# The size of graphs with given edge inclusive connectivity

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**Abstract**

In [Discrete Math. 46 (1983) 191 - 198], the concept of inclusive edge connectivity was introduced and discussed. Given a vertex $v \in V(G)$, the *inclusive edge connectivity of* $v$, denoted by $\lambda_i(v, G)$, is the minimum number of edges whose deletion results in a subgraph of $G$ in which $v$ is a cut-vertex. Define

$$\lambda_i(G) = \min\{\lambda_i(v, G) : \ v \in V(G), \text{ and } d_G(v) \geq 2\}$$

to be the *inclusive edge connectivity* of $G$. Extremal problems on $\lambda_i(G)$ are studied in this paper.

**1. Introduction.** We follow the notation of Bondy and Murty [2], unless otherwise noted. The complement of a simple graph $G$ is denoted by $G^c$, and the edge connectivity of $G$ is $\kappa'(G)$. The concept of *inclusive edge connectivity* was first introduced in 1979 [3] under the name *cohesion*. It is renamed inclusive connectivity recently in [1]. The motivation of this concept and further discussion on it can be found in [1], [3] - [7], among others. Given a vertex $v \in V(G)$ with degree at least 2, the *inclusive edge connectivity of* $v$, denoted by $\lambda_i(v, G)$, is the minimum number of edges whose deletion results

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in a subgraph of $G$ in which $v$ is a cut-vertex. When no confusion arises, we shall write $\lambda_i(v)$ for $\lambda_i(v, G)$. Define

$$\lambda_i(G) = \min\{\lambda_i(v) : v \in V(G) \text{ and } d_G(v) \geq 2\}$$

to be the *inclusive edge connectivity* of $G$. Thus when $G$ is connected, $\lambda_i(G) = 0$ if and only if $G$ has a cut-vertex. Another way to define global inclusive edge connectivity can be found in [5].

Other inclusive connectivities are also discussed in [1]. One can define, for each $e \in E(G)$, $\lambda_i(e, G)$ to be the minimum number of edges whose deletion results in a subgraph of $G$ in which $e$ is a cut edge. It is observed in [1] that $\min\{\lambda_i(e, G) : e \in E(G)\} + 1$ is just the edge-connectivity.

Given natural numbers $k$ and $n$, what are the extremal sizes of simple graphs with order $n$ and inclusive connectivity $k$? We shall investigate this problem in this paper. Let $\mathcal{G}$ denote the collection of simple connected graphs with at least 3 vertices. Define

$$f(n,k) = \min\{|E(G)| : |V(G)| = n, \lambda_i(G) \geq k, G \in \mathcal{G}\},$$

and

$$F(n,k) = \max\{|E(G)| : |V(G)| = n, \lambda_i(G) \leq k, G \in \mathcal{G}\}.$$

We shall show that

$$\lim_{n\to\infty} \frac{f(n,k)}{n} = \frac{k+1}{2},$$

and that for $n \geq k+2$,

$$F(n,k) = \binom{n-1}{2} + (k+1),$$

such that $G$ is an extremal graph for $F(n,k)$ if and only if $G$ has a vertex $v$ of degree $k+1$ with $G - v \cong K_{n-1}$.

Definition of $\mathcal{M}(k)$: Let $\mathcal{M}(k)$ denote the family of graphs $G$ with $|V(G)| \geq k+2$ such that for every subgraph $H$ of $G$, $\lambda_i(H) \leq k$, but for any edge $e \in E(G^c)$, $G + e$ has a subgraph $L$ with $\lambda_i(L) \geq k+1$. Define

$$g(n,k) = \max\{|E(G)| : |V(G)| = n, G \in \mathcal{M}(k)\},$$

and

$$\mathcal{E}(k) = \{G \in \mathcal{M}(k) : |V(G)| = n, |E(G)| = g(n,k)\}.$$

We shall show that

$$g(n,k) = \binom{k+1}{2} + (n-k-1)(k+1),$$

and characterize the graphs in $\mathcal{E}(k)$.

**2. Lemmas and examples.** Let $X \subseteq E(G)$ be an edge-cut of $G$ and let $e \in X$. Then one of the ends of $e$ will be a cut-vertex of $G - (X - \{e\})$, and so we have

Lemma 1 For any connected graph $G$ with $\kappa(G) \geq 2$,

$$\delta(G) \geq \kappa'(G) \geq \lambda_i(G) + 1.$$

Given a vertex $v \in V(G)$, if $v$ is a cut-vertex of $G$, then $G$ has some edge-disjoint connected subgraphs $G_1, G_2 \cdots, G_m$ with $V(G_i) \cap V(G_j) = \{v\}$ $(i \neq j)$ and $G = \cup_{i=1}^m G_i$. These subgraphs are called *v-components* of $G$. Fix a vertex $v \in V(G)$. A minimal edge subset $X \subseteq E(G)$ is called a *v-cut* of $G$ if $G - X$ has $v$ as a cut-vertex. By the minimality of a $v$-cut, if $X$ is a $v$-cut, then $G - X$ has exactly two $v$-components. A $v$-cut $X$ is *trivial* if $G - X$ has a pair of $v$-components $G_1$ and $G_2$ such that one of them is a $K_2$ (say $G_2 = K_2$) and such that $X \cup E(G_2)$ are all the edges incident with a single vertex in $G$. □

Lemma 2 Let $L$ be a subgraph of $H$ with $\lambda_i(L) > k$, and let $v \in V(H)$. If $X$ is a $v$-cut of $H$ with $|X| \leq k$, then $E(L) \cap X = \emptyset$.

Proof: By contradiction, we assume that $E(L) \cap X \neq \emptyset$. If $v \in V(L)$, then either $E(L) \cap X$ is a $v$-cut of $L$, contrary to the hypothesis of $\lambda_i(L) > k$; or $v$ is not a cut-vertex of $L-(X\cap E(L))$, in which case $H-(X-E(L))$ would have $v$ as a cut-vertex, contrary to the minimality of a $v$-cut. Hence we may assume that $v \notin V(L)$. If $X\cap E(L)$ is an edge cut of $L$, then by Lemma 1, $k \geq |X\cap E(L)| \geq \kappa'(L) > \lambda_i(L)$, a contradiction again. Thus $L-(X\cap E(L))$ is connected. Let $G_1$ and $G_2$ be the two $v$-components of $G-X$. Without loss of generality, we may assume that $V(G_1)\cap V(L) \neq \emptyset$. Since $L-(X\cap E(L))$ is connected and since $v \notin V(L)$, $L-(X\cap E(L))$ is a subgraph of $G_1$ and so $X-E(L)$ is also a $v$-cut of $G$, contrary to the minimality of a $v$-cut. □

For positive integers $n, k$ with $n$ sufficiently larger than $k$, construct a graph $G(n,k)$ as follows. Write

$$n-(4k+2) = s(2k+2)+t, \tag{1}$$

where $s$ and $t$ are integers with $0 \leq t < 2(k+1)$. Let $G_0$ and $G_{s+1}$ be two copies of $K_{k,k+1}$ with bipartitions $V(G_0) = \{u_1^0,\cdots,u_k^0\} \cup \{v_1^0,\cdots,v_{k+1}^0\}$ and $V(G_{s+1}) = \{u_1^{s+1},\cdots,u_{k+1}^{s+1}\} \cup \{v_1^{s+1},\cdots,v_k^{s+1}\}$, respectively. Let $G_1, G_2, \cdots, G_s$ be $s$ copies of $K_{k+1,k+1}-M$, where $M$ is a perfact matching of $K_{k+1,k+1}$, with bipartitions $V(G_i) = \{u_1^i,\cdots,u_{k+1}^i\} \cup \{v_1^i,\cdots,v_{k+1}^i\}$, $(1 \leq i \leq s)$. Let $G_{s+2} \cong K_t$. (Thus $G_{s+2}$ can be empty since $t=0$ is possible.) For $0 \leq i \leq s$, let

$$L_i = \{v_j^i u_j^{i+1} : 1 \leq j \leq k+1\}.$$

If $t=0$, then let $L_{s+1} = \emptyset$; if $t > 1$, then let $L_{s+1}$ be the set of edges joining every vertex in $G_{s+2}$ to each of $\{v_1^{s+1}, v_2^{s+1},\cdots,v_k^{s+1}\}$; and if $t=1$, the let $L_{s+1}$ be the set of edges joining the vertex in $G_{s+2}$ to each of $\{v_1^{s+1}, v_2^{s+1},\cdots,v_k^{s+1}, u_1^{s+1}\}$. Define $G(n,k)$ by

$$V(G(n,k)) = \bigcup_{i=0}^{s+2} V(G_i),$$

and

$$E(G(n,k)) = (\bigcup_{i=0}^{s+2} E(G_i)) \bigcup (\bigcup_{j=1}^{s+1} L_j).$$

See Figures 1 and 2 for examples of $G(n,k)$.

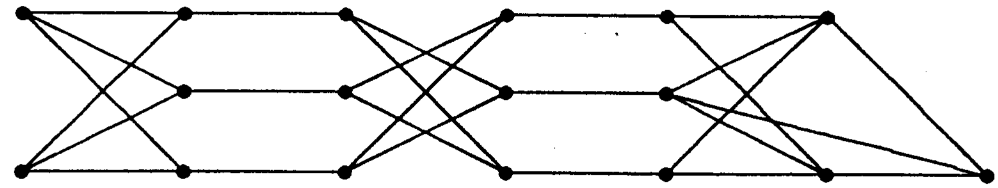

Figure 1 : Graph $G(17,2)$

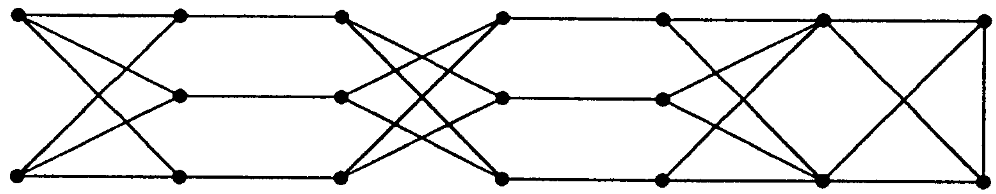

Figure 2 : Graph $G(18,2)$

Lemma 3 If $n$ is sufficiently larger than $k$, then $\lambda_i(G(n,k)) = k$.

Proof: Let $G = G(n,k)$. We shall use the same notation in the definition of $G(n,k)$. Since $\delta(G) = k+1$, by Lemma 1, $\lambda_i(G) \leq k$. Suppose that $G$ has a $v$-cut $X$ with $|X| < k$ for some vertex $v \in V(G)$.

We shall first show that $X \cap E(G_j) = \emptyset$, for all $j$.

By contradiction, we assume that for some fixed $j$, $X \cap E(G_j) \neq \emptyset$. Thus $X \cap E(G_j)$ is either an edge-cut or a $v$-cut of $G_j$. Since $G_j$ is either a complete bipartite graph or a complete bipartite graph minus a perfect matching with sufficiently many vertices on each side, $\kappa'(G_j) \geq k$. It follows that $X \subseteq E(G_j)$, $|X| = k-1$ and all edges in $X$ must be incident with a single vertex $w$ (say) and $wv \in E(G_j)$. But then the structure of $G$ shows that both $v$ and $w$ are in a cycle of $G-X$ and so $v$ cannot be a cut-vertex of $G-X$, a contradiction.

A similar arguement shows that $X \cap L_i = \emptyset$, for all $i$. Hence $X$ does not exist and so $\lambda_i(G) = k$. □

**3. Determination of $f(n,k)$ and $F(n,k)$.**

Theorem 1 For any fixed $k \geq 1$, we have

$$\lim_{n\to\infty} \frac{f(n,k)}{n} = \frac{k+1}{2}. \tag{2}$$

Proof: For any graph $G \in \mathcal{G}$ with $\lambda_i(G) \geq k$ and $|V(G)| = n$. By Lemma 1, we have

$$2|E(G)| \geq n\delta(G) \geq n(k+1),$$

and so

$$\frac{f(n,k)}{n} \geq \frac{k+1}{2}. \tag{3}$$

But then by Lemma 3, we have, when $n$ is sufficiently larger than $k$,

$$f(n,k) \leq |E(G(n,k))| = \frac{(n-t)(k+1)}{2} + \binom{k+t}{2}, \tag{4}$$

where $t$ is defined in (1). Theorem 1 follows by combining (3) and (4). □

Lemma 4 Suppose that $\lambda_i(H) = k$ and that $H$ has a $v$-cut $X$ with $|X| = k$ for some vertex $v \in V(H)$ such that $H_1$ and $H_2$ are a pair of $v$-components of $H - X$ with $|V(H_1)| \geq |V(H_2)|$. If $H_2$ is a complete subgraph, then either $H_2 \cong K_2$ or $|V(H_2)| \geq k+2$.

Proof: Assume that $H_2 = K_m$ for some $m \geq 2$. By the minimality of a $v$-cut, no edges in $X$ incident with $v$. Thus by Lemma 1, every vertex in $H_2 - v$ has degree at least $k+1$, and $v$ has degree $m-1$ in $G[X \cup E(H_2)]$. Hence if $3 \leq m \leq k+1$, then by $|X| = k$, we count the incidences of $V(H_2)$ in $G[X \cup E(H_2)]$ to get

$$(m-1)(k+1) + (m-1) \leq \sum_{x \in V(H_2)} d_{G[X\cup E(H_2)]}(x) = \frac{m(m-1)}{2} + k. \tag{5}$$

By (5), we have $(m-1)(2k+4-m) \leq 2k$. This, together with $3 \leq m \leq k+1$, implies $2(k+3) \leq (m-1)(2k+4-m) \leq 2k$, a contradiction, and so we must have $m = 2$. □

Theorem 2 For $k \geq 1$ and $n \geq k+2$,

$$F(n,k) = \binom{n-1}{2} + (k+1), \tag{6}$$

and $G$ is a simple graph with $n$ vertices and with $\lambda_i(G) \leq k$ such that $|E(G)| = F(n,k)$ if and only if $G$ has a vertex $v$ of degree $k+1$ such that $G - v \cong K_{n-1}$.

Proof: Suppose that $H$ has a vertex $v$ of degree $k+1$ such that $H - v \cong K_{n-1}$. Then by Lemma 1, $\lambda_i(H) \leq k$ and so

$$F(n,k) \geq |E(H)| = \binom{n-1}{2} + (k+1). \tag{7}$$

To show that

$$F(n,k) \leq \binom{n-1}{2} + (k+1). \tag{8}$$

we consider a simple graph $G$ with $n$ vertices and with $\lambda_i(G) \leq k$ suth that $|E(G)| = F(n,k)$. If $n = k+2$, then Theorem 2 follows by Lemma 4. Thus we argue by induction on $|V(G)|$ to assume that Theorem 2 holds for smaller values of $|V(G)|$. Since $\lambda_i(G) \leq k$, $G$ has a $v$-cut $X$ with $|X| \leq k$. Let $G_1$ and $G_2$ be a pair of $v$-components of $G - X$. We claim that $X$ is a trivial $v$-cut.

By contradiction, we assume that $n_1 = |V(G_1)| \geq n_2 = |V(G_2)| \geq 3$, and so by Lemma 4, both $n_1 \geq k+2$ and $n_2 \geq k+2$. By induction,

$$|E(G_1)| \leq \binom{n_1-1}{2} + k + 1 \text{ and } |E(G_2)| \leq \binom{n_2-1}{2} + k + 1.$$

Note that by $n_1 + n_2 = n+1$,

$$\begin{aligned} |E(G_1)| + |E(G_2)| &= \frac{(n_1-1)(n_1-2)}{2} + \frac{(n-n_1)(n-n_1-1)}{2} + 2k + 2 \\ &= n_1^2 - (1+n)n_1 + \frac{n^2-n+2}{2} + 2k + 2. \end{aligned}$$

This function of $n_1$ gets its maximum at either $n_1 = k+2$ or $n_1 = n - (k+2)$ (which means $n_2 = k+2$). By symmetry of $n_1$ and $n_2$

and by $k \geq 1$, we have

$$
\begin{aligned}
|E(G_1)| + |E(G_2)| &= (k+2)^2 - (1+n)(k+2) + \frac{n^2-n+2}{2} + 2k + 2 \\
&= \frac{n^2-3n+2}{2} + n + (k+2)(k+3-n) - 2 \\
&< \frac{n^2-3n+2}{2} + n + (k+2)(k+3-n) \\
&= \frac{n^2-3n+2}{2} + (k+2)(k+3) - n(k+1) \\
&\leq \frac{n^2-3n+2}{2} + (k+2)(k+3) - 2(k+2)(k+1) \\
&\leq \frac{n^2-3n+2}{2} - (k+2)(k-1) \leq \frac{(n-1)(n-2)}{2}.
\end{aligned}
$$

It follows by $|E(G)| = F(n,k)$ that

$$
\begin{aligned}
F(n,k) &= |E(G_1)| + |E(G_2)| + k \\
&\leq \binom{n_1 - 1}{2} + k + 1 + \binom{n_2 - 1}{2} + k + 1 + k \\
&< \binom{n-1}{2} + k + 1,
\end{aligned}
$$

contrary to (7). Thus $X$ must be a trivial $v$-cut, and so (8) holds. Therefore the conclusion of Theorem 2 follows. □

**4. Determination of $g(n,k)$.** To establish a lower bound for $g(n,k)$, we shall construct a family of graphs (called $\mathcal{F}(k)$) for each $k$, and verify that every such constructed graph is in fact in $\mathcal{M}(k)$ (Lemmas 8 and 9).

Lemma 5 If $H \in \mathcal{M}(k)$, then $\lambda_i(H) = k$.

Proof: By definition of $\mathcal{M}(k)$, we have $\lambda_i(H) \leq k$. Lemma 5 is trivial if $H = K_{k+2}$ and so we assume that $H \neq K_{k+2}$. Assume that $\lambda_i(H) \leq k-1$. Let $X$ with $|X| \leq k-1$ be a $v$-cut of $H$ for some $v \in V(H)$ and let $H_1$ and $H_2$ be a pair of $v$-components of $H - X$.

Since $H \neq K_{k+2}$, there must be an edge $e \in E(H^c)$ such that the one end of $e$ lies in $H_1$ and the other in $H_2$. Since $H \in \mathcal{M}(k)$, $H+e$ has a subgraph $L$ with $\lambda_i(L) > k$. Since $X \cup \{e\}$ is a $v$-cut of size at most $k$ in $H+e$, it follows by Lemma 2 that $E(L) \cap (X \cup \{e\}) = \emptyset$, and so $L$ is a subgraph of $H$, contrary to the assumption that $H \in \mathcal{M}(k)$. □

Corollary 6 Suppose that $H \in \mathcal{M}(k)$. Let $X$ be a $v$-cut with $|X| = k$, for some $v \in V(H)$ and let $H_1$ and $H_2$ be a pair of $v$-components of $H - X$. If for some $i \in \{1,2\}, |V(H_i)| \leq k+1$, then $H_i \cong K_2$.

Proof: This is trivial if $k = 1$ and so we assume that $k \geq 2$. Since $H \in \mathcal{M}(k)$, $\lambda_i(H) = k$ by Lemma 5. Thus if $H_i$ is a complete subgraph, then by Lemma 4, we must have $H_i \cong K_2$. If $H_i$ if not complete, then since $H \in \mathcal{M}(k)$, $H_i \in \mathcal{M}(k)$ also (by Lemma 2). By Lemma 5, $\lambda_i(H_i) = k$ and so by Lemma 1, $\delta(H_i) \geq k+1$. By the fact that $H$ is simple, we must have $|V(H_i)| \geq k+2$, contrary to the asumption of $|V(H_i)| \leq k+1$. □

Lemma 7 If for some $u \in V(G)$ with degree $d_G(u) = k+1, G-u \in \mathcal{M}(k)$, then there is no subgraph $H$ of $G$ satisfying $\lambda_i(H) \geq k+1$.

Proof: By contradiction, we assume that there is a subgraph $H$ of $G$ with $\lambda_i(H) \geq k+1$. Since $G - u \in \mathcal{M}(k)$, $H$ must contain the vertex $u$. By Lemma 1, $\delta(H) \geq k+2$ and so $k+1 = d_G(u) \geq d_H(u) \geq k+2$, a contradiction. □

Definition of $\mathcal{F}(k)$: For a given integer $k > 0$, define the family $\mathcal{F}(k)$ of simple graphs by the following inductive way:

(F1) $K_{k+2} \in \mathcal{F}(k)$.

(F2) For any $H \in \mathcal{F}(k)$, let $u \notin V(H)$ be a vertex and choose any $k+1$ distinct vertices $v_1, v_2, \cdots, v_k, v_{k+1} \in V(H)$. Form a new member $G \in \mathcal{F}(k)$ with

$$V(G) = V(H) \cup \{u\} \text{ and } E(G) = E(H) \cup \{uv_1, uv_2, \cdots, uv_k, uv_{k+1}\}.$$

Observation: Every graph $G \in \mathcal{F}(k)$ can be built up by starting with $K_{k+2}$ and by succesively adding $|V(G)| - (k+2)$ vertices of degree $k+1$ in a one-at-a-time way. Note that if $u_1$ and $u_2$ are two newly added vertices of degree $k+1$ in a graph $G \in \mathcal{F}(k)$, then $u_1$ and $u_2$ are not adjacent in $G$. Therefore the order of adding these vertices of degree $k+1$ is immaterial, and if $G \in \mathcal{F}(k) - \{K_{k+2}\}$, then the vertices of degree $k+1$ of $G$ form an independent set of $G$.

Lemma 8 Let $G \in \mathcal{F}(k)$ be a graph. Then each of the following holds.

(i) $\lambda_i(G) = k$.

(ii) If $X$ is a $v$-cut of $G$ with $|X| \leq k$ for some $v \in V(G)$, then $|X| = k$ and $X$ is trivial.

Proof: If $G = K_{k+2}$, then Lemma 8 holds trivially. Therefore we assume that $G \neq K_{k+2}$. By induction, we suppose that Lemma 8 holds for graphs in $\mathcal{F}(k)$ with smaller order than $|V(G)|$ and that $|V(G)| \geq k+3$. By the definition of $\mathcal{F}(k)$, $G$ has a vertex $u$ of degree $k+1$ such that $G - u \in \mathcal{F}(k)$. Let $E(u)$ denote the edges incident with $u$ in $G$.

If $u = v$, then since $v = u$ is a cut vertex of $G - X$, $X - E(v)$ is an edge-cut of $G - u$. By induction and since $G - u \in \mathcal{F}(k)$, we have $\lambda_i(G - u) = k$, and so by Lemma 1,

$$k = \lambda_i(G-u) \leq \kappa'(G-u) - 1 \leq |X - E(v)| - 1 \leq |X| - 1 \leq k - 1,$$

a contradiction.

Therefore we must have $v \neq u$, and so $X - E(u)$ contains a $v$-cut of $G - u$. By induction, $\lambda_i(G - u) = k$. It follows that $k \geq |X| \geq |X - E(u)| = k$, and so $X - E(u) = X$ and $|X| = k$. (By now, (i) of Lemma 8 is proved by induction.) Hence $X$ is a $v$-cut of $G - u$. As $G - u \in \mathcal{F}(k)$, by induction, $X$ is a trivial $v$-cut of $G - u$. It follows that $|X| = k$ and $X$ is a trivial $v$-cut of $G - u$. Therefore, there is a vertex $w \in V(G - u)$ such that $vw \in E(G - u)$, and such that $X \cup \{wv\}$ are all the edges incident with $w$ in $G - u$. In other words, $G[\{wv\}]$ is one of the two $v$-components of $G - X$. If $u$ is adjacent to $w$ in $G$, then $X$ is not a $v$-cut of $G$, since there is a path using $u$ in $G - X$ connecting the two $v$-components without passing through $v$. Therefore, $u$ and $w$ are not adjacent in $G$, and so $X$ is a trivial $v$-cut of $G$. □

Lemma 9 $\mathcal{F}(k) \subseteq \mathcal{M}(k)$.

Proof: By contradiction, we assume that $G \in \mathcal{F}(k) - \mathcal{M}(k)$ such that

$$|V(G)| \text{ is minimized.} \tag{9}$$

Clearly $G \neq K_{k+2}$ and so by the definition of $\mathcal{F}(k)$, $G$ has a vertex $u$ with $G - u \in \mathcal{F}(k)$. By (9), $G - u \in \mathcal{M}(k)$, and so by Lemma 7, $G$ does not have a subgraph $H$ with $\lambda_i(H) \geq k+1$.

Since $G \notin \mathcal{M}(k)$, there is an edge $e \in E(G^c)$ such that $G + e$ has no subgraph $H$ with $\lambda_i(H) \geq k+1$. Thus $G + e$ must have a $v$-cut $X \subseteq E(G+e)$ with $|X| \leq k$ for some vertex $v \in V(G)$.

If $e \in X$, then $X - e$ is a $v$-cut of $G$, by Lemma 8, $|X - e| = k$, contrary to the assumption that $|X| \leq k$. Thus $e \notin X$.

Since $e \notin X$, $X$ is a $v$-cut of $G$, and so by Lemma 8, $X$ is trivial with $|X| = k$. Therefore, there is a vertex $w \in V(G)$ of degree k+1 in $G$. Note that this implies that $e \in E(G - w)^c$. By the Observation after the definition of $\mathcal{F}(k)$, $G - w \in \mathcal{F}(k)$ also (since the order of adding the vertices of degree k+1 is immaterial). It follows that $(G - w) + e$ should have a subgraph $H$ with $\lambda_i(H) \geq k+1$, a contradiction. □

Lemma 10 Suppose that $G \in \mathcal{M}(k)$ and that $X \subseteq E(G)$ is a $v$-cut of $G$ with $|X| = k$ for some vertex $v \in V(G)$. If $G_1$ and $G_2$ are a pair of $v$-components of $G - X$, and if $|V(G_1)| > k+1$, then $G_1 \in \mathcal{M}(k)$.

Proof: We may assume that $G_1 \neq K_{k+2}$, and so $|V(G_1)| \geq k+3$ and is not complete. For any $e \in E(G_1^c)$, since $G \in \mathcal{M}(k)$, $G$ has a subgraph $L$ with $\lambda_i(L) \geq k+1$. By Lemma 2, $E(L) \cap X = \emptyset$, and so $L$ must be a subgraph of $G_1$. Thus $G_1 \in \mathcal{M}(k)$, by definition. □

Theorem 3 For $n \geq k+2 \geq 3$,

$$g(n,k) = \binom{k+1}{2} + (n-k-1)(k+1), \tag{10}$$

and $G \in \mathcal{E}(k)$ if and only if one of the following holds:

(i) $G = K_{k+2}$;

(ii) there is some vertex $v \in V(G)$ with $d_G(v) = k+1$ such that $G - v \in \mathcal{E}(k)$;

(iii) $k = 1$, $\delta(G) \geq 3$ and $G$ has a $v$-cut $X = \{e\}$, for some vertex $v \in V(G)$, such that both $v$-components $G_1$ and $G_2$ of $G - e$ are in $\mathcal{E}(1)$.

Proof: We augue by induction on $n$. If $n = k+2$, then Theorem 3 follows from Lemma 1. Hence we assume that $n \geq k+3$. By Lemma 9, we have

$$g(n,k) \geq \binom{k+1}{2} + (n-k-1)(k+1). \tag{11}$$

Let $G$ be a graph in $\mathcal{E}(k)$. If $G$ has a vertex $w$ of degree $k+1$, then by Lemma 10, $G - w \in \mathcal{M}(k)$, and so by induction,

$$\begin{aligned} g(n,k) &= |E(G-w)| + k + 1 \\ &\leq g(n-1,k) + k + 1 \\ &= \binom{k+1}{2} + (n-k-1)(k+1). \end{aligned} \tag{12}$$

It follows by (11) and (12) that $G - w \in \mathcal{E}(k)$ and so Theorem 3 follows by induction in this case.

Hence we assume that $\delta(G) \geq k+2$. Since $G \in \mathcal{E}(k)$, by Lemma 5, $G$ has a $v$-cut $X \subseteq E(G)$ with $|X| = k$ for some vertex $v \in V(G)$. Let $G_1$ and $G_2$ be a pair of $v$-components of $G - X$ and let $n_i = |V(G_i)|$, $(1 \leq i \leq 2)$. Since $\delta(G) \geq k+2$, we have by Corollary 6 that $n_i \geq k+2$. By induction and noting that $n_1 + n_2 = n+1$, we have

$$\begin{aligned} g(n,k) &\leq g(n_1,k) + g(n_2,k) + k \\ &= \binom{k+1}{2} + (n_1-k-1)(k+1) + \binom{k+1}{2} \\ &\quad + (n_2-k-1)(k+1) + k \\ &= 2\binom{k+1}{2} + (n+1-2k-2)(k+1) + k \\ &= (n-k-1)(k+1) + k \end{aligned} \tag{13}$$

$$\leq \binom{k+1}{2} + (n-k-1)(k+1),$$

where equality holds in (13) if and only if $k = 1$. It follows the when $k > 1$ or when $k = 1$ and $G$ has a vertex of degree 2, (ii) of Theorem 3 must hold. It also follows that

Lemma 11 If $G \in \mathcal{E}(1)$, $\delta(G) \geq 3$, $k = 1$ and if $X = \{e\}$ is a $v$-cut of $G$ with $G_1$ and $G_2$ being the $v$-components of $G - e$, then

$$G_1, G_2 \in \mathcal{E}(1). \tag{14}$$

What left is to show the following Lemma 12. By Lemmas 11 and 12, if $k = 1$ and $\delta(G) \geq 3$, then (iii) of Theorem 3 must hold and so the proof of Theorem 3 will be completed.

Lemma 12 Let $G$ be a 2-connected graph that has an edge $e$ such that $G - e$ has a cut-vertex $v$ with $v$-components $G_1$ and $G_2$. If $\delta(G) \geq 3$ and if $G_1, G_2 \in \mathcal{E}(1)$, then $G \in \mathcal{E}(1)$.

Proof: Let $G$ be a smallest counterexample. We first show that $G$ has no subgraph $H$ with $\lambda_i(H) \geq 2$. Assume, to the contrary, that $G$ has a subgraph $H$ with $\lambda_i(H) \geq 2$. Then by Lemma 2, $E(H) \cap \{e\} = \emptyset$ and so $H$ must be a subgraph of either $G_1$ or $G_2$ (say $G_1$). It follows that $\lambda_i(G_1) \geq 2$, contrary to the assumption of $G_1 \in \mathcal{E}(1)$.

Since $G$ is a counterexample, for some edge $e' \in E(G^c)$,

$$G + e' \text{ has no subgraph } L \text{ with } \lambda_i(L) \geq 2. \tag{15}$$

Claim 1: One end of $e'$ lies in $V(G_1) - \{v\}$ and the other end lies in $V(G_2) - \{v\}$.

If both ends of $e'$ are in $V(G_1)$, then since $G_1 \in \mathcal{E}(1) \subseteq \mathcal{M}(1)$, $G_1 + e'$ must have a subgraph $L_1$ with $\lambda_i(L_1) \geq 2$, contrary to (15). This proves Claim 1.

Denote $e = u_1u_2$ and $e' = v_1v_2$. By Claim 1, by the choice of $e$ and by $\kappa(G) \geq 2$, we may assume that

$$v_2, u_2 \in V(G_2) - \{v\} \text{ and } v_1, u_1 \in V(G_1) - \{v\}. \qquad (16)$$

By (15) and by $\kappa(G) \geq 2$, $\lambda_i(G + e') = 1$. Thus there is an edge $e'' \in E(G+e')$ such that $G+e'-e''$ has a cut-vertex $w$. By $\kappa(G) \geq 2$, $e'' \neq e'$ and so by $e'' \in E(G) = E(G_1) \cup E(G_2)$, we may assume that $e'' \in E(G_1)$. Thus $w \in V(G_1)$ and so we may assume that $G_1 - e''$ has two $w$-components $G_1'$ and $G_1''$. By (12) and (13), either $G_1' \cong K_2$ or $G_1' \in \mathcal{E}(1)$, and either $G_1'' \cong K_2$ or $G_1'' \in \mathcal{E}(1)$.

Claim 2: Neither $G_1' \cong K_2$ nor $G_1'' \cong K_2$.

By contradiction, we assume that $G_1' \cong K_2$. Then one of the vertices of $V(G_1')$ is $w$ and the other vertex of $G_1'$ must have degree 2 in $G_1$. It follows by $\delta(G) \geq 3$ that $V(G_1') = \{w, v\}$. But then by Claim 1, $e$ and the edge $vw$ will be in a cycle of $G - e''$, contrary to the fact that $w$ is a cut-vertex of $G + e' - e''$. Similarly, $G_1'' \not\cong K_2$.

Hence by Claim 2, $G_1', G_1'' \in \mathcal{E}(1)$. By (16), we consider the following cases.

Case 1 $v_1, v, u_1 \in V(G_1')$ or $v_1, v, u_1 \in V(G_1'')$.

Suppose that $v_1, v, u_1 \in V(G_1'')$. Then since $G_1', G_2 \in \mathcal{E}(1)$, by the minimality of $G$, $L = G[V(G_1') \cup V(G_2)] \in \mathcal{E}(1)$. Since $v_1, v_2 \in V(L)$, $e' \in E(L^c)$ and so by the definition of $\mathcal{E}(1) \subseteq \mathcal{M}(1)$, $L$ has a subgraph $H$ such that $\lambda_i(H) \geq 2$. It follows that $G + e'$ has a subgraph $H$ with $\lambda_i(H) \geq 2$, contrary to the assumption that $G$ is a conterexample.

Case 2 $v_1, v \in V(G_1'')$ and $u_1 \in V(G_1')$, or $v_1, v \in V(G_1')$ and $u_1 \in V(G_1'')$, or $u_1, v \in V(G_1'')$ and $v_1 \in V(G_1')$, or $u_1, v \in V(G_1')$ and $v_1 \in V(G_1'')$.

Assume first that $v_1, v \in V(G_1'')$ and $u_1 \in V(G_1')$. Note that all $G_1', G_1''$ and $G_2$ are all connected subgraphs of $G$. Note also that

$e = u_1u_2$ and $e' = v_1v_2$ are two edges joining $G_1'$ with $G_2 - v$ and $G_1'' - v$ with $G_2 - v$, respectively. It follows that $w$ is not a cut-vertex of $G + e' - e''$, contrary to the hypothesis on $w$. The proofs for the other cases are similar.

These contradictions establish Lemma 12 and so the proof of Theorem 3 is completed. □

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