u_1. Without loss of generality, we may assume that $u_i = u_2$. By Claim 2, for all i $(3 \leq i \leq 10)$, $|V(U_i)| = 1$, and so $u_3, \ldots, u_{10}$ are vertices in G. Thus $v_1, v_2 \in V(U_2)$ and $v_3, \ldots, v_d \in V(U_1) \cup \{u_5, u_6\}$. Let $G_2 = G_{23}$ (Figure 4). Note that any nonperipheral edge cut of size 3 in G_2 separating the vertices in $\{u_3, u_4, \ldots, u_{10}\}$ is also an edge cut of G. Thus by Claim 2, no such edge cut exists. By Claim 5, either the vertices $u_3, \ldots, u_{10}$ are all contained in the preimage of just one vertex in G_2', or these vertices are all vertices in G_2'.

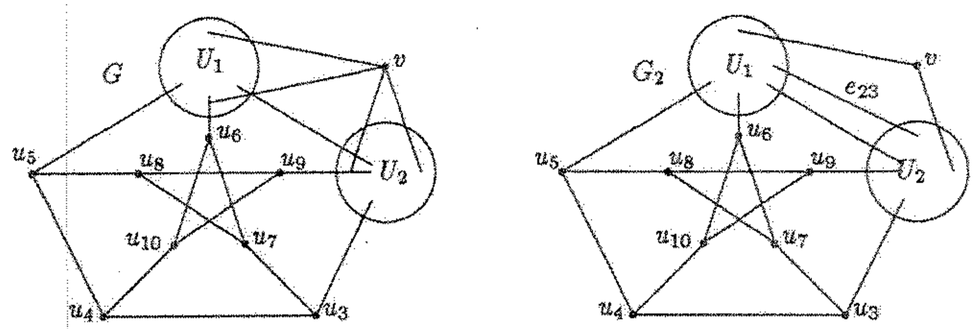

FIGURE 4. The graph G_2 in Subcase 2A.

If the former holds, then G_2' has a vertex whose preimage contains at least 8 vertices and so by $|V(G)| \leq 17$, all other vertices of G_2' must be trivial. It follows that $e_{23} \in E(G_2')$, and so we are back to Case 1.

Hence we assume the latter that all the vertices $u_3, u_4, \ldots, u_{10}$ are in $V(G_2')$. Since G_2' is the Petersen graph, either U_1 has an edge e' so that u_1u_5, u_1u_6, and e' will form an edge cut of size 3 in G_2, or U_2 has an edge e'' so that u_2u_9, u_2u_3, and e'' will form an edge cut in G_2. In the former, e' and u_2u_3, u_2u_9 is a peripheral edge cut of G, and in the latter, e'' and u_1u_5, u_1u_6 is a peripheral edge cut of G. Therefore, Claim 2 is violated in either case, and so a contradiction is obtained.

Subcase 2B. Suppose that the distance between u_i and u_1 is 2. Without loss of generality, we may assume that $u_i = u_3$ (Figure 5). Arguing as in Subcase 1B, we again have $u_4, \ldots, u_{10} \in V(G_2')$. If $u_2 \notin V(G_2')$ (i.e., u_2 is contained in a nontrivial preimage of some vertex in G_2'), then since $u_4, u_7 \in V(G_2')$, the common neighbor of u_4 and u_7 in G_2' must be a vertex $w_3 \in V(U_3)$. Since w_3 would be a vertex of degree 3 in G_2', G_2 must have an edge e_3 that separates w_3 from $V(U_3) - \{w_3\}$ in U_3. Similarly, the common neighbor of u_5 and u_6 in G_2' must be a vertex $w_1 \in V(U_1)$, and G_2 must have an e_1 that separates w_1 from $V(U_1) - \{w_1\}$ in U_1. It follows then that e_1, e_3 and u_9u_2 would form a nonperipheral edge cut in G, contrary to

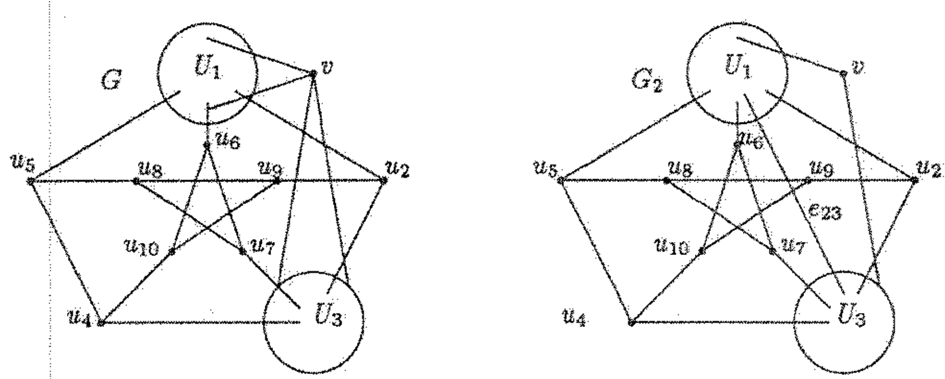

FIGURE 5. The graph G_2 in Subcase 2B.

Claim 2. Hence we may assume that $u_2 \in V(G_2')$. In this case, we again denote the common neighbor of u_4 and u_7 in G_2' by w_3 and that of u_5 and u_6 by w_1. Since $u_2 \in V(G_2')$, it is inevitable that $u_2w_1, u_2w_3 \in E(G_2')$, and so G_2' cannot be the Petersen graph, as there is a vertex v in G_2 that joins $V(U_1)$ and $V(U_3)$, a contradiction.

These contradictions arising in different cases establish the proposition. ∎

4. PROOF OF THEOREM 1.3

We start with a lemma, which is motivated by Theorem 1 of [7].

Lemma 4.1. Let G be a 2-edge-connected simple graph of order n and let $p \geq 2$ be an integer. If

$$|E(G)| \geq \binom{n-p+1}{2} + 2p - 2, \tag{9}$$

then the reduction of G has at most $p - 1$ vertices.

Proof. Let G' denote the reduction of G and let $V(G') = \{v_1, v_2, \ldots, v_c\}$. By contradiction, we assume $c \geq p$ and the choice of G maximizes $|E(G)|$. For each $1 \leq i \leq c$, let H_i denote the preimage of v_i in G. Since $|E(G)|$ is maximized, all the H_i's are complete subgraphs. Thus

$$|E(G)| = \sum_{i=1}^{c} \binom{|V(H_i)|}{2}, \qquad \text{with } \sum_{i=1}^{c} |V(H_i)| = n. \tag{10}$$

By the maximality of $|E(G)|$ and by (10), at most one H_i $(1 \leq i \leq c)$ is a nontrivial subgraph of G (since the maximum of $|E(G)|$ in (10) is obtained when all but one $|V(H_i)| = 1$), and this H_i is a complete subgraph of order $n - c + 1$. Therefore

$$|E(G)| \leq |E(H_i)| + |E(G')| = \binom{n-c+1}{2} + |E(G')|. \tag{11}$$

Since G is 2-edge-connected, and since $c \geq p \geq 2$, G' is nontrivial, and so by Theorem 2.3,

$$|E(G')| \leq 2c - 4. \tag{12}$$

Combine (9), (11), and (12) to get

$$\binom{n-p+1}{2} + 2p - 2 \leq \binom{n-c+1}{2} + 2c - 4. \tag{13}$$

Simplify (13) to get

$$2n(c - p) \le (c - p)(c + p + 3) - 4.$$

Thus we must have $c > p$, and so

$$2n < c + p + 3 - \frac{4}{c - p},$$

By $n > c$, we get

$$n \le p + 3 - \frac{4}{c - p}. \tag{14}$$

Thus by (14) and since $4/(c - p) > 0$, we have $n \le p + 2$. If $n = p + 1$, then $n = c$ and by (14), we have $n \le p - 1 < n$, a contradiction. If $n = p + 2$, then $c - p \le 2$ and so by (14), we get $p + 2 = n \le p + 3 - 2 = p + 1$, a contradiction. The proof of Lemma 4.1 is completed. ∎

Proof of Theorem 1.3. By Lemma 4.1 with $p = 18$, we conclude that G', the reduction of G, has order at most 17. By Proposition 3.2, either G' is in F_4, whence by Corollary 2.2, G is in F_4, or G' is contractible to the Petersen graph, whence G is contractible to the Petersen graph. Therefore Theorem 1.3 must hold. ∎

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