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Graph whose edges are in small cycles

Hong-Jian Lai*
West Virginia University, Morgantown, WV 26506, USA

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Abstract

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Paulraja (1987) conjectured the following:
(i) If every edge of a 2-connected graph G lies in a cycle of length at most 4 in G, then G has a dominating closed trail.
(ii) If, in addition, $\delta(G) \geqslant 3$, then G has a closed spanning trail.
Collapsible graphs are defined and studied by Catlin (1988). Catlin showed that if H is a collapsible subgraph of G, then G has a spanning closed trail if and only if G/H, the graph obtained from G by contracting H, has a spanning closed trail. Catlin (1987) conjectured that a graph satisfying the hypothesis of (ii) is collapsible. In this paper, all three conjectures are proved.

Introduction

We shall use the notation of Bondy and Murty [1] except for contractions. A graph may have multiple edges but not loops. A spanning closed trail of G is called a *spanning eulerian subgraph* (*SES*) of G. The collection of graphs that have an SES is denoted by $\mathcal{SL}$. Note that $K_1 \in \mathcal{SL}$. If a closed trail C of G satisfies $E(G-V(C)) = \phi$, then C is called a *dominating eulerian subgraph* (*DES*) of G.

Let G be a graph. An block of G that has exactly one cut-vertex of G is called an *end block* of G. A block B is *acyclic* if $B \cong K_2$. For a subset $X \subset E(G)$, the *contraction* G/X is the graph obtained from G by identifying the ends of each edge in X and then deleting the resulting loops. If H is a subgraph of G, then we write G/H for $G/E(H)$.

Let W be a subgraph of G. If for some $t \geqslant 2$, $W \cong K_{2,t}$, then W is called a *$W_{2,t}$-subgraph* of G. Let $C = v_1v_2v_3v_4v_5v_6v_1$ be a 6-cycle. Define Θ to be the graph with $V(\Theta) = V(C)$ and $E(\Theta) = E(C) \cup \{v_2v_5\}$.

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Let H be a subgraph of G. Define $H^c = G[E(G) - E(H)]$ and $A_G(H) = V(H) \cap V(H^c)$. The vertices in $A_G(H)$ are called the *vertices of attachment* of H in G. If $\kappa(H) \geq 2$ and $|A_G(H)| = 2$, then H is called a *2-block* of G.

A graph G is an *edge-disjoint union of* $K_{2,t}$'s if there is a partition $E_1, E_2, \ldots, E_k$ of $E(G)$ such that $G[E_i] \cong K_{2,n_i}$'s ($n_i \geq 2$ and $1 \leq i \leq k$). Note that when $i \neq j$, we may have $n_i \neq n_j$.

We consider the following conditions.

$$\text{Any edge of } G \text{ is an } m\text{-cycle of } G,\ m \leq 4; \tag{1}$$

$$G \text{ is an edge-disjoint union of } K_{2,t}\text{'s}. \tag{2}$$

Let $\mathcal{G} = \{G\colon G \text{ satisfies (1) with } \kappa(G) \geq 2\}$ and $\mathcal{G}_1 = \{G \in \mathcal{G}\colon \delta(G) \geq 3\}$. In [6–8], Paulraja raised the following two conjectures.

Conjecture 1. If $G \in \mathcal{G}_1$, then $G \in \mathcal{SL}$.

Conjecture 2. If $G \in \mathcal{G}$, then G has a DES.

A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, G has a subgraph Γ_R (called the *R-subgraph* of G) such that $G - E(\Gamma_R)$ is connected and R is the set of odd-degree vertices of Γ_R. The collection of all collapsible graphs is denoted by $\mathcal{CL}$. Let $H_1, H_2, \ldots, H_c$ be all the maximal collapsible subgraphs of G. Denote by $(G)_1$ the graph of order c obtained from G by contracting $H_1, \ldots, H_c$ to c distinct vertices. We call $(G)_1$ the *reduction* of G and $H_1, H_2, \ldots, H_c$ the *preimages* of the vertices of $(G)_1$. A graph G is *reduced* if $G = (G)_1$. In [2], Catlin showed that $(G)_1$ is well defined and unique. He also proved the following theorem.

Theorem A (Catlin [2]). *Let G be a graph.*

(i) *If $G \in \mathcal{CL}$, then $G \in \mathcal{SL}$.*

(ii) *$G \in \mathcal{SL}$ if and only if $(G)_1 \in \mathcal{SL}$.*

(iii) *If $H \in \mathcal{CL}$ is a subgraph of G, then $G \in \mathcal{CL}$ if and only if $G/H \in \mathcal{CL}$.*

(iv) *G is reduced if and only if G has no nontrivial subgraphs in $\mathcal{CL}$. In particular, G has no 3-cycles.*

(v) *If $H_1, H_2 \in \mathcal{CL}$ are two subgraphs of G and if $V(H_1) \cap V(H_2) \neq \phi$, then $H_1 \cup H_2 \in \mathcal{CL}$.*

(vi) *For any graph G, $(G)_1$ is reduced.*

Catlin [3] made the following conjecture.

Conjecture 3. If $G \in \mathcal{G}_1$, then $G \in \mathcal{CL}$.

By (i) of Theorem A, we have the following proposition.

Proposition 1. *Conjecture* 3 *implies Conjecture* 1.

For each $i = 1, 2, \ldots,$ define

$$D_i(G) = \{v \in V(G): \deg_G(v) = i\},$$

and

$$D_i^*(G) = \{v \in V(G): \deg_G(v) \geq i\}.$$

Proposition 2. *Conjecture* 1 *implies Conjecture* 2.

Proof. Assume that Conjecture 1 holds. Let $G \in \mathcal{G}$. Since an SES is always a DES, we may assume that $G \notin \mathcal{SL}$ and $G \in \mathcal{G} - \mathcal{G}_1$. Thus $\delta(G) = 2$. Let $D_2(G) = \{x_1, x_2, \ldots, x_m\}$. Since $\kappa(G) \geq 2$,

$$\text{no vertex in } D_2(G) \text{ is the common end of multiple edges;} \tag{3}$$

and by (1) and $\kappa(G) \geq 2$,

$$\text{no paths in } G \text{ contain three consecutive vertices in } D_2(G). \tag{4}$$

Choose $y_1, y_2, \ldots, y_m$ such that $e_i = x_i y_i \in E(G)$, $(1 \leq i \leq m)$, and such that if for some i, j with $x_i x_j \in E(G)$ then $y_i = x_j$ and $x_i = y_j$. Note that the latter can happen only when x_i and x_j are the internal vertices of a path of G of length 3. By (3) and (4), the multiplicity of each e_i is one in G. Let e_i' be an edge with the same ends as e_i but $e_i' \notin E(G)$, $(1 \leq i \leq m)$. Obtain a new graph G' by adding $\{e_1', e_2', \ldots, e_m'\}$ to G. It follows that $G' \in \mathcal{G}$ and so $G' \in \mathcal{SL}$, by Conjecture 1. Let C' be an SES of G'. Since $G \notin \mathcal{SL}$, we may assume that for some $k \leq m$,

$$\{e_1', e_2', \ldots, e_k'\} \cup \{e_1, e_2, \ldots, e_k\} \subseteq E(C'); \tag{5}$$

and that $e_j' \notin E(C')$, $(j > k)$. Choose C' so that (5) is satisfied and that k is as small as possible. Let $C = C' - \{e_1, \ldots, e_k, e_1', \ldots, e_k'\}$. Then every vertex of C has even degree in C. If $e_s' = x_s x_t \in E(C')$, for some $1 \leq s, t \leq m$, then since x_s and x_t are in $D_2(G)$, x_s and x_t are in $D_2(C')$ also. It follows that $e_s \notin E(C')$, contrary to the choice of C'. So if $x_s x_t \in E(C')$, then $e_s \in E(C')$ and $e_s' \notin (C')$. This implies that C is connected, and that $x_1, x_2, \ldots, x_k$ form an independent set in G and are the only vertices not in $V(C)$. Thus C is a DES of G. □

Reductions

Notation 1 (see Fig. 1). Let W be a $K_{2,t}$-subgraph of G and let H be a subgraph of G containing W. Denote

$$D_2(W) = \{y_1, y_2, \ldots, y_t\}.$$

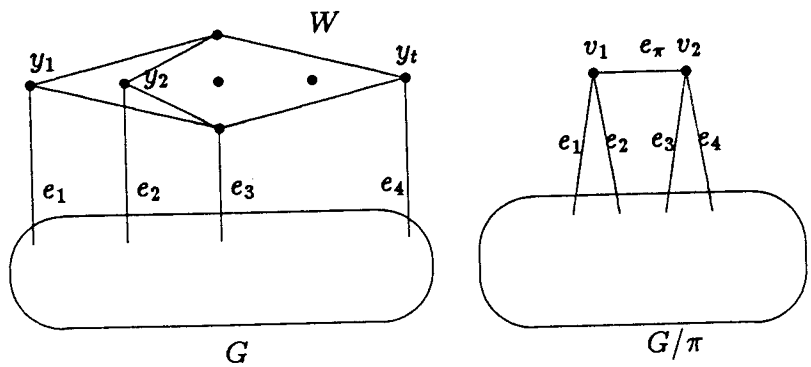

Fig. 1. Notation 1 with $V_1 = \{y_1, y_2\}$.

Let V_1 be a subset of $D_2(W)$ such that $|V_1| = 2$, let $V_2 = V(W) - V_1$ and let $\pi = \langle V_1, V_2 \rangle$ denote this partition of $V(W)$. Denote by H/π the graph obtained from H by identifying all vertices of V_1 to form a single vertex v_1, by identifying all vertices of V_2 to form a single vertex v_2, and by joining v_1, v_2 with a new edge $e_\pi = v_1v_2$, so that

$$E(H) - E(W) = E(H/\pi) - \{e_\pi\}.$$

And finally let $H' = (H/\pi) - e_\pi$.

Theorem B (Catlin [3]). *Let W, π and G/π be defined in Notation 1. If $G/\pi \in \mathscr{CL}$, then $G \in \mathscr{CL}$.*

Lemma 1. *If $G' = (G/\pi) - e_\pi \in \mathscr{CL}$, then $G \in \mathscr{CL}$.*

Proof. It follows from Theorem B and (iii) of Theorem A. □

It is easy to see the following.

Lemma 2 ([5]). *If $G \in \mathscr{G}$, then $(G)_1$ has girth 4 and $(G)_1$ satisfies (1).*

Lemma 3 ([5]). *Let B be a block of $(G)_1$.*
(i) *If $G \in \mathscr{G}_1$, then $B \in \mathscr{G}_1$;*
(ii) *If $G \in \mathscr{G}$, then $B \in \mathscr{G}$.*

Lemma 4. *If G is a counterexample to Conjecture 3 with $|V(G)|$ minimized then G is reduced.*

Proof. Let G satisfy the hypothesis of Lemma 4. If G is not reduced, then $(G)_1$, the reduction of G, has smaller order than G. By Lemmas 2 and 3, each block of $(G)_1$ is in $\mathscr{G}_1$ and so by the minimality of G, each block of $(G)_1$ is in $\mathscr{CL}$. Thus by (v) and (iii) of Theorem A, $G \in \mathscr{CL}$, a contradiction. □

Notation 2. Let $C = x_1x_2x_3x_4x_1$ be a 4-cycle of G. Define:

$$G^* = [G - \{x_1x_2, x_3x_4\}]/\{x_1x_4, x_2x_3\},$$

and denote by v_1 and v_2 the vertices of G^* to which x_1x_4 and x_2x_3 are contracted, respectively. For convenience, we regard

$$V(G^*) = [V(G) - V(C)] \cup \{v_1, v_2\}, \qquad E(G^*) \cup E(C) = E(G).$$

Lemma 5 [4–5]. *If $G^* \in \mathscr{CL}$, then $G \in \mathscr{CL}$.*

Lemma 6. *Let G be a counterexample to Conjecture* 3 *with $|V(G)|$ minimized. Then G does not have a subgraph isomorphic to Θ.*

Proof. By contradiction, suppose that G has Θ as a subgraph. Let $C = x_1x_2x_3x_4x_5x_6x_1$ denote the 6-cycle of Θ and let $C_1 = x_1x_2x_5x_6x_1$ and $C_2 = x_2x_3x_4x_5x_2$ be the two 4-cycles contained in Θ. Define

$$\pi(1) = \langle \{x_1, x_5\}, \{x_2, x_6\} \rangle \quad \text{and} \quad \pi(2) = \langle \{x_2, x_4\}, \{x_3, x_5\} \rangle.$$

Define $G^1 = G/\pi(1)$ and $G^2 = G/\pi(2)$. Let v_1^1 and v_2^1 denote the vertices of G^1 to which $\{x_1, x_5\}$ and $\{x_2, x_6\}$ are mapped, respectively; and let v_1^2, v_2^2 be the vertices of G^2 to which $\{x_3, x_5\}$ and $\{x_2, x_4\}$ are mapped, respectively. Let $e_1 = v_1^1 v_2^1$ and $e_2 = v_1^2 v_2^2$. We shall regard $C_1 = x_1v_2^2v_1^2x_6x_1$ in G^2 and $C_2 = x_3v_2^1v_1^1x_4x_3$ in G^1 throughout the proof of this lemma.

Since $G \in \mathscr{G}_1$, we have $\delta(G^1) \geqslant 3$ and $\delta(G^2) \geqslant 3$, and both G^1 and G^2 satisfy (1).

Claim 1. $\kappa(G^1) = \kappa(G^2) = 1$, *and all cut vertices of G^1 are in $\{v_1^1, v_2^1\}$ and all cut-vertices of G^2 are in $\{v_1^2, v_2^2\}$.*

Proof. Clearly G^1 is connected. If $\kappa(G^1) \geqslant 2$, then $G^1 \in \mathscr{CL}$ follows by the minimality of G. Hence $G \in \mathscr{CL}$ by Theorem B, a contradiction. Thus $\kappa(G^1) = 1$. Since $\kappa(G) = 2$, the cut-vertices of G^1 must be in $\{v_1^1, v_2^1\}$. The proof for G^2 is similar. □

Claim 2 (see Fig. 2). *For $i \in \{1, 2\}$, if v_1^i is a cut-vertex of G^i and if L^i is an end-block of G^i containing v_1^i but not v_2^i, then $L^i \in \mathscr{CL}$. A similar result holds when we replace v_1^i by v_2^i.*

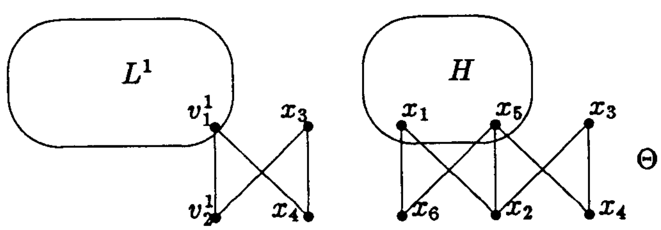

Fig. 2. Claim 2 of Lemma 6.

Proof. Without loss of generality, we assume $i=1$ and v_1^1 is cut-vertex of G^1 with L^1 an end-block of G^1 containing v_1^1 but not v_2^1. Let $H=G[E(L^1)]$. Then by $\kappa(G)\geq 2$, we have $A_G(H)=\{x_1, x_5\}$. Since G satisfies (1), either H satisfies (1) or there is a vertex $y\in V(H)$ with $yx_1, yx_5\in E(H)$.

If H satisfies (1), then $L^1\in \mathcal{G}_1$ and so $L^1\in \mathcal{CL}$, by the minimality of G.

Hence we assume that there is a vertex $y\in V(H)$ such that $yx_1, yx_5\in E(H)$. By $\kappa(G)\geq 2$, y is not a cut-vertex of G. Hence either x_1 or x_5 is adjacent to some vertex of $V(H)-\{y\}$. It follows that $\delta(L^1)\geq 3$. Since (1) holds for G, (1) holds for L^1 also. Thus $L^1\in\mathcal{CL}$ by the minimality of G. □

For $i\in\{1, 2\}$, let B^i be the block of G^i containing both v_1^i and v_2^i.

Claim 3. *Either v_1^i or v_2^i has degree less than 3 in B^i.*

Proof. By Claim 2, all blocks of G^i other than B^i are in $\mathcal{CL}$. Clearly B^i satisfies (1) and $\kappa(B^i)\geq 2$. If $\delta(B^i)\geq 3$, then $B^i\in\mathcal{CL}$ follows from the minimality of G and so by (v) of Theorem A and by Theorem B, $G\in\mathcal{CL}$, a contradiction. Hence $\delta(B^i)<3$. Since $\delta(G)\geq 3$, every vertex in $V(B^i)-\{v_1^i, v_2^i\}$ has degree at least 3 in B^i. Hence either v_1^i or v_2^i has degree less than 3 in B^i. □

Claim 4. *For $i, j\in\{1, 2\}$, if v_j^i is not a cut-vertex of G^i, then v_j^i has degree at least 3 in B^i.*

Proof. Without loss of generality, we assume that $i=j=1$. Since the degree of x_1 is at least 3 in G, x_1 is incident with an edge $e'\in E(G)-E(\Theta)$. Since v_1^1 is not a cut-vertex of G^1, all the edges incident with v_1^1 in G^1 are in B^1. Since we map x_1 and x_5 to v_1^1, and since $e'\in E(G)-E(\Theta)$, v_1^i has degree at least 3 in B^1. □

The proof of Lemma 6 will now be divided into the following cases.

Case 1: *For some $i\in\{1, 2\}$, v_1^i, v_2^i are cut vertices of G^i.*

Without loss of generality, we assume $i=1$ and that there are end blocks L_1^1 and L_2^1 in G^1, where $B^1\notin\{L_1^1, L_2^1\}$, with $A_{G^1}(L_j^1)=\{v_j^i\}$ $(1\leq j\leq 2)$. Let $H_j=G[E(L_j^1)]$, $(1\leq j\leq 2)$. Then H_1 and H_2 are connected, and $A_G(H_1)=\{x_1, x_5\}$ and $A_G(H_2)=\{x_2, x_6\}$. Note that H_1 and H_2 also induce subgraphs in G^2 (which we also call H_1 and H_2), with $A_{G^2}(H_1)=\{x_1, v_1^2\}$ and $A_{G^2}(H_2)=\{x_6, v_2^2\}$. Therefore, the block B^2 of G^2 containing v_1^2 and v_2^2 also contains C_1, H_1 and H_2, and so v_1^2 and v_2^2 have degree at least 3 in B^2, contrary to Claim 3.

Case 2: *The only cut-vertex of G^i is v_1^i, $(1\leq i\leq 2)$.*

Let L^1 be an end block of G^1 that does not contain v_2^1 and has v_1^1 as its only vertex of attachment in G^1, and let $H^1=G[E(L^1)]$. Then H^1 is connected with $A_G(H^1)=\{x_1, x_5\}$. Thus x_5 is incident with an edge $e''\in E(H^1)$. Since H^1 can be regarded as a subgraph of B^2, e'' is an edge incident with v_1^2 in B^2. Hence v_1^2 is incident with e'' and two edges in C_1 and so v_1^2 has degree at least 3 in B^2. Since

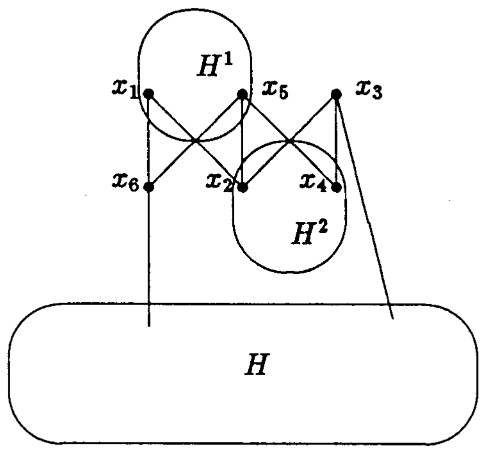

Fig. 3. Case 3 of Lemma 6.

v_2^2 is not a cut-vertex of G^2, it follows by Claim 4 that v_2^2 has degree at least 3 in B^2, contrary to Claim 3.

Case 3 (see Fig. 3): *The only cut-vertex of* G^1 *is* v_1^1 *and the only cut-vertex of* G^2 *is* v_2^2.

Let L^i be an end block of G^i that contains v_i^i but not v_{3-i}^i and let $H^i = G[E(L^i)]$, $(1 \leqslant i \leqslant 2)$. Since L^i is an end block of G^i, $(1 \leqslant i \leqslant 2)$, and since $\kappa(G) \geqslant 2$,

$$A_G(H^1) = \{x_1, x_5\} \quad \text{and} \quad A_G(H^2) = \{x_2, x_4\}.$$

Now we show that $G - E(\Theta)$ has a component H (say) with $A_G(H) = \{x_3, x_6\}$.

Since $\delta(G) \geqslant 3$, x_3 is incident with an edge that is not in $E(\Theta)$. By $\kappa(G) \geqslant 2$, there is an (x_3, x_j)-path P in $G - E(\Theta)$ for some $j \in \{1, 2, 4, 5, 6\}$. If $j = 1$ or $j = 5$, then in G^1, v_1^1 has degree at least 3 in B^1. By Claim 4, v_2^1 has degree at least 3 in B^1, contrary to Claim 3. If $j = 2$ or $j = 4$, then in G^2, v_2^2 has degree 3 in B^2, and a similar contradiction arises. Hence $j = 6$ and so $A_G(H) = \{x_3, x_6\}$.

Claim 5. *There is no* $y' \in V(G) - \{x_2, x_5\}$ *such that either* $y'x_6, y'x_e \in E(G)$ *or* $y'x_1, y'x_3 \in E(G)$.

Proof. If $y'x_4, y'x_6 \in E(G)$ and $y' \neq x_5$, then since G has a connected subgraph H with $A_G(H) = \{x_3, x_6\}$, in G^1, both v_1^1 and v_2^1 have degree at $\geqslant 3$ in B^1, violating Claim 3. The proof when $y'x_1, y'x_3 \in E(G)$ is similar. □

Subcase 3.1: *There is a vertex* $y \in V(H^1)$ *with* $yx_1, yx_5 \in E(G)$.

Then $x_1 y x_5 x_2 x_1$ is a 4-cycle in G. Let $\pi(3) = \langle \{y, x_2\}, \{x_1, x_5\} \rangle$ and let $G^3 = G/\pi(3)$ Denote by v_1^3 and v_2^3 the vertices of G^3 to which $\{x_1, x_5\}$ and $\{y, x_2\}$ are mapped, respectively. Since $x_3 v_2^3 v_1^3 x_4 x_3$ is a 4-cycle in G^3, and since $\Theta_1 = G[\{x_1, x_2, x_3, x_4, x_5, y\}] \cong \Theta$, we can apply the previous proofs to G^3 to make the following.

Claim 3′. *Let B^3 be the block in G^3 that contains v_1^3 and v_2^3. Then either v_1^3 or v_2^3 has degree less than* 3 *in B^3.*

Proof. Note that the path $P' = x_6v_1^3v_2^3x_3$ and the (x_3, x_6)-path P in $G - E(\Theta)$ form a cycle in G^3 containing v_1^3 and v_2^3. The block B^3 in G^3 containing v_1^3 and v_2^3 contains P, P', H^2 and the edge $v_1^3x_4$. Hence v_1^3 and v_2^3 have degree ≥ 3 in B^3, contrary to Claim 3′. □

Subcase 3.2: *There is no $y \in V(H^1)$ with $yx_1, yx_5 \in E(H^1)$.*

Let $G^* = (G - \{x_1x_2, x_5x_6\})/\{x_1x_6, x_2x_5\}$, and let v_1^*, v_2^* denote the vertices of G^* to which x_1x_6 and x_2x_5 are contracted, respectively. The component H of $G - E(\Theta)$ containing x_6 does not contain x_2. Hence, by the assumption of this subcase and by Claim 5, no 4-cycle of G contains exactly 2 edges of $E(C_1)$. Since G is reduced (and hence is simple), no 4-cycle of G has exactly 3 edges in $E(C_1)$. By Claim 3, no 4-cycle of G contains x_5x_6 and no other edges in $E(C_1)$; and no 4-cycle of G contains x_1x_2 and no other edges of $E(C_1)$. Hence G^* satisfies (1). Since $\kappa(G) \geq 2$, every cut-vertex of G^* other than v_1^* and v_2^* must separate v_1^* and v_2^*. Since H^1 is connected, there is a (x_1, x_5)-path Q in H^1 that is disjoint from the (x_3, x_6)-path P. These paths Q and P, together with $v_2^*x_3$, form a cycle in G^* containing v_1^* and v_2^*. Hence only v_1^* or v_2^* can be a cut-vertex of G^*.

If v_1^* and v_2^* are not cut-vertex of G^*, then $\kappa(G^*) \geq 2$. By the assumption of this subcase and by $A_G(H^1) = \{x_1, x_5\}$, (1) holds for H^1. Hence x_1 has degree ≥ 2 in H^1 and so v_1^* has degree ≥ 3 in G^*. Similarly, x_5 has degree ≥ 2 in H^1 and so v_2^* has degree ≥ 3 in G^*. It follows by the minimality of G that $G^* \in \mathscr{CL}$ and so by Lemma 5, $G \in \mathscr{CL}$, a contradiction.

If v_1^* is a cut-vertex of G^*, then G^* has an end block B_1^* containing v_1^* but not v_2^*. Let $B_1 = G[E(B_1^*)]$. Since $\kappa(G) \geq 2$, $A_G(B_1) = \{x_1, x_6\}$. Hence x_1 is incident with an edge $e_1 \in E(B_1)$ and x_6 is incident with an edge $e_6 \in E(B_1)$. Since $e_1, e_6 \notin E(\Theta)$, so in G^1, the vertices v_1^1, v_2^1 have degree ≥ 3 in B^1, contrary to Claim 3.

If v_2^* is a cut-vertex of G^*, then G^* has an end block B_2^* that contains v_2^* but not v_1^*. Let $B_2 = G[E(B_2^*)]$. Since $\kappa(G) \geq 2$, $A_G(B_2) = \{x_2, x_5\}$ and so in G^1, v_1^1 and v_2^1 have degree ≥ 3 in B^1 contrary to Claim 3.

This proves Lemma 6. □

Main result

Theorem 1. *If $G \in \mathscr{G}_1$, then $G \in \mathscr{CL}$.*

By contradiction, let G be a counterexample to Theorem 1 with $|V(G)|$ minimized. Immediately from Lemmas 4 and 6 and from definitions, we have the following observations.

Lemma 7 ([5]). *Each of the following holds.*

(i) *G is reduced and satisfies* (2).

(ii) *If W is a maximal $K_{2,t}$-subgraph of G, $(t \geq 2)$, then*

$$D_2(W) \subseteq D_4^*(G). \tag{6}$$

(iii) *If L and J are two subgraphs of G with $|V(J) \cap V(L)| \leq 1$ and $J \cup L = G$, then both J and L satisfy* (2).

(iv) *If W is a maximal $K_{2,t}$-subgraph of G and if L is a connected subgraph of $G - E(W)$ with $A_G(L) \subseteq V(W)$, then* (2) *hold for L.*

(v) *If $C = x_1x_2x_3x_4x_1$ is a maximal $K_{2,2}$-subgraph of G and if $G - E(C)$ has an (x_1, x_4)-path P_1 and an (x_2, x_3)-path P_2, then $V(P_1) \cap V(P_2) \neq \emptyset$.*

Lemma 8. *G cannot have a maximal $K_{2,2}$-subgraph.*

Proof. By contradiction, let $C = x_1x_2x_3x_4x_1$ be a maximal $K_{2,2}$-subgraph of G. Define G^*, v_1, v_2 as in Notation 2. Then G^* satisfies (2) and by (6), $\delta(G^*) \geq 3$. By Lemma 5 and by the minimality of G, $\kappa(G^*) \leq 1$. By (v) of Lemma 7, G^* is connected. Hence $\kappa(G^*) = 1$.

Case 1: *There is a cut-vertex $z \notin \{v_1, v_2\}$ in G^*.*

Then by $\kappa(G) \geq 2$, z must separate v_1 and v_2 in G^*. By (v) of Lemma 7, we may assume that z is in every (x_1, x_4)-path and in every (x_2, x_3)-path of $G - E(C)$. Hence there are connected subgraphs H_1, H_2, H_3, H_4 of $G - E(C)$ such that $A_G(H_i) = \{z, x_i\}$ and

$$G - E(C) = H_1 \cup H_2 \cup H_3 \cup H_4.$$

Let L^* be the graph obtained from $H_1 \cup H_4$ by identifying x_1 and x_4, J^* be the graph obtained from $H_2 \cup H_3$ by identifying x_2 and x_3. Note that $G^* = J^* \cup L^*$. By (iv) of Lemma 7, $G - E(C)$ satisfying (2), and so by (iii) of Lemma 7, both $H_1 \cup H_4$ and $H_2 \cup H_3$ satisfy (2). Thus L^* and J^* satisfy (2). By (2) and by the fact that z is a cut-vertex of $H_1 \cup H_4$, the degree of z in L^* is at least 4 and so by (6), $\delta(L^*) \geq 3$. Similarly, $\delta(J^*) \geq 3$. It follows that J^*, $L^* \in \mathcal{G}_1$ and so by the minimality of G, J^*, $L^* \in \mathcal{CL}$. By (v) of Theorem A, $G^* \in \mathcal{CL}$ and so by Lemma 5, $G \in \mathcal{CL}$, a contradiction.

Case 2: *There is no cut-vertex in G^* not in $\{v_1, v_2\}$.*

Without loss of generality, let v_1 be a cut-vertex of G^*. Then G^* has a nontrivial connected subgraph B^* with $A_{G^*}(B^*) = \{v_1\}$ and $v_2 \notin V(B^*)$. Let $B = G[E(B^*)]$. Then $A_G(B) = \{x_1, x_4\}$. By $\kappa(G) \geq 2$, B is connected. By (v) of Lemma 7,

$$\text{every } (x_2, x_3)\text{-path in } G - E(C) \text{ uses } x_1 \text{ or } x_4. \tag{7}$$

Since G^* has no cut-vertex not in $\{v_1, v_2\}$, $G^* - E(B^*)$ has a cycle containing v_1 and v_2. Let H^* be the block of G^* that contains both v_1 and v_2 and let $H = G[E(H^*)]$. Then $A_G(H) \subseteq V(C)$ and so by (iv) of Lemma 7, H satisfies (2). Without loss of generality we assume that $x_2 \in V(H)$.

Claim. $x_3 \in V(H)$.

Proof. Suppose not, we assume that $x_3 \notin V(H)$. By $\delta(G) \geq 3$, x_3 is incident with an edge in $G - E(C)$. Note that this edge is an edge on G^*. Let K^* be a block of G^* containing v_2 but not v_1, and let $K = G[E(K^*)]$. By $\kappa(G) \geq 2$, $x_2, x_3 \in V(K)$. Hence there is an (x_2, x_3)-path in K, contrary to (7). This proves the Claim. □

Since H^* contains v_1, either x_1 or x_4 is in $V(H)$. If $x_4 \notin V(H)$, then by (7), x_1 is a cut-vertex of H and so $x_1 \in D_3^*(H)$; if $x_4 \in V(H)$, then $V(C) \subseteq D_2^*(H)$. Hence in either case, v_1 and v_2 have degree ≥ 3 in H^* and so by the minimality of G, $H^* \in \mathscr{C}\mathscr{L}$. Note that $(G[E(H) \cup E(C)])^* = H^*$. By Lemma 5, G is not reduced, contrary to Lemma 4.□

Lemma 9 ([5]). *If G has a $K_{2,t}$-subgraph W with $t \geq 3$, then $t = 3$ and $D_3(W) \subseteq D_3(G)$.*

Proof. The proof uses essentially the same technique as in the proof of Lemma 8 and is routine. □

Notation 3 (see Fig. 4). Define D to be the union of two copies of $K_{2,3}$'s W_1 and W_2 such that

$$V(W_1) \cap V(W_2) = D_2(W_1) \cap D_2(W_2) = \{y\} \subseteq D_4(G).$$

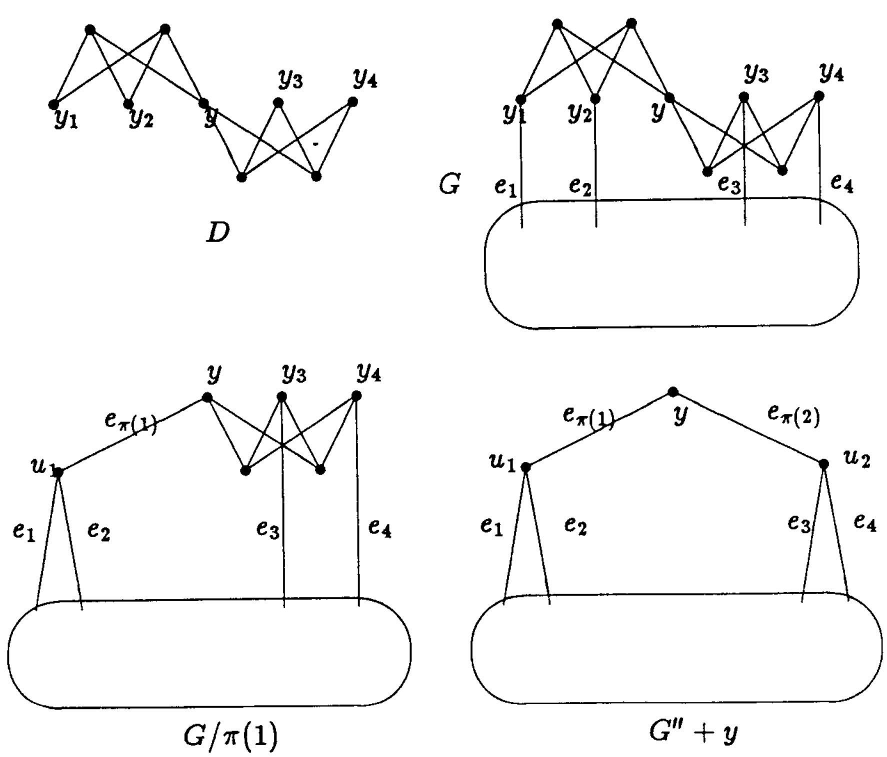

Fig. 4. Notation 3.

Denote

$$D_1(W_1)=\{y_1, y_2, y\} \quad \text{and} \quad D_2(W_2)=\{y_3, y_4, y\}.$$

Suppose that H is a subgraph of G containing D. Let

$$V_{11}=\{y_1, y_2\}, \qquad V_{12}=V(W_1)-V_{11} \quad \text{and} \quad \pi(1)=\langle V_{11}, V_{12}\rangle.$$

For convenience, we let u_1, y denote the vertices of $G/\pi(1)$ to which V_{11} and V_{12} are mapped, respectively. Denote $e_{\pi(1)}=u_1y$ and $H'=(H/\pi(1))-e_{\pi(1)}$. We regard W_2 as a subgraph of H'. Let

$$V_{21}=\{y_3, y_4\}, \qquad V_{22}=V(W_2)-V_{21} \quad \text{and} \quad \pi(2)=\langle V_{21}, V_{22}\rangle.$$

Let u_2, y denote the vertices of $H'/\pi(2)$ to which V_{21} and V_{22} are mapped, respectively and let $e_{\pi(2)}=u_2y$. Define H'' to be the nontrivial component of $H'/\pi(2)-e_{\pi(2)}$. Note that

$$G''=(G/\pi(1))/\pi(2)-\{y\}.$$

By (iii) of Theorem A, by Theorem B, and by $\kappa(G)\geq 2$, we have the following.

Lemma 10 ([5]). *Let D, G'', u_1, u_2 be defined in Notation* 3. *Then*:

(i) *If $G''\in\mathscr{CL}$, then $G\in\mathscr{CL}$.*

(ii) *Neither u_1 nor u_2 is a cut-vertex of G''.*

Since $G\in\mathscr{G}_1$, G'' satisfies (1). By (6), $\delta(G'')\geq 3$. Since G is a minimum counterexample to Theorem 1, by (i) of Lemma 11, $\kappa(G'')\leq 1$. This, together with $\kappa(G)>2$, implies

$$\kappa(G'')=1. \tag{8}$$

Lemma 11 ([5]). *If H is a subgraph of G with $|A_G(H)|=2$, then either $H\cong K_2$ or H contains a 2-block of G.*

Proof. This follows from the fact that $G\in\mathscr{G}_1$. □

Lemma 12 ([5]). *If H is a minimal 2-block of G, then*:

(i) *H cannot have D as a subgraph.*

(ii) *H cannot have a $K_{2,3}$-subgraph W with*

$$D_2(W)\subseteq D_5^*(G). \tag{9}$$

Proof. The proof is routine with the help of Lemmas 10 and 11, and the minimality of H. □

Lemma 13. *G has a 2-block.*

Proof. By (i) of Lemma 7, and by Lemmas 8 and 9, G is an edge-disjoint union of $K_{2,3}$'s such that if W is a $K_{2,3}$-subgraph of G, then $D_3(W) \subseteq D_3(G)$. Hence G either has D as a subgraph or has a subgraph $W \cong K_{2,3}$ satisfying (9).

Case 1: *G has D as a subgraph.*

We shall use the notations in Notation 3 and let G'', u_1, u_2 be defined as in Notation 3. By (8), by (ii) of Lemma 10, and by $\kappa(G) \geq 2$, G'' has a cut-vertex $z \notin \{u_1, u_2\}$ separating u_1 and u_2. Let B'' be an end block of G'' that contains u_1 but not u_2, and let $B = G[E(B'')]$. Then $A_G(B \cup W_1) = \{z, y\}$ and so by Lemma 11 with $H = B \cup W_1$, G has a 2-block. □

Case 2: *G has a subgraph $W \cong K_{2,3}$ satisfying* (9).

Define G', v_1, v_2 as in Notation 1 with $t = 3$. By (9), $\delta(G') \geq 3$. It is easy to see that $\kappa(G') = 1$ and G' has a cut-vertex $z \notin \{v_1, v_2\}$ that separates v_1 and v_2 in G'. Let B' be an end block of G' that contains v_1 but not v_2 and let $B = G[E(B')]$. Then $A_G(B) = \{z, y_3\}$ and so by Lemma 11, G has a 2-block. □

Proof of Theorem 1. By Lemma 13, G has a minimal 2-block H. By Lemma 7, 8 and 9, H must be an edge-disjoint union of $K_{2,3}$'s. Since H is a 2-block, and since $\delta(G) \geq 3$, $H \not\cong K_{2,3}$. Hence either H has D as a subgraph or H has a subgraph $W \cong K_{2,3}$ that satisfies (9), contrary to Lemma 12. □

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