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JACKSON'S CONJECTURE ON EULERIAN SUBGRAPHS

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Abstract

In [*Discrete Math.* 101 (1992), 351-360], Jackson conjectured that if G is a 2-edge-connected graph, then G has an eulerian subgraph H with $|V(H)| \geq 2$ such that for each component F of $G - V(H)$, there are at most three edges between F and H.

In [*J. Graph Theory* 10 (1986), 309-324], Thomassen conjectured that all 4-connected line graphs are hamiltonian.

In this note, we show the following: (1) Jackson's conjecture implies Thomassen's conjecture. (2) Jackson's conjecture holds for graphs having no pseudo-peripheral cuts of size 3. If a connected graph G does not have vertices of degree 3, and if $L(G)$ is 4-connected, then $L(G)$ is hamiltonian. (4) For any graph G, if $L^2(G)$ is 4-connected, the $L^2(G)$ is hamiltonian. (This result is best possible.)

I. Introduction

We follow the notation of [2] except otherwise noted. Graphs may have multiple edges but loops are prohibited. Let G be a graph. The set of all odd vertices of G is denoted by $O(G)$. If $O(G) = \emptyset$, then G is an *even graph*. If G is even and connected, then G is *eulerian*. Note that K_1 is an eulerian graph. The *line graph* of G, written $L(G)$, has $E(G)$ as its vertex set, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. We denote $L^2(G) = L(L(G))$. Let $D_1(G)$ denote the set of vertices of G of degree 1 (*pendant vertices*) in G.

In [8], Thomassen poses the following conjecture.

Conjecture 1 (Thomassen, [8]). If $L(G)$ is 4-connected, then $L(G)$ is hamiltonian.

The following theorem was proved by Zhan [10] and, independently, by Jackson [6].

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Theorem 1.1 (Zhan, [10]). If $L(G)$ is 7-connected, then $L(G)$ is hamiltonian.

In fact, Zhan [10] proves a stronger result that if $L(G)$ is 7-connected, then $L(G)$ is hamiltonian connected. Conjecture 1 also holds for planar graphs.

Theorem 1.2 (Lai, [7]). If G is a simple planar graph, then if $L(G)$ is 4-connected, then $L(G)$ is hamiltonian.

In [1], Jackson has conjectured:

Conjecture 2 (Jackson, [1]). If G is a 2-edge-connected graph, then G has an eulerian subgraph H with $V(H) \neq \emptyset$ such that for each component F of $G - V(H)$, there are at most three edges between F and H.

We shall call an eulerian subgraph H of G satisfying the conclusion of Conjecture 2 a *J-subgraph* of G. A minimal edge cut $X \subset E(G)$ is *pseudo-peripheral* if all the edges in X are incident with a single vertex v such that one component of $G - X$ is an edge-disjoint union of paths $P_1, P_2, \ldots P_m$ with $V(P_i) \cap V(P_j) = \{v\}$, for any $1 \leq i < j \leq m$. Here we regard the single vertex graph K_1 as a path of length zero, and so the three edges incident with a vertex of degree 3 also form a peripheral edge cut of size 3. In this note, we shall prove the following results.

(a) Jackson's conjecture implies Thamassen's conjecture.
(b) Jackson's conjecture holds for graphs having no pseudo-peripheral cuts of size 3.
(c) As a corollary of (2), if a connected graph G does not have vertices of degree 3, and if $L(G)$ is 4-connected, then $L(G)$ is hamiltonian.
(d) For any graph G, if $L^2(G)$ is 4-connected, then $L^2(G)$ is hamiltonian.

The result in part (d) is best possible in the following sense. Let P_{10} denote the Petersen graph and let SP_{10} denote the graph obtained from P_{10} by replacing each edge of P_{10} by a path of length 2. Then $L^2(SP_{10})$ is 3-connected, but $L^2(SP_{10})$ is not hamiltonian. One can easily obtain an infinite family of such examples based on SP_{10}.

2. Jackson's conjecture implies Thomassen's conjecture.

An eulerian subgraph H of G is *dominating* if $G - V(H)$ is edgeless. The following are needed.

Theorem 2.1 (Harary and Nash-Williams, [5]). Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph H.

Lemma 2.2. Let G be a connected graph. If G has a J-subgraph, and if $L(G)$ is 4-connected, then $L(G)$ is hamiltonian.

Proof. By Theorem 2.1, it suffices to show that the J-subgraph H of G is a dominating eulerian subgraph of G. If not, then there is an edge $e \in E(G)$ lying in $G - V(H)$. Let F be the component of $G_1 - V(H)$ containing e. Then there is a set X, say, of at most three edges between F and H in G. Thus $|X| \leq 3$. By the definition of L(G), X is a vertex cut of L(G) contrary to the assumption that L(G) is 4-connected. Therefore, H must be a dominating eulerian subgraph of G. This completes the proof. ∎

Theorem 2.3. Jackson's conjecture implies Thomassen's conjecture.

Proof. Let G be a graph with L(G) 4-connected, and suppose that Conjecture 2 is true. Let $G_1 = G - D_1(G)$. Since L(G) is 4-connected, there is no edge cut $X \subset E(G)$ of G with $|X| \leq 3$ such that both sides of $G - X$ have edges. Therefore $G - D_1(G)$ is 2-edge-connected.

By the truth of Conjecture 2, G_1 has a J-subgraph H. Note that H is also an eulerian subgraph of G. It suffices, by Lemma 2.2, to show that for every component F' of $G - V(H)$, G has at most 3 edges between F' and H. Since every such F' is either a component of $G_1 - V(H)$, or obtained from a component F of $G_1 - V(H)$ by adding some pendant vertices to F, H is also a J-subgraph of G. By Lemma 2.2, L(G) is hamiltonian, and so Conjecture 1 holds. ∎

3. The existence of J-subgraphs and hamiltonian line graphs.

We need a few more terms. For a graph G with $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then by deleting the resulting loops. We write G/e for G/{e} when X = {e}. A graph G is *collapsible* if for every subset $R \subseteq V(G)$ with |R| even, G has a spanning connected subgraph H_R such that $O(H_R) = R$. Catlin introduced collapsible graphs in [3] mainly for finding graphs with spanning eulerian subgraphs. Catlin in [3] showed that every graphs G has a unique collection of maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The *reduction* of G is the graph G' obtained from G by contracting all nontrivial maximal collapsible subgraphs of G. If a vertex v in the reduction G' is the contraction image of a nontrivial subgraph H of G, then we say that v is a *nontrivial* vertex.

Theorem 3.1 (Catlin [3]). Let G be a connected graph. Each of the following holds:

(i) If G is collapsible, then G has a spanning eulerian subgraph.

(ii) If G' is the reduction of G, then either $G' = K_1$, or $\delta(G') \leq 3$.

(iii) Cycles of length at most 3 are collapsible.

Lemma 3.2. Let G be a connected graph, and let $e = uv \in E(G)$ be a cut edge of G. If one component of $G - e$ has a J-subgraph, then G has a J-subgraph.

Proof. Let G_1 be a component of $G - e$ such that $v \in V(G_1)$. By assumption, G_1 has a J-subgraph H. As v is adjacent to at most one component F of $G_1 - V(H)$ in G, H is a J-subgraph of G also. ∎

Corollary 3.3. Let G be a connected graph, and let $v \in D_1(G)$. If $G - v$ has a J-subgraph, then G has a J-subgraph.

Lemma 3.4. Let G be a connected graph and let $v \in V(G)$ be a vertex of degree 2 that is incident with edges $e, e' \in E(G)$. If G/e has a J-subgraph, then G has a J-subgraph.

Proof. Let $G_2 = G/e$. By assumption, G_2 has a J-subgraph H'. Let

$$H = \begin{cases} G[E(H')] & \text{if } e' \notin E(H') \\ G[E(H') \cup \{e\}] & \text{if } e' \in E(H'). \end{cases}$$

Note that H is an eulerian subgraph of G and that $v \in V(H)$ if and only if F is also a component of $G_2 - V(H')$, therefore there are at most 3 edges between F and H, and so H is a J-subgraph of G. If $v \notin V(H)$, then either $e' \in E(G_2[V(H')])$ and so $\{v\}$ is a component of $G - V(H)$, or $e' \in E(F')$ for some component F' of $G_2 - V(H')$ and so v is in the component $F = G[E(F') \cup \{e\}]$ of $G - V(H)$. In any case, since H' is a J-subgraph of G_2, H is a J-subgraph of G. ∎

For a graph G, we define $\tilde{G}$ to be the graph obtained from G by repeatedly deleting all vertices of degree 1 in G until noe are left (the resulting graph at this stage will be called $\tilde{G}_1$), and then by eliminating all vertices of degree 2 by repeatedly contracting as edge that is incident with a vertex of degree 2 until none are left. Note that if $\tilde{G}_1$ is 2-edge-connected and is not a cycle, then $\tilde{G}$ has minimum degree 3.

Proposition 3.5. If $\tilde{G}$ has a J-subgraph, then G has a J-subgraph.

Proof. If follows by combining Corollary 3.3 and Lemma 3.4. ∎

Proposition 3.6. For any graph G, if the reduction $\tilde{G}'$ of $\tilde{G}$ has a nontrivial vertex of degree at most 3 in $\tilde{G}'$, then G has a J-subgraph.

Proof. By Proposition 3.5, it suffices to show that $\tilde{G}$ has a J-subgraph. Thus we may assume that $G = \tilde{G}$. Let $u \in V(\tilde{G}') = V(G')$ be a nontrivial vertex that has degree at most 3 in

Therefore we as assumed that G is not a tree, and so by Corollary 3.7, G has a J-subgraph H. By Lemma 2.2, L(G) is hamiltonian. This proves Corollary 3.8. ■

Corollary 3.9. For any graph G, if $L^2(G)$ is 4-connected, then $L^2(G)$ is hamiltonian.

Proof. We shall show that L(G) has a J-subgraph. We claim first that any vertex of degree at most 3 in $L(\tilde{G})'$ (the reduction of $L(\tilde{G})$) must be nontrivial. Let $v \in V(L(\tilde{G})')$ be a vertex of degree at least 3. If v has degree at most 2 and is tricial, then v should have been deleted, or eliminated, in forming $L(\tilde{G})$, a contradiction. Hence we may assume that v has degree 3 in $L(\tilde{G})'$. If v is trivial, then $v \in V(L(G))$ is a vertex of degree 3. Since L(G) does not have an induced K(1,2) (see [4], page 74), the two edges incident with v must be in a 3-cycle C (say) of L(G). By (iii) of Theorem 3.1, v must be lying in a maximal collapsible subgraph H containing C in L(G), contrary to the assumption that v is a trivial vertes in the reduction $L(\tilde{G})'$. Hence we have proved the claim that any vertex of degree at most 3 in $L(\tilde{G})'$ must be nontrivial. By Proposition 3.6, L(G) has a J-subgraph. By Lemma 2.2 and by the assumption that $L^2(G)$ is 4-connected, $L^2(G)$ must be hamiltonian. ■

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$\tilde{G}'$. Let H'_u denote the preimage of u in G. Note that by the definition of a reduction, H'_u is a collapsible subgraph of G. By (i) of Theorem 3.1, H'_u has a nontrivial eulerian subgraph H_u spanning H'_u. Since the degree of u is at most 3 in G', and since H_u spans $H_{u'}$, for any component F of $G - V(H_u) = G - V(H'_u)$, there are at most 3 edges between F and $H_{u'}$, and so H_u is a J-subgraph of G. This proves Proposition 3.6. ∎

Corollary 3.7. Let G be a connected graph. If G is not a tree and if G does not have a pseudo-peripheral edge cut of size 3, then G has a J-subgraph.

Proof. By Lemma 3.2, we assume that $\tilde{G}_1$ is 2-edge-connected. If $\tilde{G}_1$ is a cycle, then this cycle is a J-subgraph of T. Thus $\tilde{G}$ has minimum degree at least 3.

Since G does not have any pseudo-peripheral edge cut of size 3, if $\tilde{G}$ has a vertex u of degree 3, then in $\tilde{G}_1$, there must be a cycle C_u containing u such that all vertices in $V(C_u - u)$ have degree 2 in G. Such a cycle C_u is a J-subgraph of G, and so we are done.

Therefore we assume that $\delta(\tilde{G}) \geq 4$. Let $\tilde{G}'$ denote the reduction of $\tilde{G}$. By (ii) of Theorem 3.1, either $\tilde{G}$ is collapsible (in that case $\tilde{G}' = K_1$), or $\delta(\tilde{G}') \geq 3$. If $\tilde{G}$ is collapsible, then by (i) of Theorem 3.1, $\tilde{G}$ has a spanning eulerian subgraph H, which is a J-subgraph of G. If $\tilde{G}$ is not collapsible, then since $\delta(\tilde{G}) \geq 4$ and $\delta(\tilde{G}') \leq 3$, the preimage of a vertex u of degree at most 3 in $\tilde{G}'$ must be a nontrivial collapsible sugraph H'_u of G. By Proposition 3.6, G has a J-subgraph. ∎

Corollary 3.8. If a connected graph G does not have vertices of degree 3, and if L(G) is 4-connected, then L(G) is hamiltionian.

Proof. If G has a pseudo-peripheral edge cut X of size 3, then the three edges in X must be incident with a vertex v such that the component of G – X containing v is a path P_v with v being an end of the path. If P_v has length zero, then v is a vertex of degree 3 in G, contrary to the assumption tht G does not have any vertices of degree 3. If P_v has length at least 1, then X would be a vertex cut of L(G), contrary to the assumption that L(G) is 4-connected. Therefore G does not have any pseudo-peripheral edge cut of size 3.

If G is a tree, then any edge of G is a cut edge of G. Since L(G) is 4-connected, G does not have an edge e such that both components of G – e have an edge. Therefore, G $\cong K(1,n-1)$ where $n = |V(G)| \geq 4$. In this case, L(G) is a complete graph and so is hamiltonian.