Cycles in Line Graphs

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Abstract

Let G be a graph with $E(G) \neq \emptyset$. The line graph of G, written $L(G)$, has $E(G)$ as its vertex set, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. In [J. Graph Theory, 10 (1986) 309 - 324] Thomassen conjectured that all 4-connected line graphs are hamiltonian. We show that this conjecture holds for projective planar graphs. Dominating cycles in line graphs will also be discussed.

I. Introduction

We follow the notation of [1] except otherwise noted. Graphs in this note are finite and simple. Let G be a graph. The line graph of G, written $L(G)$, has $E(G)$ as its vertex set, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. A cycle C in G is a dominating cycle of G if $G - V(C)$ is edgeless, and C is a Hamilton cycle if $V(C) = V(G)$. A graph is called a hamiltonian graph if it has a Hamilton cycle. A connected graph is called an eulerian graph if it has no odd degree vertices. An eulerian subgraph H is a dominating eulerian subgraph of G if $G - V(H)$ is edgeless. In [5], Thomassen poses the following conjecture.

Conjecture 1 [5]. If $L(G)$ is 4-connected, then $L(G)$ is hamiltonian.

In this note, we will prove this conjecture for projective planar graphs. Relationships between dominating cycles in line graphs and dominating eulerian subgraphs in graphs will also be studied.

II. Hamilton cycles in line graphs of projective planar graphs

Several authors have proved conjecture 1 for some special classes of graphs. The following theorem was proved by Zhan [7] and, independently, by Jackson [3].

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Theorem 2.1 [7] If $L(G)$ is 7-connected, then $L(G)$ is hamiltonian. □

In [7], Zhan proves a stronger result that if $L(G)$ is 7-connected, then $L(G)$ is hamiltonian connected.

Theorem 2.2 [4]. Every 4-connected line graph of a planar graph is hamiltonian. □

Following closely the method of [4], we will extend Theorem 2.2 for projective planar graphs. Theorem A will be used.

Theorem A [6]. Every 4-connected projective planar graph is hamiltonian. □

Our main result is:

Theorem 2.3. Let G be a projective planar simple graph. If $L(G)$ is 4-connected, then $L(G)$ is hamiltonian.

In the following, we construct a new projective planar graph $W(G)$ so that if $W(G)$ is hamiltonian, then $L(G)$ is hamiltonian. We shall show that if $L(G)$ is 4-connected, then $W(G)$ is also 4-connected. Thus, by Theorem A, $W(G)$ is hamiltonian, and so $L(G)$ is hamiltonian.

2.1. Definition of a new graph from a projective planar graph

Let G be a simple connected projective planar graph. For each vertex $u \in V(G)$, denote

$$E_G(u) = \{e \in E(G) : \ e \text{ is incident with } u \text{ in } G\}.$$

When there is no confusion, we use $E(u)$ for $E_G(u)$. We assume that G is embedded in the projective plane. Define a new graph $W(G)$ using the following 5 steps:

(W1) Replace each $e \in E(G)$ by a path P_e of length 2 with a new vertex $v(e)$ as the only internal vertex of P_e.

(W2) For each vertex $u \in V(G)$ with $d_G(u) = m \geq 3$, let $e_1, e_2, \cdots, e_m$ denote the edges in $E(u)$, where the subscripts indicate the cyclic ordering of the planar embedding of G. Add new edges $v(e_i)v(e_{i+1})$, for all $1 \leq i \leq m$ (mod m).

(Thus each u together with the new vertices $\{v(e) : e \in E(u)\}$ will induce a wheel with a rim cycle of length $d_G(u)$. We call this wheel the <u>wheel associated with u</u>, and u the <u>center</u> of the wheel).

(W3) For each $u \in V(G)$ with $d_G(u) = 3$, delete the center of the wheel associated with u, (namely, the vertex u).

(W4) For each $u \in V(G)$ with $d_G(u) = 2$, join $v(e)$ and $v(e')$ and delete u, where e and e' are the edges of G incident with u in G.

(W5) For each $u \in V(G)$ with $d_G(u) = 1$, delete u.

The resulting graph will be denoted by $W(G)$. Figure 1 shows some local correspondence between G and $W(G)$.

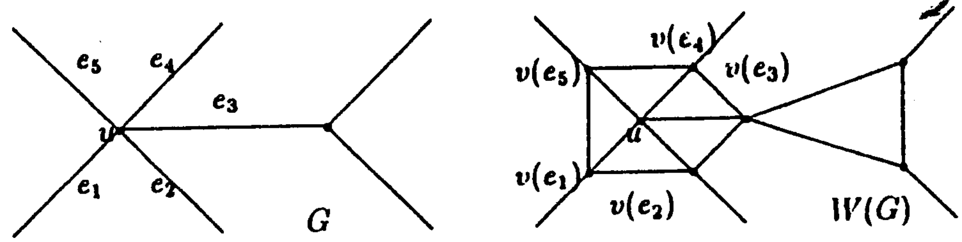

Figure 1: The graphs G and $W(G)$

Lemma 2.1. Let G be a projective planar graph. Each of the following holds:

(i) $V(W(G))$ is the disjoint union of $V(L(G))$ and $\{u \in V(G) : d_G(u) \geq 4\}$.

(ii) The line graph $L(G)$ can be obtained from $W(G)$ by the following operation: For each $u \in V(G)$ with $d_G(u) = m \geq 4$, remove u from $W(G)$, and add $m(m-1)/2$ new edges joining the vertices $\{v(e) : e \in E(u)\}$ so that $\{v(e) : e \in E(u)\}$ induces a complete subgraph K_m in the resulting graph.

(iii) If G is 2-connected and if $L(G)$ is 4-connected, then $W(G)$ is 4-connected.

(iv) If $W(G)$ has a Hamilton cycle, then $L(G)$ has a Hamilton cycle C such that for every vertex $u \in V(G)$ with $d_G(u) \geq 4$, there are two edges $e, e' \in E(u)$ that are consecutive vertices in C.

Proof. A proof of these conclusions can be found in [4].

Lemma 2.2. $W(G)$ is projective planar.

Proof. This follows from the facts that G is a projective planar graph and that the operations (W1) - (W5) preserve projective planarity. □

Lemma 2.3. Let G be a simple 2-connected projective planar graph. If $L(G)$ is 4-connected, then $L(G)$ has a Hamilton cycle C such that for every vertex $u \in V(G)$ with $d_G(u) \geq 4$, there are two edges $e, e' \in E(u)$ that are consecutive vertices in C.

Proof. By Lemma 2.2, $W(G)$ is projective planar. By (iii) of Lemma 2.1, $W(G)$ is 4-connected. Therefore by Theorem A, $W(G)$ has a Hamilton cycle, and so Lemma 2.3 now follows from (iv) of Lemma 2.1. □

2.2. Proof of Theorem 2.3

Let v be a cut vertex of G. Let $V_1, V_2, \cdots, V_c$ be the vertex sets of the components of $G - v$. Each subgraph $G[V_i \cup \{v\}]$ is called a <u>v-component</u> of G. We apply induction on $|V(G)|$ to prove the following theorem that implies Theorem 2.3:

Theorem 2.4. Let G be a simple projective planar graph with $\kappa(L(G)) \geq 4$. Then $L(G)$ has a Hamilton cycle C such that for every vertex $v \in V(G)$ with $d_G(v) \geq 4$, if v is not a cut vertex of G, then there are two distinct edges $e, e' \in E(v)$ that are consecutive vertices of C in $L(G)$.

Proof. If G does not have a cut vertex, then $\kappa(G) \geq 2$, and so Theorem 2.4 follows from (iv) of Lemma 2.1. Therefore we assume that G has a cut vertex v.

Let G_i $(1 \leq i \leq c)$ be the v-components of G. Since $L(G)$ is 4-connected, for each

$i, 1 \leq i \leq c$, either $G_i \cong K_2$, or v has degree at least 4 in G_i. (1)

By (2), by the definition of v-components of G, and by $\kappa(L(G)) \geq 4$, we have $\kappa(L(G_i)) \geq 4$ for each G_i where $G_i \not\cong K_2$.

Therefore by (2) and by induction, for each G_i where $G_i \not\cong K_2$; $L(G_i)$ has a Hamilton cycle C_i satisfying the conclusion of Theorem 2.4. Since G_i is a v-component of G, v is not a cut vertex of G_i, $(1 \leq i \leq c)$. Therefore, for each $G_i \not\cong K_2$, the v-component G_i has two distinct edges $e_i', e_i'' \in E_{G_i}(v)$ with $v(e_i')$ and $v(e_i'')$ being consecutive vertices in C_i. Let $v(e')P_iv(e'') = C_i - v(e_i')v(e_i'')$ denote the Hamilton $(v(e_i'), v(e_i''))$-path in $L(G_i)$.

For each G_j where $G_j \cong K_2$, let e_j denote the only edge in $E(G_j)$, let $v(e_j') = v(e_j'') = v(e_j)$, and for convenience regard the notation $v(e_j')P_jv(e_j'')$ as the single vertex path $v(e_j)$ (only when $G_j \cong K_2$).

Recall that for all $1 \leq i \leq c$, e_i', e_i'' are in $E_G(v)$, and so in $L(G)$, all the $v(e_i')$'s and the $v(e_j'')$'s are vertices in a complete subgraph of $L(G)$. Thus one can glue these cycles and vertices together as follows

$$v(e_1')P_1v(e_1'')v(e_2')P_2v(e_2'')v(e_3')\cdots v(e_c')P_cv(e_c'')v(e_1'),$$

and so a Hamilton cycle of $L(G)$ that satisfies the conclusion of Theorem 2.4 is obtained. Theorem 2.4 is now proved by induction. □

III. Dominating cycles in line graphs

As shown by Harary and Nash-Williams [2], there is a close relationship between dominating eulerian subgraphs and Hamilton cycles in $L(G)$.

Theorem B [2]. The line graph $L(G)$ of a connected graph G is hamiltonian if and only if G has a dominating eulerian subgraph and $G \notin \{K_1, K_2, K_{1,2}\}$. □

In the following, we present a theorem for dominating cycles in line graphs which is analogous to Theorem B. A path with lenght i is denoted by P_i. Note that $P_1 = K_2$ and $P_2 = K_{1,2}$.

Theorem 3.1. Let G be a connected graph and $G \notin \{K_1, K_2, P_2, P_3, P_4\}$. Then the line graph $L(G)$ of G has a dominating cycle if and only if G has an eulerian subgraph H such that each component of $G - V(H)$ is either K_1 or K_2.

Proof. Let H be an eulerian subgraph of G such that each component of $G - V(H)$ is either K_1 or K_2. Since G is connected, every K_1 component of $G - V(H)$ is adjacent to some vertices in $V(H)$, and each K_2 has at least one vertex that is adjacent to some vertices in $V(H)$. Let E_1 be the set of all K_2 components of $G - V(H)$. Let V_1 be the set of vertices in $V(G[E_1])$ which are not adjacent to any vertices in $V(H)$. Let G_e be the graph obtained from G by deleting the edges of E_1 and the resulting isolated vertices, (i.e. $G_e = (G - E_1) - V_1$). Then G_e is a subgraph of G which contains H as a dominating eulerian subgraph, and $V(L(G)) - V(L(G_e)) = E_1$. Since $G \notin \{K_1, K_2, P_2, P_3, P_4\}$, it is easy to check that $G_e \notin \{K_1, K_2, K_{1,2}\}$. By Theorem B, $L(G_e)$ is hamiltonian. Let C_L be a Hamilton cycle in $L(G_e)$. Note that E_1 is a matching in G, and so E_1 is an independent vertex set in $L(G)$. Therefore, since $V(L(G)) - V(C_L) = V(L(G)) - V(L(G_e)) = E_1$, C_L is a dominating cycle in $L(G)$.

Conversely, let $C_L = e_1e_2\cdots e_k$ be a dominating cycle in $L(G)$. Then $E_1 = \{e_1, e_2, \cdots, e_k\}$ is the edge set in G corresponding to the vertices of C_L in $L(G)$. Let $G_1 = G[E_1]$. Then G_1 is a subgraph of G such that $L(G_1)$ has a Hamilton cycle C_L. By Theorem B, G_1 has a dominating eulerian subgraph H_1. Let $V_e = V(L(G)) - V(C_L) = \{e'_1, e'_2, \cdots, e'_r\}$. Then V_e is an independent set in $V(L(G))$ since C_L is a dominating cycle in $L(G)$, and so V_e viewed as an edge set in G is a matching in G. Therefore, each component of $G - V(H_1)$ is either a single vertex K_1 (in $V(G_1) - V(H_1)$) or an edge K_2 (in V_e). The proof is complete. □.

References

[1] J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications". Americn Elsevier (1976).

[2] F. Harary and C.S.J.A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs, Canad. Math. Bull. 8 (1965) 701-710.

[3] B. Jackson, private communication.

[4] H.-J. Lai, Every 4-connected line graph of a planar graph is hamiltonian, Graphs and Combinatorics, accepted.

[5] C. Thomassen, Reflections on graph theory, J. Graph Theory, 10 (1986) 309 - 324.

[6] R. Thomas and X. Yu, 4-connected projective planar graphs are hamiltonian, to appear.

[7] S. M. Zhan, On hamiltonian line graphs and connectivity. Discrete Math. 89 (1991) 89 - 95.