Nowhere-zero flows in some regular graphs

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Abstract

In [Congressus Numerantium, 44 (1984) 33 - 40], Ellingham proved that if $r \geq 3$ and if G is a r-regular graph with a 2-factor consisting of two chordless cycles A and B, then either G is hamiltonian (therefore G has a nowhere-zero 4-flow) or G has a subdivision of the Petersen graph. In this note, we shall show that all such graphs have nowhere-zero 5-flows and when $r \geq 4$, all such graphs have nowhere-zero 3-flows, regardless of the existence of the subdivision of the Petersen graph.

1. Introduction. We follow the notation of [1] except otherwise noted. Graphs may have multiple edges but loops are prohibited. For an integer $k \geq 1$, a k-factor M of G is a k-regular spanning subgraph of G. Let C be a cycle of G, a chord of C is an edge $e \in E(G[V(C)]) - E(C)$. In [3], Ellingham proved the following theorem.

Theorem 1.1 (M. Ellingham, [3]) Let $r \geq 3$ be an integer. If G is a r-rugular graph with a 2-factor consisting of two chordless cycles A and B, then either G is hamiltonian or G has a subdivision of the Petersen graph.

A nowhere-zero k-flow ($k \geq 2$) of a graph G is an orientation of $E(G)$ together with a mapping from $E(G)$ to the integer weights $\{1, 2, \cdots, k-1\}$ such that, at every vertex, the sum of the weights on the edges directed out equals the sum of the

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weights on the edges directed in. Following Jaeger [4], we let F_k denote the collection of graphs that have nowhere-zero k-flows, $(k \geq 2)$. From the definition, we have

$$F_k \subseteq F_{k'}, \text{ for all } k' \geq k. \tag{1}$$

See Jeager's survey [4] on nowhere-zero flows for more on them. Tutte has made several conjectures on flows.

Conjecture 1 (Tutte, [4]) If G is a 4-edge-connected graph, then G is in F_3.

Conjecture 2 (Tutte, [4]) If G is a 2-edge-connected graph and if G does not have a subdivision of the Peterson graph, then G in in F_4.

Conjecture 3 (Tutte, [4]) If G is a 2-edge-connected graph, then G is in F_5.

The motivation of Theorem 1.1 is, as indicated in [3], to find evidence to support Conjecture 2. As hamiltonian graphs always have nowhere-zero 4-flows, (see Lemma 2.1 below), Theorem 1.1 has the following immediate corollary.

Corollary 1.2 Let $r \geq 3$ be an integer. If G is a r-rugular graph with a 2-factor consisting of two chordless cycles A and B, then either G is in F_4, or G has a subdivision of the Petersen graph.

In this note, we shall show the following:

Theorem 1.3 Let $r \geq 3$ be an integer. If G is a r-rugular graph with a 2-factor consisting of two chordless cycles A and B, then G is in F_5.

Theorem 1.4 Let $r \geq 4$ be an integer. If G is a r-rugular graph with a 2-factor consisting of two chordless cycles A and B, then G is in F_3.

As the Petersen graph satisfies the hypothesis of Theorem 1.3 with $r = 3$, we can see that the condition $r \geq 4$ in Theorem 1.4 cannot be improved. At the end of this note, we shall show that the assumption that G has a 2-factor consisting of two chordless cycles is also necessary.

2. Prelimilaries.

We shall recall some prior results first. A graph G is even if every vertex of G has even degree in G; and G is eulerian if G is both even and connected. A graph is supereulerian if G has a spanning eulerian subgraph. A proof of Lemma 2.1 can be found in [2].

Lemma 2.1 ([2]) If G is supereulerian, then G is in F_4.

Let G be a graph and let e be an edge not in $E(G)$ but both ends of e are in $V(G)$. Then $G+e$ denotes the graph obtained from G by adding e to $E(G)$.

Theorem 2.2 (Jaeger, [4, Theorem 8.1]) Let G be a 2-edge-connected graph. If there is an edge e not in G whose addition to G would make $G+e \in F_4$, then G is in F_5.

Theorem 2.3 (Hall, [1, page 73]) Every regular bipartite graph has a 1-factor.

Theorem 2.4 (Jaeger, [4, Theorem 3.6]) A cubic graph is in F_3 if and only if G is bipartite.

3. The Proofs. In [5], we proved that if G has an eulerian subgraph H with at most two components such that $V(H)$ contains all odd degree vertices of G, then G has a nowhere-zero 5-flow. Thus Theorem 1.3 is a immediate consequence of this result. For the sake of completeness, we present a direct proof of Theorem 1.3 here.

Proof of Theorem 1.3 As A and B are chordless cycles and as $r \geq 3$, one can conclude that G is 2-edge-connected, and that there must be an edge $e' = xy$ with $x \in V(A)$ and $y \in V(B)$. Let e be an edge not in G but has the same ends as e' (that is, e is parallel to e'). Then $G+e$ is supereulerian and so by Lemma 2.1, $G+e$ has a nowhere-zero 4-flow. By Theorem 2.2, G must have a nowhere-zero 5-flow. □

Proof of Theorem 1.4 We argue by induction on $r \geq 4$. If $r = 4$, then G is eulerian and so G has a nowhere-zero 2-flow. By (1), $G \in F_2 \subset F_3$. If $r = 5$, the $G' = G - (E(A) \cup E(B)$ is a cubic bipartite graph and so by Theorem 2.4, $G' \in F_3$ also. Since A and B are two cycles, both A and B are in $F_2 \subset F_3$, by (1). Thus one can combine the nowhere-zero 3-flows on A, B and $G - (E(A) \cup E(B))$ to get a nowhere-zero 3-flow of G. Therefore when $r = 5$, G is also in F_3.

Now assume the truth of Theorem 1.4 for smaller values of r and assume that $r \geq 6$. Let $G' = G - (E(A) \cup E(B))$ again. Then G' is a $(r-2)$-regular bipartite graph. By using Theorem 2.3 twice, G' has a 2-factor M. Let $G'' = G - E(M)$. Then G'' is a $(r-2)$ regular graph with a 2-factor consisting of two chordless cycles A and B. As $r \geq 6$, $r - 2 \geq 4$. By induction, G'' has a nowhere-zero 3-flow. Since M, as a 2-factor, is an even graph, which has a nowhere-zero 2-flow. Combining a nowhere-zero 2-flow of M and a nowhere-zero 3-flow of $G'' = G - E(M)$, one obtains a nowhere-zero 3-flow of G, and so Theorem 1.4 is proved by induction. □

We conclude this note with an example to show that in Theorem 1.4, the assumption of having a 2-factor consisting of 2 chordless cycles is necessary. Let G and H be two vertex disjoint graphs. As in [1], the join of G and H is denoted by $G \vee H$ and defined by

$$V(G \vee H) = V(G) \cup V(H), \text{ and } E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}.$$

Let k be a positive integer. Denote by C_{2k} a cycle of order $2k$ and by M_{k-2} a graph of order $2(k-2)$ consisting of $k-2$ independent edges. Let x, y, z be three vertices that are not in $V(M_{k-2})$ and let $H(k)$ denote the graph obtained from M_{k-2} by adding x, y, z to M_{k-2} as three isolated vertices. We then define a graph $G(2k+1; x, y, z)$ with three distinguished vertices x, y and z by

$$G(2k+1; x, y, z) = C_{2k} \vee H(k).$$

Note that each vertex in $V(G(2k+1; x, y, z)) - \{x, y, z\}$ has degree $2k+1$ and the three distinguished vertices x, y, z have degree $2k$.

For $1 \le i \le 10$, let $L_i(x_i, y_i, z_i) = G(2k+1; x_i, y_i, z_i)$ such that all the L_i's are mutually vertex disjoint, and such that each L_i has x_i, y_i, z_i as three distinguished vertices. Now we obtain our final graph $G(k)$ from the disjoint union $\cup_{i=1}^{10} L_i$ by add the following 15 new edges:

$$x_1x_2, y_1y_3, z_1z_4, y_2y_5, z_2z_9, x_3x_7, z_3z_8, y_4y_6, x_4x_{10}, x_5x_6, z_5x_8, z_6z_7, y_7y_9, y_8y_{10}, x_9z_{10}.$$

Thus $G(k)$ is a $(2k+1)$-regular graph. Since contracting all edges in $\cup_{i=1}^{10} E(L_i)$ in $G(k)$ will result in a Petersen graph, $G(k)$ does not have a nowhere-zero 4-flow, and so by (1), $G(k)$ does not have a nowhere-zero 3-flow either.

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