Edge-Maximal (k, l)-Graphs

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ABSTRACT

A graph G is a (k, l)-graph if for any subgraph H of G, that $|V(H)| \geq l$ implies that $\kappa'(H) \leq k - 1$. An edge-maximal (k, l)-graph G is one such that for any $e \in E(G^c)$, $G + e$ is not a (k, l)-graph. In [F. T. Boesch and J. A. M. McHugh, "An Edge Extremal Result for Subcohesion," *Journal of Combinatorial Theory B,* vol. 38 (1985), pp. 1–7] a class of edge-maximal graphs was found and used to show best possible upper bounds of the size of edge-maximal (k, l)-graphs. In this paper, we investigate the lower bounds of the size of edge-maximal (k, l)-graphs. Let $f(n, k, l)$ denote the minimum size of edge-maximal (k, l)-graphs of order n. We shall give a characterization of edge-maximal (k, l)-graphs. This characterization is used to determine $f(n, k, l)$ and to characterize the edge-maximal (k, l)-graphs with minimum sizes, for all $n \geq l \geq k + 2 \geq 5$. Thus prior results in [F. T. Boesch and J. A. M. McHugh, op. cit.; H.-J. Lai, "The Size of Strength-Maximal Graphs," *Journal of Graph Theory,* vol. 14 (1990), pp. 187–197] are extended. © 1994 John Wiley & Sons, Inc.

1. INTRODUCTION

We follow the notation of Bondy and Murty [2] and consider simple graphs only. For a real number x, $\lfloor x \rfloor$ denotes the largest integer not bigger than x. Let $\kappa'(G)$ denote the edge-connectivity of G and let G^c be the complement of G. For convenience, we define $\kappa'(K_1) = 0$. By $H \subseteq G$ we mean that H is a subgraph of G. Generalizing a prior result of Mader [4], Boesch and McHugh called a graph G, where $|V(G)| \geq l > k \geq 2$, a *(k, l)-graph* if for any $H \subseteq G$, $|V(H)| \geq l$ implies that $\kappa'(H) \leq k - 1$. A (k, l)-graph G is an *edge-maximal* (k, l)-graph if, for any $e \in E(G^c)$, $G + e$ has a subgraph H with $|V(H)| \geq l$ and $\kappa'(H) \geq k$. Edge-maximal $(k, k + 1)$-graphs have been studied in [1]–[6], among others.

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Theorem A (Mader [4]). If G is an edge-maximal $(k+1, k+2)$-graph of order n, then

$$|E(G)| \le (n-k)k + \binom{k}{2},$$

where G is an extremal graph if and only if G has a vertex v of degree k such that $G - v$ is also an extremal graph. ∎

Theorem B (Lai [3]). If G is an edge-maximal $(k+1, k+2)$-graph of order n, then

$$|E(G)| \ge (n-1)k - \binom{k}{2}\left\lfloor \frac{n}{k+2} \right\rfloor.$$

The bound of Theorem B is also best possible. The extremal graphs for Theorem B are characterized in [3] in a way similar to but more complicated than those for Theorem A.

In [1], Boesch and McHugh constructed a class of edge-maximal (k, l)-graphs and use them to extend Theorem A to the following:

Theorem C (Boesch and McHugh [1]). Let G be a simple graph of order n and let $n \ge l \ge k$. Let $s, r \ge 0$ be integers such that $n = s(l-1) + r$ with $0 \le r < l - 1$. If G is an edge-maximal (k, l)-graph, then

$$|E(G)| \le \begin{cases} \dfrac{s(l-1)(l-2)}{2} + (s-1+r)(k-1), \\ \qquad\qquad 2(k-1) < l-1 \text{ and } r < 2(k-1), \\ \dfrac{s(l-1)(l-2)}{2} + s(k-1) + \dfrac{r(r-1)}{2}, \\ \qquad\qquad 2(k-1) < l-1 \text{ and } r \ge 2(k-1), \\ \dfrac{(l-1)(l-2)}{2} + (n-l+1)(k-1), \\ \qquad\qquad 2(k-1) \ge l-1. \end{cases}$$

The main result given in Section 2 is to do for Theorem C what Theorem B does for Theorem A. In Section 3, the structure of edge-maximal (k, l)-graphs are discussed, and the proofs of the main results are in Section 4. Au: correct as edited?

2. MAIN RESULT

Let $\mathcal{E}(k,l)$ denote the collection of all edge-maximal (k,l)-graphs and, for integers $n \geq l \geq k+1$, define

$$f(n,k,l) = \min\{|E(G)| : G \in \mathcal{E}(k,l) \text{ and } |V(G)| = n\}.$$

And define

$$S\mathcal{M}(n,k,l) = \{G : G \in \mathcal{E}(k,l), |V(G)| = n \text{ and } |E(G)| = f(n,k,l)\}.$$

Thus Theorem B gives a result on $f(n,k+1,k+2)$. In this paper, we shall determine $f(n,k,l)$ for all $n \geq l \geq k+2 \geq 5$, and characterize all (k,l)-graphs and all graphs in $S\mathcal{M}(n,k,l)$. For $k \geq 3$, we have

$$f(n,k,l) = \begin{cases} \binom{l-1}{2} + (n-l+1)(k-1), & l \leq n < 2k+2, \\ (n-1)(k-1) - \left\lfloor \frac{n}{k+1} \right\rfloor \frac{k^2-3k}{2}, & l \leq 2k+2 \leq n, \\ (n-2t+1)(k-1) + t(t-1) - \left\lfloor \frac{n-2t}{k+1} \right\rfloor \frac{k^2-3k}{2}, & n \geq l = 2t \geq 2k+3, \\ (n-2t)(k-1) + t^2 - \left\lfloor \frac{n-2t-1}{k+1} \right\rfloor \frac{k^2-3k}{2}, & n \geq l = 2t+1 \geq 2k+3. \end{cases}$$

3. THE STRUCTURES OF EDGE-MAXIMAL *(k, l)*-GRAPHS

Let m,k,l be positive integers. By Theorem B, we shall assume, throughout the paper, that $l \geq k+2$. Let T be a tree of order $m \geq 2$ with $V(T) = \{v_1, v_2, \ldots, v_m\}$. For positive integers $r_1, r_2, \ldots, r_m$, define $T(r_1, r_2, \ldots, r_m; k, l)$ to be the collection of simple graphs obtained from T by replacing each vertex v_i by a complete graph $R_i = K_{r_i}$, $(1 \leq i \leq m)$, and by replacing each edge v_iv_j (say) of T by a set of $k-1$ edges with one end in R_i and the other end in R_j in such a way that the resulting graph is simple. When no confusion arises, we also use $T(r_1, \ldots, r_m; k, l)$ to denote a graph in the collection. We consider the following constraints to the parameters r_i's:

$$\text{for each } i, \quad r_i = 1 \quad \text{or} \quad l-1 \geq r_i \geq k+1; \tag{1}$$

$$\text{if } v_iv_j \in E(T), \quad \text{then } r_i + r_j \geq l. \tag{2}$$

Let $\mathcal{T}(k, l)$ denote the collection of all $T(r_1, r_2, \ldots, r_m; k, l)$, $(l > k)$, such that (1) and (2) are satisfied. Thus if $G \in \mathcal{T}(k, l)$, then

$$|V(G)| \geq l. \tag{3}$$

Lemma 1. Let $T = v_1 v_2 \ldots v_m$ be a (v_1, v_m)-path with $m \geq 2$, and let $G = T(r_1, r_2, \ldots, r_m; k, l)$ be a graph in $\mathcal{T}(k, l)$. Then each of the following holds:

(a) $\kappa'(G) = k - 1$ and every edge-cut X with $|X| = k - 1$ consists of all edges from R_i to R_{i+1}, for some $i \in \{1, 2, \ldots, m - 1\}$.
(b) If $e \in E(G^c)$ is an edge that joins a vertex in R_1 to one in R_m, then $\kappa'(G + e) = k$.

Proof. It follows directly from the definitions. ∎

Note that all edge-maximal (k, l)-graphs constructed in [1] are in $\mathcal{T}(k, l)$. We therefore generalize a result in [1] to the following:

Corollary 1 ([1]). Every member in $\mathcal{T}(k, l)$ is in $\mathcal{E}(k, l)$.

Proof. Let $G = T(r_1, r_2, \ldots, r_p; k, l) \in \mathcal{E}(k, l)$. Let $e \in E(G^c)$. We shall show that $G + e$ has a subgraph H with $|V(H)| \geq l$ and $\kappa'(H) \geq k$. Since $e \in E(G^c)$ and since every R_i is a complete graph, e must join some R_i to some R_j, $i \neq j$. Since T is a tree, there is a unique path from the vertex representing R_i to that representing R_j. Without loss of generality, we may assume that $i = 1$ and $j = m$, and that $P = v_1 v_2 \ldots v_m$ is that path. Thus by (b) of Lemma 1, G has a subgraph $H = P(r_1, \ldots, r_m; k, l)$ with the desired conditions. ∎

Lemma 2. If $|V(G)| \geq l > k$ and if $G \in \mathcal{E}(k, l)$, then $\kappa'(G) = k - 1$.

Proof. Obvious. ∎

Lemma 3. Let $G \in \mathcal{E}(k, l)$ and let X be an edge-cut of G with $|X| = k - 1$ and with G_1 being a component of $G - X$.

(a) If $|V(G_1)| \geq l$, then $G_1 \in \mathcal{E}(k, l)$.
(b) If $|V(G_1)| \leq l - 1$ and $l \geq k + 2$, then G_1 is a complete graph; moreover, when $k \geq 3$, either $|V(G_1)| = 1$ or $|V(G_1)| \geq k + 1$.

Proof. If $|V(G_1)| \geq l$, then G_1 is not a complete graph. Since $G \in \mathcal{E}(k, l)$, for any $e \in E(G_1^c)$, $G + e$ has a subgraph L with $|V(L)| \geq l$ and $\kappa'(L) \geq k$. By Lemma 2, L must be a subgraph of G_1 and so (a) of Lemma 3 follows.

Now suppose that $|V(G_1)| \leq l - 1$. If G_1 is not complete, then there is an edge $e \in E(G_1^c) \subseteq E(G^c)$ and so $G + e$ has no subgraph H with $|V(H)| \geq l$ and $\kappa'(H) \geq k$, contrary to the assumption that $G \in \mathcal{E}(k, l)$. Hence G_1 must be complete.

In the following, we shall show that if $1 < |V(G_1)| \leq k$, then $k = 2$. Assume that r_1 is an integer so that $G_1 = K_{r_1}$ and suppose that

$$1 < r_1 \leq k\,. \tag{4}$$

Claim. There is a vertex $u \in V(G_1)$, not incident with at least one edge joining G_1 to G_2, such that there is a vertex $u' \in V(G_2)$ with $e = uu' \in E(G^c)$.

If there is a vertex in G_1 incident with all edges in X, then the claim holds since $r_1 > 1$. Therefore we assume that no vertex in G_1 that is incident with all edges in X.

Let $r_2 = |V(G_2)|$. If no such vertices satisfying the claim exist, then since every vertex in G_1 is adjacent to all vertices in G_2, we have $k - 1 = r_1 r_2$. By (4) and by $r_1 + r_2 = |V(G)| \geq l \geq k + 2$, we have $r_2 \geq l - r_1 \geq 2$. Since $r_1 \geq 2$, and we have $r_2 \geq 2$, $r_1 r_2 \geq r_1 + r_2$, and so $k - 1 \geq r_1 r_2 \geq r_1 + r_2 = |V(G)| \geq l$, a contradiction. This proves the claim. ∎

Let $e = uu' \in E(G^c)$ be the edge defined in the claim above. Since $G \in \mathcal{E}(k, l), G + e$ has a subgraph H with $|V(H)| \geq l$ and $\kappa'(H) \geq k$. Let $H_i = H \cap G_i$, $(1 \leq i \leq 2)$. Since $\kappa(H) \geq k$, all the $k - 1$ edges joining G_1 and G_2 are in $E(H)$ and $e \in E(H)$. By the choice of e and by $k \geq 3$, there is at least one vertex $v \in V(H_1) - \{u\}$. Hence

$$r_1 \geq |V(H_1)| \geq 2\,. \tag{5}$$

Note that $\delta(H) \geq \kappa'(H) \geq k$. Counting the incidents in H_1, we get

$$k|V(H_1)| \leq \sum_{v \in V(H_1)} \deg_H(v) \leq |V(H_1)|\,(|V(H_1)| - 1) + k\,,$$

and so $(|V(H_1)| - 1)k \leq (|V(H_1)| - 1)|V(H_1)|$. By (5) and (4),

$$|V(H_1)| = k = r_1 = |V(G_1)|, \qquad \text{and so } H_1 = G_1. \tag{6}$$

It follows that every vertex in $G_1 - u$ is incident with exactly one of the $k - 1$ edges joining G_1 to G_2. Since $r_2 \geq 2$, there is an edge $e' \in E(G^c)$ not incident with u. By $G \in \mathcal{E}(k, l)$, $G + e'$ has a subgraph H' with $|V(H')| \geq l$ and $\kappa'(H') \geq k$. Let $H_i' = H' \cap G_i$, $(1 \leq i \leq 2)$. Note that then u has degree $k - 1$ in $G + e'$ and so $u \notin V(H_1')$. Since every vertex in $G_1 - u$ is incident with one of the $k - 1$ edges joining G_1 to G_2,

$H_1' = G_1 - u$ and so $|V(H_1')| = k - 1$, by (5). It follows by counting the incidences in H_1' that

$$\begin{aligned} k(k-1) = k|V(H_1')| &\le \sum_{v \in V(H_1')} \deg_{H'}(v) \\ &\le |V(H_1')|(|V(H_1')| - 1) + k \\ &= (k-1)(k-2) + k. \end{aligned}$$

Thus $k = 2$, as desired. ∎

Corollary 2. For $l \ge k + 2 \ge 5$, every graph $G \in \mathcal{I}(k,l)$ contains a $K_2(r_1, r_2; k, l)$ with

$$l \le r_1 + r_2, \qquad \text{and } r_i = 1 \quad \text{or} \quad k + 1 \le r_i \le l - 1, \qquad (1 \le i \le 2). \tag{7}$$

Corollary 3. For $k \ge 3$ and for every $G \in \mathcal{I}(k,l)$,

(a) G has a complete subgraph K_r with $3 \le r \le (l-1)/2$; and
(b) every complete subgraph in G has order at most $l - 1$.

Proof. Corollary 3 follows from Corollary 2 and definition of (k,l)-graphs. We argue by induction to show Corollary 2. By Lemma 2, G has an edge-cut X with $|X| = k - 1$. Let G_1 and G_2 be the two components of $G - X$. If $|V(G_i)| \ge l$ for some $i \in \{1,2\}$, then we are done by induction. Therefore we assume that $|V(G_i)| \le l - 1$ $(1 \le i \le 2)$. Thus Corollary 2 follows from (b) of Lemma 3. ∎

Definition. Let H_1 and H_2 be vertex-disjoint graphs with $\max\{|V(H_1)|, |V(H_2)|\} \ge k$. A *$(k,l)$-joint* of H_1 and H_2 is a simple graph obtained from the disjoint union of H_1 and H_2 by adding k new edges $e_1, e_2, \ldots, e_k$ to $H_1 \cup H_2$ such that each e_i is incident with a vertex in $V(H_1)$ and a vertex in $V(H_2)$, and such that if the new edges $e_1, e_2, \ldots, e_k$ are joining two maximal complete subgraphs $K_{r_1} \subseteq H_1$ and $K_{r_2} \subseteq H_2$, then the orders of these subgraphs must satisfy $r_1 + r_2 \ge l$. Denote by $[H_1, H_2]_k^l$ the set of all (k,l)-joints of H_1 and H_2. Clearly $[H_1, H_2]_k^l = [H_2, H_1]_k^l$.

Lemma 4. Let H_1 and H_2 be two vertex disjoint graphs such that one of the follows holds:

(a) $H_1 = K_{r_1}$, $H_2 = K_{r_2}$ with $r_1 + r_2 \ge l \ge k + 2$ and either $r_i = 1$ or $k + 1 \le r_i \le l - 1$, $i \in \{1,2\}$,
(b) $H_1 \in \mathcal{I}(k,l)$, and $H_2 = K_{r_2}$ with $r_2 = 1$ or $k + 1 \le r_2 \le l - 1$,
(c) $H_1, H_2 \in \mathcal{I}(k,l)$,

then $[H_1, H_2]^l_{k-1} \subseteq \mathcal{E}(k,l)$.

Proof. If (a) holds, then any graph in $[H_1, H_2]^l_{k-1}$ is a $K_2(r_1, r_2; k, l)$ and so Lemma 4 follows from Lemma 1. Hence we assume that (b) or (c) holds.

Let $G \in [H_1, H_2]^l_{k-1}$ and let E' denote the $k-1$ edges joining H_1 and H_2. By contradiction, we assume that G is a counterexample to the lemma with as few vertices as possible. By definition of $[H_1, H_2]^l_{k-1}$ and Lemma 2, $\kappa'(G+e) \leq k-1$ and so $G+e$ has a bond X with $|X| \leq k-1$. By Lemma 2, G is a (k,l)-graph. Since $G \notin \mathcal{E}(k,l)$, there is an edge $e \in \mathcal{E}(G^c)$ such that

$$G + e \text{ is also a } (k,l)\text{-graph.} \tag{8}$$

By (8), and by (b) or (c) of Lemma 4, we may assume that $e = x_1x_2$ with $x_1 \in V(H_1)$ and $x_2 \in V(H_2)$. By (8) and Lemma 2, $\kappa'(H_i) \geq k-1$, and so $e \notin X$. It follows that $(E' \cup \{e\}) \cap X = \emptyset$ and so we may assume that $X \subseteq E(H_1)$.

Let H_1' and H_1'' be the two components of $H_1 - X$. Since X is an edge-cut of $G+e$, exactly one of H_1' and H_1'' is incident with all edges in $E' \cup \{e\}$. We assume that H_1' is incident with all edges in $E' \cup \{e\}$ and let $G' = G - V(H_1'')$. Clearly $G' \in [H_1', H_2]^l_{k-1}$. Since $H_1 \in \mathcal{E}(k,l)$, by Lemma 3, H_1' is either a K_r with $r = 1$ or $k+1 \leq r \leq l-1$, or $H_1' \in \mathcal{E}(k,l)$. Note that by the definition of the (k,l)-joints, when $H_1' = K_r$ and $H_2 = K_{r_2}$, we have $r + r_2 \geq l$. It follows by the minimality of G and by (a) of Lemma 4 that $G' \in \mathcal{E}(k,l)$ and so $G' + e$ has a subgraph H with $|V(H)| \geq l$ and $\kappa'(H) \geq k$. But $H \subseteq G' + e \subseteq G + e$, contrary to the assumption that G is a counterexample. This completes the proof of Lemma 4. ∎

Definition. Let $\mathcal{M}(k,l)$ denote the collection of graphs that contains all the graphs $K_2(r_1, r_2; k, l)$ with r_1, r_2 satisfying (2) and (3), such that a graph G of order at least $2l-1$ is in $\mathcal{M}(k,l)$ if and only if there exist graphs H_1 and H_2, where H_1 and H_2 satisfy one of the hypotheses of Lemma 4, such that $G \in [H_1, H_2]^l_{k-1}$.

Theorem 1. For $k \geq 3$ and $l \geq k+2$, $\mathcal{E}(k,l) = \mathcal{M}(k,l)$.

Proof. By Lemma 4, $\mathcal{M}(k,l) \subseteq \mathcal{E}(k,l)$, for any $k \geq 2$. To see $\mathcal{E}(k,l) \subseteq \mathcal{M}(k,l)$, let $G \in \mathcal{E}(k,l)$ be a minimum counterexample. By Lemma 2, G has an edge-cut X with $|X| = k-1$ such that $G - X$ has two components, H_1, H_2 (say). Choose X so that

$$|V(G_1)| \text{ is minimized.} \tag{9}$$

If one of the H_i's is a complete subgraph, say $H_1 = K_r$, and if not all the edges in X are incident with vertices of a complete subgraph in H_2, then by Lemma 3, either $r = 1$ or $r \geq k + 1$ and so $G \in [H_1, H_2]_{k-1}^l$, a contradiction.

Suppose then that $H_1 = K_r$ and all edges in X are incident with vertices of a maximal complete subgraph $K_t \subseteq H_2$ $(t > 1)$. If $H_2 = K_t$, then since $G \in \mathcal{E}(k,l)$, we must have $r + t \geq l$ and so $G \in [H_1, H_2]_{k-1}^l$, a contradiction. Hence by (a) of Lemma 3, $H_2 \in \mathcal{E}(k,l)$ and so by Lemma 2, H_2 has an edge-cut X' with $|X'| = k - 1$. Let H_2' and H_2'' be the two components of $H_2 - X'$. By Lemma 3, we may assume that the intersection of K_t and H_2' is a complete subgraph $K_{t'}$ and $|V(H_2')| \geq k + 1$.

Since $|V(H_2')| \geq k + 1$, there must be an edge $e \in E(G^c)$ that joins H_1 to H_2'. Since $G \in \mathcal{E}(k,l)$, $G + e$ has a subgraph L with $|V(L)| \geq l$ and $\kappa'(L) \geq k$. Since $|X'| = k - 1$, H_2'' is vertex-disjoint from L. Thus by $\kappa'(L) \geq k$ and by $|X| = k - 1$, all edges in X must be incident with vertices in $K_{t'}$ and so by the minimality of G, $G - V(H_2'') \in [H_1, H_2']_{k-1}^l$ and so $r + t \geq r + t' \geq l$. Therefore $G \in [H_1, H_2]_{k-1}^l$, a contradiction.

Finally we assume that neither H_1 nor H_2 is a complete subgraph, and so by (a) of Lemma 3, $H_i \in \mathcal{E}(k,l)$ $(1 \leq i \leq 2)$. By Lemma 2, H_1 has an edge-cut X'' with $|X''| = k - 1$. Let H_1' and H_1'' be the two components of $H_1 - X''$. By (9), some edges in X must be incident with vertices in H_1' and some edges in X must be incident with vertices in H_1''. By Lemma 3, we may assume that $|V(H_1')| \geq k + 1$ and so there must be an edge $e' \in E(G^c)$ that joins H_2 and H_1'. Since $G \in \mathcal{E}(k,l)$, $G + e'$ has a subgraph L' such that $|V(L')| \geq l$ and $\kappa'(L') \geq k$. Since $|X''| = k - 1$, L' and H_1'' are vertex-disjoint and so all edges in X must be incident with vertices in H_1'. However, one can then replace X by X'', contrary to (9). This completes the proof of Theorem 1. ∎

4. THE PROOFS OF THE MAIN RESULTS

We need two more lemmas.

Lemma 5. Let $G \in \mathcal{SM}(n,k,l)$ and let $X \subseteq E(G)$ be an edge-cut with $|X| = k - 1$ and with H_1 and H_2 being the two components of $G - X$. If $n_i = |V(H_i)| \geq l$, then $H_i \in \mathcal{SM}(n_i,k,l)$.

Proof. Suppose that $n_1 = |V(H_1)| \geq l$ but $H_1 \notin \mathcal{SM}(n_1,k,l)$. By Lemma 3, $H_1 \in \mathcal{E}(k,l)$. Hence there is some $H' \in \mathcal{SM}(n_1,k,l)$ with

$$|E(H')| < |E(H_1)|. \tag{10}$$

Choose some $G' \in [H', H_2]_{k-1}^l$. By Lemma 4, $G' \in \mathcal{E}(k,l)$ and so

$$f(n,k,l) \leq |E(G')| = |E(H')| + |E(H_2)| + (k - 1). \tag{11}$$

Combine (10) and (11) to get $f(n,k,l) < |E(H_1)| + |E(H_2)| + (k-1) = |E(G)|$, contrary to the fact that $G \in S\mathcal{M}(n,k,l)$. ∎

Lemma 6. Suppose that $k + 2 \leq l \leq 2k + 2 \leq n$. If $G \in S\mathcal{M}(n,k,l)$, then G does not have a complete subgraph of order at least $k + 2$.

Proof. By contradiction, we assume that there is a minimum counterexample G that satisfies the hypothesis of the lemma but has a subgraph $L \cong K_r$ with

$$r \geq k + 2. \tag{12}$$

By Theorem 1, $G \in [H_1, H_2]^l_{k-1}$ for some graphs H_1 and H_2. Recall that there are only $k - 1$ edges joining H_1 to H_2 in G. This implies that

$$\text{either } L \subseteq H_1 \quad \text{or} \quad L \subseteq H_2. \tag{13}$$

By (13), we may assume that $L \subseteq H_1$.

Case 1. H_1 is a complete graph. Then we may assume that $H_1 = L = K_r$. By (12) and (13), there is a vertex $v \in V(H_1)$ that is not incident with any edges joining H_1 to H_2.

If $n = |V(G)| > l$, then by Lemma 4, $G - v \in \mathcal{E}(k,l)$ and so any $G' \in [K_1, G - v]^l_{k-1}$ is in $\mathcal{E}(k,l)$. But then by (12), $|E(G')| = |E(G - v)| + k - 1 = |E(G)| - (r - 1) + k - 1 < |E(G)|$, contrary to the assumption that $G \in S\mathcal{M}(n,k,l)$.

Hence $n = 2k + 2 = l$. By (b) of Lemma 3 and by (12), we must have $H_2 = K_1$. But then by Lemma 4, any $G'' \in [K_{k+1}, K_{k+1}]^l_{k-1}$ is in $\mathcal{E}(k,l)$ and satisfies

$$|E(G'')| = (k+1)k + (k-1) < k(2k+1) + (k-1) = |E(G)|,$$

contrary to the assumption that $G \in S\mathcal{M}(n,k,l)$.

Case 2. H_1 is not a complete graph. Then by (b) of Lemma 3, $H_1 \in \mathcal{E}(k,l)$ and so $n_1 \geq l$. By Lemma 5, $H_1 \in S\mathcal{M}(n_1,k,l)$. This violates the minimality of G. ∎

The following fact will be used in the proofs below: for positive real numbers x and y,

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor. \tag{14}$$

Theorem 2. If $n \geq l \geq k + 2 \geq 5$, then

$$f(n,k,l) = \begin{cases} \binom{l-1}{2} + (n - l + 1)(k - 1), & l \leq n < 2k + 2, \\ (n - 1)(k - 1) - \left\lfloor \frac{n}{k+1} \right\rfloor \frac{k^2 - 3k}{2}, & l \leq 2k + 2 \leq n, \\ (n - 2t + 1)(k - 1) + t(t - 1) - \left\lfloor \frac{n - 2t}{k+1} \right\rfloor \frac{k^2 - 3k}{2}, & n \geq l = 2t \geq 2k + 3, \\ (n - 2t)(k - 1) + t^2 - \left\lfloor \frac{n - 2t - 1}{k+1} \right\rfloor \frac{k^2 - 3k}{2}, & n \geq l = 2t + 1 \geq 2k + 3, \end{cases}$$

where $G \in S\mathcal{M}(n,k,l)$ if and only if one of the following holds:

(i) $5 \leq k + 2 \leq l \leq n < 2k + 2$, and $G = K_2(l - 1, 1; k, l)$ or G has a vertex v of degree $k - 1$ such that $G - v \in S\mathcal{M}(n - 1, k, l)$. (In other words, G has a unique nontrival complete subgraph K_r and r must be $l - 1$.)

(ii) $k + 2 \leq l \leq 2k + 2 \leq n$, and $G = K_2(k + 1, k + 1; k, l)$ or $G \in [H_1, H_2]_{k-1}^{l}$ such that $H_1 \in \{K_1, K_{k+1}\} \cup S\mathcal{M}(n,k,l)$ and $H_2 \in H_1 \in \{K_1, K_{k+1}\} S\mathcal{M}(n,k,l)$ and such that

$$\left\lfloor \frac{|V(H_1)|}{k+1} \right\rfloor + \left\lfloor \frac{|V(H_2)|}{k+1} \right\rfloor = \left\lfloor \frac{n}{k+1} \right\rfloor.$$

(iii) $n \geq l = 2t \geq 2k + 3$, and $G = K_2(t, t; k, l)$ or $G \in [H_1, H_2]_{k-1}^{l}$ such that $H_1 \in \{K_1, K_{k+1}\}$ and $H_2 \in S\mathcal{M}(|V(H_2)|, k, l)$.

(iv) $n \geq l = 2t + 1 \geq 2k + 3$, and $G = K_2(t + 1, t; k, l)$ or $G \in [H_1, H_2]_{k-1}^{l}$ such that $H_1 \in \{K_1, K_{k+1}\}$ and $H_2 \in S\mathcal{M}(|V(H_2)|, k, l)$.

Proof. For $k + 2 \leq l \leq n \leq 2k + 1$, it follows by Theorem 1 and Lemma 3 that G has one and only one complete subgraph K_r with $r \geq k + 1$. Since K_r is the only complete subgraph of G of order at least $k + 1$, and by Corollary 2, we have $r = l - 1$ and so G has a $K_2(l - 1, 1; k, l)$. Hence if $n = l$, then $G = K_2(l - 1, 1; k, l)$ and we are done. When $n > l$, (i) of Theorem 2 follows by Theorem 1 and by induction on n.

We now assume that $k + 2 \leq l \leq 2k + 2 \leq n$. We first show that

$$f(n,k,l) \leq (n - 1)(k - 1) - \left\lfloor \frac{n}{k+1} \right\rfloor \frac{k^2 - 3k}{2}. \tag{15}$$

Let $m = \lfloor n/(k+1) \rfloor \geq 2$ and let T be a tree of m vertices. By Corollary 1, $T(k+1, \ldots, k+1; k, l) \in \mathcal{E}(k, l)$. Form a graph $G(n, k, l)$ of order n from $T(k+1, \ldots, k+1; k, l)$ by joining each of the other $n - m(k+1)$ vertices via the way of (b) of Lemma 4. (By $m \geq 2$, this can be done). By Lemma 4, $G(n, k, l) \in \mathcal{E}(k, l)$ and $|E(G(n, k, l)|$ equals the right-hand side of (15).

Let $G \in S\mathcal{M}(n, k, l)$. If $n = 2k + 2$, then by Corollary 2, G contains a $K_2(r_1, r_2; k, l)$. By Lemma 7, $r_i \leq k + 1$ $(1 \leq i \leq 2)$. Thus by Corollary 2, either a K_{k+1} is the unique complete subgraph of G with order at least 3, or $G \in K_2(k+1, k+1; k, l)$. If the former holds, then since $l \geq k + 2$, Corollary 2 and the fact that G has only one complete subgraph of order at least 3 imply that this complete subgraph must have order at least $l - 1 \geq k + 1$. Thus this can happen only when $l = k + 2$ and G has a vertex v of degree $k - 1$ such that $G - v \in S\mathcal{M}(n, k, l)$. Hence (ii) of Theorem 2 holds for $n = 2k + 2$. We shall show (ii) of Theorem 2 by induction on n.

Assume that $n \geq 2k + 3$. By Theorem 1, $G \in [H_1, H_2]^l_{k-1}$. We may assume that $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$ with $n_1 \leq n_2$. If $n_1 \geq l$, then by induction,

$$\begin{aligned} f(n,k,l) &\geq \sum_{i=1}^{2} f(n_i, k, l) + (k-1) \\ &\geq (n-1)(k-1) - \left(\left\lfloor \frac{n_1}{k+1} \right\rfloor + \left\lfloor \frac{n_2}{k+1} \right\rfloor \right) \frac{k^2 - 3k}{2}. \end{aligned} \tag{16}$$

Hence (ii) of Theorem 2 follows by (14), (16), (15), and by induction. The case when $n_1 = 1$ can be done similarly.

Thus by (b) of Lemma 3 and by Lemma 6, $H_1 = K_{k+1}$. If $n_2 \leq l - 1$, then by (b) of Lemma 3, $G = K_2(n_1, n_2; k, l)$. But by Lemma 6, $n_1 = n_2 = k + 1$, contrary to $n \geq 3k + 3$. Hence $n_2 \geq l$ and so $G \in [K_{k+1}, H_2]^l_{k-1}$, for some $H_2 \in \mathcal{E}(k, l)$. By induction,

$$\begin{aligned} f(n,k,l) &\geq f(n-k-1, k, l) + \frac{k(k+1)}{2} + (k-1) \\ &\geq (n-k-2)(k-1) - \left\lfloor \frac{n-k-1}{k+1} \right\rfloor \frac{k^2 - 3k}{2} \\ &\quad + \frac{(k+1)k}{2} + (k-1) \\ &= (n-1)(k-1) - \left\lfloor \frac{n}{k+1} \right\rfloor \frac{k^2 - 3k}{2}. \end{aligned} \tag{17}$$

By (15) and (17), (ii) of Theorem 2 follows.

We then assume that $n \geq l = 2t \geq 2k + 3$. Let $m = \lfloor (n - 2t)/(k + 1) \rfloor$. Obtained a graph L' of order $m(k + 1) + 2t$ from $K_2(t, t; k, l)$ by adding m distinct K_{k+1}'s, disjoint from the $K_2(t, t; k, l)$, and joining each of these K_{k+1}'s with $k - 1$ new edges to the two K_t's so that each K_t is incident with at least one new edge. By Lemma 4, $L' \in \mathcal{E}(k, l)$. Form a simple graph $L(n, k, l)$ from L' by adding $n - 2t - m(k + 1)$ new vertices and joining each new vertices with $k - 1$ edges so that not all of the new edges are incident with the same complete subgraph. By Lemma 4, $L(n, k, l) \in \mathcal{E}(k, l)$. Hence

$$f(n,k,l) \leq |E(L(n,k,l))| \leq (n - 2t + 1)(k - 1) + t(t - 1) - \left\lfloor \frac{n - 2t}{k + 1} \right\rfloor \frac{k^2 - 3k}{2}. \tag{18}$$

Let $G \in \mathcal{SM}(n, k, l)$. By Theorem 1, $G \in [H_1, H_2]^l_{k-1}$ for some H_1 and H_2. Choose H_1 and H_2 so that

$$|V(H_1)| \text{ is minimized.} \tag{19}$$

Let $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$ and assume that $n_1 \leq n_2$.

If $n = l$, then $n_1 \leq n_2 \leq l - 1$ and so by (b) of Lemma 3, we have $H_i = K_{n_i}$ $(1 \leq i \leq 2)$, and so $|E(G)| = n_1(n_1 - 1)/2 + (n - n_1)(n - n_1)/2 + k - 1$. As a function of n_1, the minimum is reached at $n_1 = n_2 = t$ and so G must be in $K_2(t, t; k, l)$. Thus we assume that $n > l$.

If $n_1 = 1$, then since $n \geq l + 1$, we have $n_2 \geq l$ and so by induction,

$$|E(G)| \geq f(n - 1, k, l) + k - 1 = (n - 2t + 1)(k - 1) + t(t - 1) - \left\lfloor \frac{n - 2t - 1}{k + 1} \right\rfloor \frac{k^2 - 3k}{2},$$

and so by (18), equality must holds and (iii) of Theorem 2 follows.

If $n_1 = k + 1$ and $n_2 \geq l$, then by induction,

$$\begin{aligned}|E(G)| &\geq f(n - k - 1, k, l) + k - 1 + \frac{(k + 1)k}{2} \\ &= (n - 2t + 1)(k - 1) + t(t - 1) - \left\lfloor \frac{n - 2t}{k + 1} \right\rfloor \frac{k^2 - 3k}{2},\end{aligned}$$

and so by (18), (iii) of Theorem 2 must hold.

If $n_1 = k + 1$ and $n_2 < l$, then by (b) of Lemma 3, $H_2 = K_{n_2}$. Since $n \geq 2k + 3$, $n_2 \geq k + 2$ and so there must be a vertex $v \in V(H_2)$ such that v is not incident with any edge joining H_1 to H_2. Thus $H_2 - v = K_{n_2 - 1}$ and so by $n > l$, and by Lemma 5, $G - v \in \mathcal{SM}(n - 1, k, l)$, contrary to (19).

If $k+2 \leq n_1 < l$, then by (b) of Lemma 3, $H_1 = K_{n_1}$. Since $n_1 > k-1$, there is one vertex v in H_1 not incident with the $k-1$ edges joining H_1 to H_2. Form G' from G by deleting all but one edges joining v to $H_1 - v$ and joining to H_2 with $k-2$ edges. By Lemma 4, $G' \in \mathcal{E}(k,l)$. But $|E(G)| - |E(G')| = (n_1 - 1) - (k-1) > 0$, contrary to the assumption that $G \in S\mathcal{M}(n,k,l)$.

Finally we assume that $n_1 \geq l$. Since $2t \geq 2k+3$, $t \geq k+2$. By induction, by (14), by $f(n,k,l) = |E(G)|$, and by $t \geq k+2$, we have

$$\begin{aligned} f(n,k,l) &\geq f(n_1,k,l) + f(n_2,k,l) + k - 1 \\ &\geq (n-2t+1)(k-1) + 2t(t-1) - 2(t-1)(k-1) \\ &\quad - \left\lfloor \frac{n-4t}{k+1} \right\rfloor \frac{k^2-3k}{2} \\ &\geq (n-2t+1)(k-1) + t(t-1) - \left\lfloor \frac{n-2t}{k+1} \right\rfloor \frac{k^2-3k}{2} \\ &\quad + (t-1)(t-2k+2) + \left\lfloor \frac{2t}{k+1} \right\rfloor \frac{k^2-3k}{2} \\ &\geq (n-2t+1)(k-1) + t(t-1) - \left\lfloor \frac{n-2t}{k+1} \right\rfloor \frac{k^2-3k}{2} \\ &\quad + \frac{(t-1)(k^2-5k+8)}{2} \\ &> (n-2t+1)(k-1) + t(t-1) - \left\lfloor \frac{n-2t}{k+1} \right\rfloor \frac{k^2-3k}{2}, \end{aligned}$$

contrary to (18). Therefore the proof for (iii) of Theorem 2 is completed. The proof for (iv) of Theorem 2 is similar to that for (iii) of Theorem 2, and so is omitted. ∎

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