Graphs without K_4-minors

Hong-Jian Lai
West Virginia University
Morgantown, WV 26506

Hongyuan Lai
Wayne State University
Detroit, MI 48202

Abstract. In [Discrete Math. 75 (1989) 69 - 99], Bondy conjectured that if G is a 2-edge-connected simple graph with n vertices, then G admits a double cycle cover with at most $n-1$ cycles. In this note we prove this conjecture for graph without subdivision of K_4 and characterize all the extremal graphs.

Introduction

Graphs in this note are finite and loopless. For all undefined terms, see Bondy and Murty [BM]. Let G be a graph and $e \in E(G)$. The *contraction* G/e is the graph obtained from G by identifying the two ends of e and deleting the resulting loops. A *subdivision* of a graph H is a graph obtained from H by subdividing some edges of H, and will be denoted by TH. As in [BM], $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and the edge-connectivity of G, respectively. In 1952, Dirac showed the following:

Theorem A (Dirac [D]). *If G is a nontrivial simple graph without TK_4, then G has a vertex of degree at most 2.* ∎

An *arc* of G is an (x, y)-path P of G with $x, y \in V(G)$, such that all the internal vertices of P have degree 2 in G. A *maximal* arc is one that cannot be extended in G. The *length* of an arc P is $|E(P)|$. We regard K_1 as an arc of length 0 (with identical ends) and K_2 as an arc of length 1. Let k be a nonnegative integer. Given graphs G_1 and G_2, if for $i \in \{0, 1, 2\}$, G_i has an arc P_i with $|E(P_i)| = k$ and with the ends of P_i being $x_i, y_i \in V(G_i)$, then one can define the *k-arc-sum* of G_1 and G_2 to be the graph obtained from the vertex disjoint union of G_1 and G_2 by deleting all the internal vertices of P_2 and identifying x_1 with x_2 and y_1 with y_2. Thus the k-arc-sum of G_1 and G_2 contains G_1 and G_2 as subgraphs. If G is a k-arc-sum of G_1 and G_2 with

$$|E(G_i)| < |E(G)|, \ (1 \le i \le 2), \tag{1}$$

then G is called a *proper* k-arc-sum of G_1 and G_2.

Remark. *The definition of the k-arc-sums here is motivated by and similar to the $(k+1)$-sums of Bondy* [B], *but is different from the k-sums of Seymour* [S].

Let G be a simple graph. An edge $e \in E(G)$ is called a *sum-edge* of G, if G is a proper 1-arc-sum of two subgraphs G_1 and G_2 with $E(G_1) \cap E(G_2) = \{e\}$. For each integer $i \geq 3$, define $\mathcal{K}(i)$ to be the family of simple graphs G such that either G is a cycle of length at most i, or G is a 0-arc-sum or a 1-arc-sum of G_1 and G_2 for some $G_1, G_2 \in \mathcal{K}(i)$, such that every k-cycle of G, $3 \leq k \leq i$, has at most two sum-edges of G, and such that if a k-cycle C has exactly two sum-edges in G, $3 \leq k \leq i$, then these two sum-edges are adjacent in C.

Define $\mathcal{K} = \cup_{i \geq 3} \mathcal{K}(i)$. By definition of the k-arc-sum, the following Proposition is immediate.

Proposition 1. *Suppose that G is a k-arc-sum of G_1 and G_2. If each of G_1 and G_2 has no TK_4, then G has no TK_4. In particular, every graph in $\mathcal{K}$ has no TK_4.*

Main Results

Theorem 1 *Let G be a nontrivial 2-edge-connected graph. If G contains no TK_4, then either G is a cycle or G is a proper k-arc-sum of some graphs G_1 and G_2, for some $k \geq 0$, with $\kappa'(G_i) \geq 2, (1 \leq i \leq 2)$. Moreover, if G is simple and not a cycle, then G_1 and G_2 are simple graphs.*

Let $sc(G)$ denote the minimum number of cycles of G that are needed to cover $E(G)$ exactly twice. In [B], Bondy conjectured that any 2-edge-connected simple graph with n vertices satisfies $sc(G) \leq n-1$, where equality holds if and only if G has a spanning tree T such that for every edge $e \in E(G) - E(T)$, $T + e$ has a 3-cycle, (such a tree T is called a *tritree* of G, and such a graph G is called a *trigraph*).

Theorem B. *(Bondy* [B]*) Let G be a graph with n vertices.*

(i) *If G is a trigraph, then $sc(G) \geq n-1$.*
(ii) *If G is a 0-arc-sum of two trigraphs, then G is a trigraph.*
(iii) *Suppose that G is a 1-arc-sum of trigraphs G_1 and G_2 and that e is the sum edge shared by G_1 and G_2. If each of G_1 and G_2 has a tritree that uses e, then G is also a trigraph.* ∎

Proposition 2. *If $G \in \mathcal{K}(3)$, then G is a trigraph.*

Proof: We argue by induction on $|V(G)|$. By (ii) of Theorem B, we may assume that G is not a 3-cycle nor a 0-arc-sum of some graphs in $\mathcal{K}(3)$. Thus by definition of $\mathcal{K}(3)$, G is a 1-arc-sum of G_1 and G_2, for some $G_1, G_2 \in \mathcal{K}(3)$. Choose G_1 and G_2 so that $|E(G_2)|$ is minimized. We claim that G_2 is a 3-cycle.

Let e denote the edge shared by G_1 and G_2. If G_2 is not a 3-cycle, then since $G_2 \in \mathcal{K}(3)$, G_2 is a 1-arc-sum of some $G'_2, G''_2 \in \mathcal{K}(3)$. If $e \in E(G'_2) \cap E(G''_2)$, then let $G'_1 = G_1 \cup G''_2$, and so the minimality of G_2 is violated, since G'_2 is a proper subgraph of G_2 and since G is a 1-arc-sum of G'_1 and G'_2. Hence we may assume that $e \in E(G'_2) - E(G''_2)$. Let $G''_1 = G_1 \cup G'_1$, then the minimality of G_2 is violated again, since G''_2 is a proper subgraph of G_2 and G is a 1-arc-sum of G''_1 and G''_2. Hence G_2 must be a 3-cycle.

By induction hypothesis, G_1 is a trigraph. Let C_1 be a 3-cycle of G_1 that contains e. Since C_1 has at most two sum-edges, and since e is a sum-edge of G, C_1 contains an edge $e_1 \in E(C_1) - \{e\}$ that is not a sum-edge of G. Let T_1 be a tritree of G_1. If $e \notin E(T_1)$, then since C_1 is a 3-cycle, $e_1 \in E(T_1)$. Let $T'_1 = T_1 + e - e_1$. Since e_1 is not a sum-edge of G, and since $|E(C_1)| = 3$, T'_1 is a tritree of G_1. It follows that G_1 has a tritree that uses e, and so by (iii) of Theorem B, G is a trigraph. ∎

Let $\mathcal{A}(G)$ denote the collection of all maximal arcs A of G with $|E(A)| \geq 2$. For any $A \in \mathcal{A}(G)$, A is called a *cyclic* arc if there is an arc A' with $E(A') \subseteq E(G) - E(A)$ such that $G[E(A) \cup E(A')]$ is a cycle of G, or if A itself is a cycle; and A is an *acyclic* arc, otherwise. For each $A \in \mathcal{A}(G)$, define $b(A)$ as follows:

$$b(A) = \begin{cases} |E(A)| - 3 & \text{if } A \text{ is a cycle} \\ |E(A)| - 2 & \text{if } A \text{ is cyclic but not a cycle} \\ |E(A)| - 1 & \text{if } A \text{ is acyclic.} \end{cases}$$

Note that by Theorem A, if G is simple and has no TK_4, then $\mathcal{A}(G) \neq 0$. Define

$$b(G) = \sum_{A \in \mathcal{A}(G)} b_G(A).$$

As examples, $b(K_{2,t}) = 0$ if $t \geq 2$; and if G is a subdivision of the Petersen graph, then $b(G)$ is equal to the number of vertices of degree 2.

Proposition 3. *Let $G \in \mathcal{K}$. Then $G \in \mathcal{K}(3)$ if and only if $b(G) = 0$.*

Proof: Suppose that $G \in \mathcal{K}(3)$. Then every arc $A \in \mathcal{A}(G)$ is in a 3-cycle and so $b(G) = 0$. Suppose then that $G \in \mathcal{K} - \mathcal{K}(3)$. Then G has a cycle C of length at least 4. Since C has at most 2 sum-edges, and since when $|E(C)| = 4$ and C has exactly two sum-edges, these two sum-edge must be adjacent, C contains an arc $A \in \mathcal{A}(G)$ such that either A is a cycle of length at least 4, or a cyclic arc of length at least 3 that is not a cycle, or an acyclic arc of length at least 2. Thus $b(G) > 0$. ∎

Theorem 2. *Let G be a simple graph with n vertices. If G has no TK_4,*

$$sc(G) \leq n - 1 - b(G), \tag{2}$$

where equality holds if and only if $G \in \mathcal{K}$. Moreover, if $b(G) = 0$, then equality holds in (2) if and only if $G \in \mathcal{K}(3)$.

Corollary 1. *Let G be a 2-edge-connected simple graph with n vertices. If G has no TK_4, then $sc(G) \leq |V(G)| - 1$, where equality holds if and only if $G \in \mathcal{K}(3)$.*

Corollary 2. *A 2-edge-connected simple graph G is a trigraph without a TK_4 if and only if $G \in \mathcal{K}(3)$.*

Proof of Corollaries 1 and 2: Corollary 1 follows from Theorem 2. Corollary 2 follows from Theorem 2, from Corollary 1 and from (i) of Theorem B. ∎

The Proofs

Let H be a subgraph of G. The set of all vertices in $V(H)$ that are incident with at least one edge in $E(G) - E(N)$, denoted by $A_G(H)$, is called the *vertices of attachment* of H in G.

Lemma 1. *Let G be a graph without TK_4 and let H be a subgraph of G with $\kappa'(H) \geq 2$ such that $A_G(H) = \{x_1, x_2\}, (x_1 \neq x_2)$ and such that G has an (x_1, x_2)-path P using no edges in $E(H)$. If for some $k \geq 1$, H is a proper k-arc-sum of some 2-edge-connected graphs H_1 and H_2, then G is a proper k-arc-sum of some graphs G_1 and G_2 with $\kappa'(G_i) \geq 2$.*

Proof: Since H is a proper k-arc-sum of H_1 and H_2, both H_1 and H_2 are subgraphs of H, and so of G.

Case 1: $x_1, x_2 \in V(H_2)$.

Let $G_1 = H_1$ and $G_2 = G[E(H_2) \cup E(G - E(H))]$, then G is a proper k-arc-sum of G_1 and G_2. When $x_1, x_2 \in V(H_1)$, the proof is the same.

Case 2: $x_1 \in V(H_i), (1 \leq i \leq 2)$.

Since $k \geq 1$, there is an edge e (say) shared by H_1 and H_2. Since both H_1 and H_2 are 2-edge-connected, there is a cycle C_i in H_i such that $e \in E(C_i)$, $(1 \leq i \leq 2)$. Let P_i denote a path in H_i that joins x_1 to exactly one vertex y_i (say) in C_i.

If $y_1, y_2 \notin V(P)$, then C_1, C_2, P_1, P_2 and P form a TK_4 in G, a contradiction. Hence we may assume that $y_1 \in V(P)$ and that any path from x_1 to C_1 in H must use y_1. Let $P^1, P^2, \ldots, P^m$ be all the (x_1, y_1)-paths in H. It follows that that subgraph

$$H' = H - (\cup_{i=1}^{m} V(P^i) - \{y_1\})$$

is a 2-edge-connected subgraph of G with $A_G(H) = \{x_2, y_1\}$, and so we are back to Case 1. ∎

Proof of Theorem 1: If G has 2 edges e_1, e_2 with the same ends, (that is, e_1, e_2 are parallel edges in G) then let $G_1 = G - e_1$ and $G_2 = G[e_1, e_2]$ and we are done.

The theorem will also be trivial if G has a cut-vertex. Hence we shall assume that G is simple and 2-connected.

We prove the general case by induction on the number of vertices of G and so we assume that G is not a cycle and Theorem 1 holds for graphs with order smaller than $|V(G)|$.

By Theorem A, G has a vertex v of degree 2. Since G is not a cycle, G has a maximal arc P of length at least 2, where the ends x and y of P have degree at least 3 in G. Since G is 2-connected, $x \neq y$. We shall show first that G has a 2-edge-connected proper subgraph H such that

$$|A_G(H)| = 2, \tag{3}$$

and, (assuming $A_G(H) = \{u, v\}$), such that

$$G - E(H) \text{ has an } (u, v) - \text{path}. \tag{4}$$

Let $G' = G-(V(P)-\{x,y\})$. If $\kappa'(G') \geq 2$, then G' is a subgraph satisfying (3) and (4). Hence by $\kappa'(G) \geq 2$, we may assume that $\kappa'(G) = 1$. Thus G' has an edge such that $G' - e$ has two components G'' and G''' with $x \in V(G'')$ and with $|V(G'')|$ being minimized. If $y \in V(G'')$, then e is a cut-edge of G, contrary to the assumption that $\kappa'(G) \geq 2$. Thus $y \in V(G''')$. Let $z \in V(G'')$ be the vertex incident with e. Since G is 2-connected, $x \neq z$. Clearly G has an (x, z)-path using no edges in $E(G'')$ and $A_G(G'') = \{x, z\}$. By the minimality of $|V(G'')|$, $\kappa'(G'') \geq 2$. Thus G'' is a 2-edge-connected proper subgraph of G satisfying (3) and (4).

Choose a minimal 2-edge-connected proper subgraph H of G satisfying (3) and (4), and assume that $A_G(H) = \{x_1, x_2\}$. By induction, H is a proper k-arc-sum of two sub-graphs H_1 and H_2. Suppose first that $k = 0$. Since G is 2-connected, we may assume that $x_i \in V(H_i), (1 \leq i \leq 2)$. But then H may contain a smaller 2-edge-connected subgraph satisfying (3) and (4), contrary to the minimality of H. Hence we must have $k \geq 1$ and so Theorem 1 follows from Lemma 1. ∎

Lemma 2. *Let G be a 2-edge-connected graph. If G has an arc A with $|E(A)| \geq 2$, then for each $e \in E(A)$,*

$$sc(G/e) = sc(G). \tag{5}$$

Proof: The proof is routine and straightforward. ∎

Lemma 3. *If G is a 1-arc-sum of H and a k-cycle H', where $k \geq 3$, then $sc(G) \leq sc(H) + 1$.*

Proof: Let $C' = \{C'_1, \dots, C'_m\}$ be a double cycle cover of H with $m = sc(H)$.

Let e be the edge commonly shared by H and H'. We may assume that $e \in E(C'_1)$. Thus let $C_1 = G[E(C'_1) \cup E(G_1) - e]$, $C_2 = C'_2, \dots, C_m = C'_m$ and $C_{m+1} = H'$. Then $\{C_1, C_2, \dots, C_{m+1}\}$ is a double cycle cover of G, and so we have $sc(G) \leq m + 1$. ∎

Lemma 4. *Let G be a simple graph of n vertices. Each of the following holds.*

(i) *If $G \in \mathcal{K}(3)$, then $sc(G) = n-1$.*

(ii) *If $G \in \mathcal{K}$, then $sc(G) = n-1-b(G)$.*

(iii) *Suppose that G is a 2-arc-sum of G_1 and G_2 where $G_1, G_2 \in \mathcal{K}$. If the common arc P shared by G_1 and G_2 is not in a K_3 of G, then*

$$sc(G) \leq n-2-b(G). \tag{6}$$

Proof: Both (i) and (ii) of Lemma 4 hold if G is a cycle. So we suppose that G is not a cycle. Assume first that $G \in \mathcal{K}(3)$. Then by definition of $\mathcal{K}(3)$, G is a k-arc-sum of G_1 and G_2, for some $G_1, G_2 \in \mathcal{K}(3)$, where $0 \leq k \leq 1$. If $k = 0$, then (i) follows easily by induction. Hence we assume that $k = 1$. Choose G_1 and G_2 so that $|E(G_2)|$ is minimized. Then by the definition of $\mathcal{K}(3)$ and since $k = 1$, G_2 must be a K_3. Thus by Lemma 3 and by induction,

$$sc(G) \leq sc(G_1) + 1 = (n-2) + 1 = n-1. \tag{7}$$

Hence (i) of Lemma 4 follows by (i) of Theorem B and by (7).

By Proposition 3, if $G \in \mathcal{K} - \mathcal{K}(3)$, then $b(G) > 0$ and so G has an arc $A \in \mathcal{A}(G)$ such that $b_G(A) > 0$. Pick $e \in E(A)$. Then $G' = G/e$ is simple and G/e is in $\mathcal{K}$. By induction, we have $sc(G/e) = (n-1) - 1 - b(G/e)$. But since $b_G(A) > 0$, we have $b(G) = b(G/e) + 1$ and so (ii) of Lemma 4 follows by induction.

Let G,G_1,G_2,P satisfy the hypothesis of (iii) of Lemma 4. We argue by induction on n and so we may assume that G is 2-connected. If G_1 is not a cycle, then since $\kappa(G) \geq 2$, G_1 is a 1-arc-sum of H_1 and H_2, for some $H_1, H_2 \in \mathcal{K}$. Since P is an arc, either $E(P) \subseteq E(H_1)$ or $E(P) \subseteq E(H_2)$. We may assume that $E(P) \subseteq E(H_1)$. Choose H_1 and H_2 so that $|E(H_2)|$ is minimized. Hence H_2 is a k-cycle, for some $k \geq 3$. Note that P is shared by G_2 and H_1. Let G denote the 2-arc-sum of G_2 and H_1. By induction,

$$sc(G') \leq |V(G')| - 2 - b(G') = (n-k+2) - 2 - b(G'). \tag{8}$$

Since H_2 is a k-cycle, H_2 contributes $k-3$ to $b(G)$. Hence by Lemma 3 and by (8),

$$sc(G) \leq sc(G') + 1 \leq n-k+1-b(G') + (k-3) \leq n-2-b(G),$$

and so (iii) follows by induction, when G_1 is not a cycle.

Hence we may assume that G_1 is a k-cycle and that G_2 is a k'-cycle. Thus $n = k+k'-3$ and $b(G) = (k-4) + (k'-4)$. Since it is clear that $sc(G) = 3$, we have $sc(G) = n-2-b(G)$, and so (iii) of Lemma 4 follows by induction. ∎

Proof of Theorem 2: We proceed by induction on $n = |V(G)|$. If G is a cycle, then Theorem 2 holds trivially. So we assume that G is not a cycle and that $|V(G)| \geq 4$. Since Theorem 2 follows easily by induction if G has a cut-vertex, we assume that G is 2-connected. It follovs that G has no arc A such that A itself is a cycle.

If $b(G) > 0$, then G has an arc A and an edge $e \in E(A)$ such that G/e is simple and such that A is either acyclic with $|E(A)| \geq 2$, or A is cyclic with $|E(A)| \geq 3$ and is not a cycle. It follows by the definition of $b(G)$ that

$$b(G) - 1 = b(G/e). \tag{9}$$

By induction, $sc(G/e) \leq (n-1) - 1 - b(G/e)$ and so by (9) and by Lemma 2, $sc(G) \leq n - 1 - b(G)$. Furthermore, if $sc(G) = n - 1 - b(G)$, then we must have $sc(G/e) = (n-1) - 1 - b(G/e)$. Thus by induction, $G/e \in \mathcal{K}$. To show that $G \in \mathcal{K}$, it remains to show that when $|E(A)| = 2$, G is not a proper 2-arc-sum of some subgraphs that share A. But this follows by (iii) of Lemm 4 and by $sc(G) = n - 1 - b(G)$.

Hence we may assume that $b(G) = 0$. We argue further that G has no A with $|E(A)| \geq 2$ such that A has an edge e with G/e simple. Suppose, to the contrary, that G has such an arc A and such an edge $e \in E(A)$ with G/e simple. Since $b(G) = 0$, we have $b(G/e) = 0$ also. Thus by induction and by (5),

$$sc(G) = sc(G/e) \leq (n-1) - 1 - b(G/e) = (n-2) - b(G),$$

and so (2) holds by induction.

Since G is not a cycle, by Theorem 1, G is a proper k-arc-sum of two 2-edge-connected subgraphs G_1 and G_2. Since G has no arc A of length at least 2 such that A has an edge e with G/e simple, and since G is 2-connected, we have

$$1 \leq k \leq 2, \text{ and every arc of length 2 is in a } K_3 \text{ of } G. \tag{10}$$

Denote $n_i = |V(G_i)|$, $(1 \leq i \leq 2)$. Note that since G is simple, $n_i < n$, $(1 \leq i \leq 2)$. For convenience, let $b_i(A) = b_{G_i}(A)$, $(1 \leq i \leq 2)$. Let P be the common arc shared by both G_1 and G_2 with $k = |E(P)|$. Then we have

$$n_1 + n_2 = n + k + 1. \tag{11}$$

By induction, there are double cycle covers C_1 and C_2 for G_1 and G_2, respectively, such that

$$sc(G_i) = |C_i| \leq n_i - 1 - b(G_i), \; (1 \leq i \leq 2). \tag{12}$$

Denote $C_i = \{C_1^i, C_2^i, \ldots, C_{m(i)}^i\}$, $(1 \leq i \leq 2)$, where $m(i) = |C_i|$. Since P is an arc, any cycle in C_i containing an edge of P will contain all edges of P. Thus we may assume that

$$E(P) \subseteq E(C_1^1) \cap E(C_1^2). \tag{13}$$

Let $C = (C_1 \cup C_2 - \{C_1^1, C_1^2\}) \cup \{G[(E(C_1^1) \cup E(C_1^2)) - E(P)]\}$. Then C is a double cycle cover of G with

$$|C| = |C_1| + |C_2| - 1.$$

It follows that

$$sc(G) \leq sc(G_1) + sc(G_2) - 1. \tag{14}$$

Suppose first that $k = 1$. Then we have

$$b(G_1) + b(G_2) \leq b(G) = 0, \tag{15}$$

since in this case, $\mathcal{A}(G) \subseteq \mathcal{A}(G_1) \cup \mathcal{A}(G_2)$. Thus by (12), (14) and (15), we have

$$sc(G) \leq n - 1.$$

Suppose further that $sc(G) = n - 1$, then by (14), equality must hold in (12), and so both G_1 and G_2 are in $\mathcal{K}(3)$, by induction. We must show that $G \in \mathcal{K}(3)$.

Since $G_1 \in \mathcal{K}(3)$ and since $\kappa(G) \geq 2$, either G_1 is a 3-cycle or G_1 is a 1-arc-sum of a 3-cycle and some other subgraph.

Suppose first that G_1 has a 3-cycle G_1'', such that G_1 is the 1-arc-sum of G_1' and G_1'', for some subgraph G_1' of G_1 and such that $e \notin E(G_1'')$. Let e' be the common edge shared by G_1' and G_1''. Then G is the 1-arc-sum of G_1'' and $H = G[E(G) - (E(G_1'') - \{e'\})]$. Thus by induction, $H \in \mathcal{K}(3)$. Since $e \notin E(G_1'')$, and since $H \in \mathcal{K}(3)$, every 3-cycle in G containing e has at most two sum-edges. Since G_1'' is a subgraph of $G_1 \in \mathcal{K}(3)$, and since $e \notin E(G_1'')$, every 3-cycle in G containing e' has at most two sum-edges. It follows that $G \in \mathcal{K}(3)$, by definition.

Thus we may assume that every 3-cycle in G_1 contains e, and so $G_1 - e = K_{2,t}$ for some $t \geq 1$. Similarly, we may assume that $G_2 - e = K_{2,s}$ for some $s \geq 1$. It follows that $G - e = K_{2,s+t}$ and so $G \in \mathcal{K}(3)$. Hence Theorem 2 follows by induction when $k = 1$.

Suppose then that $k = 2$. Recall that P is shared by G_1 and G_2. Let $x, y \in V(G)$ be the two ends of P. By (7), $xy \in E(G)$. Without loss of generality, we may assume that $x, y \in E(G_1)$. Let $G_1^1 = G[E(G_1) \cup E(P)]$ and $G_2^2 = G[E(G_2) - E(P)] + xy$. Then G is a 1-arc-sum of G_1^1 and G_2^2, and so we are back to the case when $k = 1$. This proves Theorem 2. ∎

References

[B] J.A. Bondy, *Trigraphs*, Discrete Math. 75 (1989), 69–99.

[BM] J.A. Bondy and U.S.R. Murty, "Graph Theory with Applications", American Elsevier, New York, (1976).

[D] G.A. Dirac *A property of 4-chromatic graphs and some remarks on critical graphs*, J. London Math. Soc. 27 (1952), 85–92.

[S] P.D. Seymour, *Decomposition of regular matroids*, J. Combin. Theory (B) 28 (1980), 305–359.