

# Cycle covering of plane triangulations

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**Abstract.** Bondy conjectures that if  $G$  is a 2-edge-connected simple graph with  $n$  vertices, then at most  $(2n-1)/3$  cycles in  $G$  will cover  $G$ . In this note, we show that if  $G$  is a plane triangulation with  $n \geq 6$  vertices, then at most  $(2n-3)/3$  cycles in  $G$  will cover  $G$ .

## 1. Introduction

We follow the notation of Bondy and Murty [BM], except where noted otherwise. An edge  $e$  of a graph  $G$  is called a *multiple edge* if  $G - e$  has an edge  $f$  having the same ends as  $e$ , and in this case we say that  $e$  is an *extra edge* of  $G - e$  parallel to the edge  $f$ . Graphs may have multiple edges but loops are prohibited. Let  $G$  be a graph. For  $X \subseteq E(G)$ , the *contraction*  $G/X$  is the graph obtained from  $G$  by identifying the ends of each edge in  $X$  and then deleting the resulting loops. A collection  $\mathcal{C}$  of cycles in  $G$  is called a *cycle cover* (CC) of  $G$ , if every edge of  $G$  lies in at least one cycle in  $\mathcal{C}$ . It is obvious that  $G$  has a CC if and only if  $G$  is 2-edge-connected. For a graph with  $\kappa'(G) \geq 2$ , define

$$cc(G) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is a CC of } G\}.$$

In [B], Bondy raised the following conjecture.

*Conjecture SCC:* If  $G$  is a simple 2-edge-connected graph with  $n$  vertices, then

$$cc(G) \leq \frac{2n-1}{3}.$$

If  $\mathcal{C}$  is a collection of cycles of  $G$  and if every edge in  $G$  lies in exactly 2 members of  $\mathcal{C}$ , then  $\mathcal{C}$  is called a *cycle double cover* (CDC) of  $G$ . The eminent cycle double cover conjecture, due to Seymour [S1] and Szekeres [S2], says that every 2-edge-connected graph admits a CDC. The following conjecture is also posted by Bondy in [B].

*Conjecture SCDC:* If  $G$  is a simple 2-edge-connected graph with  $n$  vertices, then  $G$  admits a CDC with at most  $n-1$  cycles.

**Theorem 1.1.** (Bondy and Seyffarth [B]) *If  $G$  is a simple plane triangulation with  $n$  vertices, then  $G$  has a CDC with at most  $n - 1$  cycles.*

In this paper, we shall show that if  $G$  is a simple plane triangulation with  $n \geq 6$  vertices, then

$$cc(G) \leq \frac{2n-3}{3}.$$

We shall prove a multigraph version of Conjecture SCC for plane triangulations. Let  $G$  be a graph. Define an equivalence relation on  $E(G)$  such that  $e$  is related to  $e'$  if and only if  $e = e'$  or  $e$  and  $e'$  share the same ends ( $e$  and  $e'$  are parallel edges). Let  $[e]$  denote the equivalence class containing  $e$  and  $[G]$  the collection of all equivalence classes. Define

$$\mu(G) = \sum_{[e] \in [G]} (|[e]| - 1).$$

Hence a graph  $G$  is simple if and only if  $\mu(G) = 0$ . We define a (multi) graph  $G$  to be a *plane triangulation* if  $G$  is a plane graph each of whose faces has degree 2 or 3. In Section 2, we develop some reduction techniques, and in Section 3, we shall show Theorem 1.2 below. Some of the routine and repeated arguments in the proofs are omitted. Interested readers may contact the authors for details.

**Theorem 1.2.** *If  $G$  is a plane triangulation of  $n \geq 6$  vertices, then*

$$cc(G) \leq \frac{2n-3}{3} + \frac{\mu(G)}{2}. \quad (1)$$

## 2. Reductions

Let  $X, Y$  be two sets. The *symmetric difference* of  $X$  and  $Y$ , denoted by  $X \Delta Y$ , is  $X \cup Y - X \cap Y$ . If  $G$  is a graph and  $H$  and  $J$  are subgraphs of  $G$ , then denote

$$H \Delta J = G[E(H) \Delta E(J)].$$

If  $G$  has 2 subgraphs  $G'$  and  $G''$  such that  $G = G' \cup G''$  and such that  $G' \cap G''$  is a 2-cycle of  $G$ , then  $G$  is called a  *$C_2$ -sum* of  $G'$  and  $G''$ .

**Lemma 2.1.** *Let  $G$  be a graph with  $\kappa'(G) \geq 2$ . If  $G$  is a  $C_2$ -sum of  $G_1$  and  $G_2$ , where  $\kappa'(G_i) \geq 2$ , then*

$$cc(G) \leq cc(G_1) + cc(G_2) - 1.$$

*Proof:* Let  $\{e_1, e_2\}$  be the edges of the 2-cycle  $C$  commonly shared by  $G_1$  and  $G_2$ . For  $i \in \{1, 2\}$ , let  $\mathcal{C}_i$  be a CC of  $G_i$ . Let  $C_j^i$  be a cycle in  $\mathcal{C}_i$  that contains the edge  $e_j$ , ( $1 \leq i, j \leq 2$ ). If  $C_1^1 = C_2^1$ , then  $C = C_1^1 = C_2^1$ , and so  $(\mathcal{C}_1 - \{C\}) \cup \mathcal{C}_2$

is a CC of  $G$  and Lemma 2.1 follows. Hence we may assume that  $C_1^i \neq C_2^i$  and so  $E(C_j^i) \cap E(C) = \{e_j\}$ . Thus

$$(C_1 - \{C_1^1, C_2^1\}) \cup (C_2 - \{C_1^2, C_2^2\}) \cup \{C_1^1 \Delta C_1^2, C_2^1 \Delta C_2^2\}$$

is a CC of  $G$  and so Lemma 2.1 follows again.  $\blacksquare$

Let  $H$  be a subgraph of  $G$ . The *vertices of attachment* of  $H$  in  $G$ , denoted by  $A_G(H)$ , are the vertices in  $V(H)$  that are incident with some edges in  $E(G) - E(H)$ .

For a graph  $H$ ,  $H^+$  denotes a graph obtained from  $H$  by adding an extra edge parallel to some edge of  $H$ .

**Lemma 2.2.** *Suppose that  $H = \Gamma_1$  or  $H = \Gamma_1^+$  (see Figure 1) with an extra edge  $e$  that is parallel to an edge in  $E(\Gamma_1) - \{v_1 v_2, v_2 v_3, v_3 v_1\}$ , such that  $H$  is a subgraph of  $G$  with  $A_G(H) \subseteq \{v_1, v_2, v_3\}$ . Let  $e_1$  be an extra edge parallel to  $v_1 v_2$ , and  $e_2$  be an extra edge parallel to  $v_2 v_3$ . Let  $V_H = V(H) - \{v_1, v_2, v_3\}$ .*

(i) *If  $H = \Gamma_1$ , then let  $G' = (G - V_H) + e_2$  and we have*

$$cc(G) \leq cc(G') + 1. \quad (2)$$

(ii) *If  $H = \Gamma_1^+$ , then let  $G'' = (G - V_H) + \{e_1, e_2\}$  and we have*

$$cc(G) \leq cc(G'') + 1. \quad (3)$$

**Proof:** We shall show (i) first. Let  $\mathcal{C}$  be a CC of  $G'$ , and let  $C \in \mathcal{C}$  be a cycle containing  $e_2$ . Let  $C' = C - e_2 + \{v_2 v_5, v_5 v_4, v_4 v_6, v_6 v_3\}$ , and  $F = v_2 v_4 v_1 v_6 v_5 v_3 v_2$ . Thus  $(\mathcal{C} - \{C\}) \cup \{C', F\}$  is a CC of  $G$  and so (2) holds.

The proof for (ii) is similar and uses the fact that we can always assume that  $e_1$  and  $e_2$  are in distinct cycles of any CC of  $G''$ .  $\blacksquare$

**Lemma 2.3.** *Suppose that  $H = \Gamma_i$  or  $H$  is  $\Gamma_i^+$  (see Figures 1 and 2) with an extra edge  $e$  that is parallel to an edge of  $E(\Gamma_i) - \{v_1 v_2, v_2 v_3, v_3 v_1\}$ , ( $2 \leq i \leq 4$ ), such that  $H$  is a subgraph of  $G$  with  $A_G(H) \subseteq \{v_1, v_2, v_3\}$ . Let  $e_i$  be an extra edge parallel to  $v_i v_3$ , ( $1 \leq i \leq 2$ ), and let  $V_H = V(H) - \{v_1, v_2, v_3\}$ .*

(i) *If  $H = \Gamma_i$ , then let  $G' = G - V_H$  and we have*

$$cc(G) \leq cc(G') + 2. \quad (4)$$

(ii) *Suppose that  $H = \Gamma_i^+$ . If  $e$  is not incident with  $v_1$ , then let  $G'' = G - V_H + e_1$ , and if  $e$  is incident with  $v_1$ , then let  $G'' = G - V_H + e_2$ . In either case, we have*

$$cc(G) \leq cc(G'') + 2.$$

Proof: We consider the following cases.

Case 1:  $i = 2$ .

Let  $\mathcal{C}$  be a CC of  $G'$  and let  $C$  be a cycle in  $\mathcal{C}$  that contains  $v_1 v_3$ . Let  $C' = C - v_1 v_3 + \{v_1 v_6, v_6 v_4, v_4 v_5, v_5 v_3\}$ , let  $F_1 = v_1 v_3 v_6 v_5 v_1$  and  $F_2 = v_1 v_2 v_5 v_4 v_1$ . Then  $(\mathcal{C} - \{C\}) \cup \{C', F_1, F_2\}$  is a CC of  $G$  and so (4) holds.

The proof for (5) is similar and uses the fact that we can assume that  $e_1$  and  $v_1 v_3$  are in distinct cycles of any CC of  $G''$ .

Case 2:  $i = 3$ .

Let  $\mathcal{C}$  be a CC of  $G'$  and let  $C_1, C_2$  be cycles in  $\mathcal{C}$  that contain  $v_1 v_3$  and  $v_2 v_3$ , respectively. (It may happen that  $C_1 = C_2$ ). Let  $C'_1 = C_1 - \{v_1 v_3\} + \{v_1 v_4, v_4 v_6, v_6 v_3\}$ ,  $C'_2 = C_2 - \{v_2 v_3\} + \{v_2 v_5, v_5 v_7, v_7 v_3\}$ , (if  $C_1 = C_2$ , then  $C'_1 = C'_2$  is obtained by replacing  $v_1 v_3, v_2 v_3$  by the above two paths, respectively), and let  $F_1 = v_1 v_3 v_2 v_7 v_6 v_5 v_1$  and  $F_2 = v_1 v_2 v_5 v_4 v_6 v_1$ . Thus  $(\mathcal{C} - \{C_1, C_2\}) \cup \{C'_1, C'_2, F_1, F_2\}$  is a CC of  $G$ , and so (4) holds.

Suppose that  $H$  is  $\Gamma_3^+$  and  $e$  is not incident with  $v_1$ . Let  $\mathcal{C}'$  be a CC for  $G''$  and let  $C_1, C_2$  be defined as above and let  $C_e$  be the cycle in  $\mathcal{C}$  containing  $e_1$ .

If  $E(C_2) \neq \{e_1, v_2 v_3\}$ , then  $C_e \neq C_2$ . Since  $e$  is not incident with  $v_1$ , there is a  $(v_2, v_3)$ -path  $P$  in  $\Gamma_3$  containing  $e$  such that the internal vertices of  $P$  are in  $V_H$ . Thus we can define  $C'_e$  to be  $C_e - e_1$  plus the  $(v_2, v_3)$ -path  $P$ , and define  $C'_1, C'_2, F_1, F_2$  as above. It follows that  $(\mathcal{C} - \{C_1, C_2, C_e\}) \cup \{C'_1, C'_2, C'_e, F_1, F_2\}$  is a CC of  $G$  and so (5) holds.

Thus we assume that  $E(C_2) = E(C_e) = \{e_1, v_2 v_3\}$ . Without loss of generality, we assume that  $e$  is not parallel to  $v_5 v_7$ . Let  $F_3 = v_1 v_4 v_6 v_3 v_7 v_2 v_5 v_1$ ,  $F_4 = v_1 v_4 v_5 v_6 v_7 v_2 v_3 v_1$ ,  $C''_1 = C_1 - \{v_1 v_3\} + \{v_1 v_6, v_6 v_3\}$ , and let  $F_5$  be any cycle containing both  $v_5 v_7$  and  $e$ . Thus  $(\mathcal{C} - \{C_2, C_1\}) \cup \{C''_1, F_3, F_4, F_5\}$  is a CC of  $G$ , and so (5) holds.

The case when  $e$  is incident with  $v_1$  can be shown similarly.

Case 3:  $i = 4$ .

Let  $\mathcal{C}$  be a CC of  $G'$  and let  $C_1, C_2$  be cycles in  $\mathcal{C}$  containing  $v_1 v_3$  and  $v_2 v_3$ , respectively. (Possibly  $C_1 = C_2$ ). Let  $C'_1 = C_1 - \{v_1 v_3\} + \{v_1 v_4, v_4 v_6, v_6 v_3\}$  and  $C'_2 = C_2 - \{v_2 v_3\} + \{v_2 v_5, v_5 v_6, v_6 v_7, v_7 v_3\}$ , and let  $F_1 = v_1 v_3 v_2 v_8 v_7 v_5 v_6 v_1$  and  $F_2 = v_1 v_2 v_7 v_6 v_4 v_5 v_1$ . Then  $(\mathcal{C} - \{C_1, C_2\}) \cup \{C'_1, C'_2, F_1, F_2\}$  is a CC of  $G$  and so (4) holds.

The proof when  $H = \Gamma_4^+$  is similar to that for the Case of  $i = 3$ . ■

**Lemma 2.4.** *Suppose that  $H = \Gamma_5$  or  $H = \Gamma_5^+$  (see Figure 3) with an extra edge  $e$  that is parallel to an edge of  $E(\Gamma_5) - \{v_1 v_2, v_2 v_3, v_3 v_1\}$  such that  $H$  is a subgraph of  $G$  with  $A_G(H) \subseteq \{v_1, v_2, v_3\}$ . Let  $V_H = V(H) - \{v_1, v_2, v_3\}$  and let  $G' = G - V_H$ . Then*

$$\text{cc}(G) \leq \text{cc}(G') + 3. \quad (6)$$

Proof: Let  $C$  be a CC of  $G'$  and let  $C_1, C_2, C_3$  be cycles in  $C$  containing  $v_1 v_3$ ,  $v_2 v_3$  and  $v_1 v_2$ , respectively.

Assume first that  $H = \Gamma_5$ . Then let  $C'_1 = C_1 - \{v_1 v_3\} + \{v_1 v_4, v_4 v_6, v_6 v_3\}$ ,  $C'_2 = C_2 - \{v_2 v_3\} + \{v_2 v_8, v_8 v_7, v_7 v_9, v_9 v_3\}$ , and  $C'_3 = C_3 - \{v_1 v_2\} + \{v_1 v_5, v_5 v_2\}$ , and let  $F_1 = v_1 v_4 v_5 v_6 v_7 v_2 v_3 v_1$ ,  $F_2 = v_1 v_6 v_9 v_7 v_5 v_8 v_2 v_1$  and let  $F_3$  be any cycle in  $G$  containing  $v_3 v_7$ . Thus  $C - \{C_1, C_2, C_3\} \cup \{C'_1, C'_2, C'_3, F_1, F_2, F_3\}$  is a CC of  $G$  and so (6) holds.

Now we assume that  $H$  has one multiple edge  $e$ . Without loss of generality, we may assume that  $e$  is not parallel to  $v_3 v_7$ . Thus one can choose  $F_3$  above so that  $e, v_3 v_7 \in E(F_3)$  and so (6) holds also. ■

**Lemma 2.5.** *Suppose that  $H = \{\Gamma(6), \Gamma(6)^+\}$  (see Figure 3) such that  $H$  is a subgraph of  $G$  with  $A_G(H) \subseteq \{w_1, w_2, w_3\}$ . Let  $e$  be an extra edge not in  $E(G)$  that is parallel to  $w_1 w_2$ . If  $H = \Gamma(6)$ , then let  $G' = G - \{w_4, w_5, w_6\}$ , and if  $H = \Gamma(6)^+$  (without loss of generality, we assume that the multiple edge in  $H$  is parallel to one of  $\{w_1 w_4, w_4 w_5, w_5 w_6, w_2 w_4, w_2 w_5, w_2 w_6\}$ ), then let  $G' = G - \{w_4, w_5, w_6\} + e$ . In any case, we have*

$$cc(G) \leq cc(G') + 2.$$

Proof: Let  $F_1 = w_1 w_4 w_2 w_5 w_3 w_1$  and  $F_2 = w_2 w_6 w_5 w_3 w_4 w_2$ . Let  $C$  be a CC of  $G'$  and let  $C_1$  be a cycle in  $C$  containing  $w_1 w_3$ . Assume first that  $H = \Gamma(6)$ . Define  $C'_1 = C_1 - w_1 w_3 + w_1 w_4 w_5 w_6 w_3$ . Then  $(C - \{C_1\}) \cup \{C'_1, F_1, F_2\}$  is a CC of  $G$ , and so (7) must hold.

Then we assume that  $H = \Gamma(6)^+$ . Since  $e$  is parallel to  $w_1 w_2$  in  $G'$ , we may assume that  $e$  does not lie in  $C_1$ . Let  $C_e \in C$  be a cycle containing  $e$ . Let  $C'_e$  be obtained from  $C_e$  by replacing  $e$  by a  $(w_1, w_2)$ -path in  $H$  that covers the multiple edge. Hence  $(C - \{C_1, C_e\}) \cup \{C'_1, C'_e, F_1, F_2\}$  is a CC of  $G$ , and so (7) holds also. ■

**Lemma 2.6.** *Let  $H$  be a subgraph of  $G$ .*

- (i) *Suppose that  $H = \Gamma_6$  (see Figure 4) or  $H = \Gamma_6^+$  with an extra edge  $e$  parallel to an edge of  $E(\Gamma_6 - \{x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_1\})$  and with  $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$ . If  $H = \Gamma_6$ , then define  $G_1 = G - \{x_5, x_6\}$ ; and if  $H = \Gamma_6^+$ , then define  $G_1 = G - \{x_5, x_6\} + e'$ , where  $e' \notin E(G)$  is an extra edge parallel to  $x_2 x_4$ . We have*

$$cc(G) \leq cc(G_1) + 1. \quad (8)$$

- (ii) *Suppose that  $H = L_6$  or  $H = L_6^+$  (see Figure 4) with an extra edge  $e$  that is parallel to an edge of  $E(\Gamma_6 - \{x_1 x_2, x_2 x_3, x_3 x_1\})$  and with  $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$  and  $e' \notin E(G)$  be an extra edge. If  $H = L_6$ , then let  $G_2 = G - \{x_5, x_6\} + x_2 x_4$ ; and if  $H = L_6^+$ , then let  $G_2 = G - \{x_5, x_6\} + x_2 x_4 + e'$ ,*

where  $e'$  is parallel to  $x_4x_5$  if  $e$  is parallel to  $x_3x_6$  or  $x_4x_6$ ; or where  $e'$  is parallel to  $x_2x_4$  if  $e$  is parallel to  $x_2x_6$  or  $x_2x_5$ ; or where  $e'$  is parallel to  $x_1x_4$  if  $e$  is parallel to  $x_1x_5$  or  $x_1x_6$ . In any case, we have

$$cc(G) \leq cc(G_2) + 1. \quad (9)$$

- (iii) Let  $L \in \{L'_8, L''_8, L'''_8\}$  (see Figure 6) and let  $e' \notin E(G)$  be an extra edge parallel to  $x_2x_4$ . If  $H = L$  or  $H = L^+$  with an extra edge  $e$  that is parallel to an edge of  $E(H - \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\})$  and with  $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$ , then defining  $G_3 = G - \{x_5, x_6, x_7, x_8\} + e'$ , we have

$$cc(G) \leq cc(G_3) + 2. \quad (10)$$

- (iv) Let  $L''_7 = L''_8 - x_8$ . If  $H = L''_7$  or  $H = L''_7^+$  with an extra edge  $e$  parallel to an edge in  $E(H) - \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$  and with  $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$ , then defining  $G_4 = G - \{x_5, x_6, x_7\}$ , we have

$$cc(G) \leq cc(G_4) + 2. \quad (11)$$

- (v) Let  $L'_6 = L''_8 - \{x_5, x_6\}$  and let  $L''_6 = L''_8 - \{x_7, x_8\}$ . Suppose that  $H \in \{L'_6, L''_6\}$  or  $H \in \{L'^+_6, L''^+_6\}$  with an extra edge  $e$  parallel to an edge in  $E(H) - \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$  and with  $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$ . If  $H \in \{L'_6, L''_6\}$ , then let  $G_5 = (G - \{x_5, x_6, x_1x_4, x_4x_3\}) / \{x_2x_4\}$ , and if  $H \in \{L'^+_6, L''^+_6\}$ , then let  $G_5 = (G - \{x_5, x_6, x_1x_4, x_4x_3\} + e') / \{x_2x_4\}$ , where  $e' \notin E(G)$  is an extra edge parallel to  $x_2x_3$ . In any case, we have

$$cc(G) \leq cc(G_5) + 2. \quad (12)$$

Proof: We shall show (i) first. Suppose that  $H = \Gamma_6$  and that  $\mathcal{C}$  is a CC of  $G_1$  and let  $C_1$  and  $C_2$  be cycles in  $\mathcal{C}$  such that  $x_1x_4 \in E(C_1)$  and  $x_3x_4 \in E(C_2)$ . Let  $C'_1 = C_1 - x_1x_4 + x_1x_5x_4$ ,  $C'_2 = C_2 - x_3x_4 + x_3x_6x_4$ , and let  $F = x_1x_5x_2x_6x_3x_4x_1$ . Then  $(\mathcal{C} - \{C_1, C_2\}) \cup \{C'_1, C'_2, F\}$  is a CC of  $G$  and so (8) holds.

Suppose that  $H = \Gamma_6^+$ . By the case when  $H = \Gamma_6$ , we may assume that  $e$  is parallel to one of  $\{x_2x_5, x_5x_4, x_2x_6, x_6x_4\}$  and so we can replace  $e'$  by a  $(x_2, x_4)$ -path that passes the multiple edge  $e$ .

To show (ii) of Lemma 2.6, we assume that  $H = L_6$  and let  $\mathcal{C}$  be a CC of  $G_1$ . Let  $C_1, C_2, C_3 \in \mathcal{C}$  be cycles containing  $x_1x_2, x_2x_4$  and  $x_2x_3$ , respectively. Since no cycle can contain  $x_1x_2, x_2x_4, x_2x_3$  simultaneously, we may assume that either  $C_1 \neq C_2$  or  $C_2 \neq C_3$ .

If  $C_1 \neq C_2$ , then let  $C'_1 = C_1 - x_1x_2 + x_1x_6x_2$  and let  $C'_2 = C_2 - x_2x_4 + x_2x_5x_6x_4$ . Let  $F = x_1x_2x_3x_6x_5x_1$ . Replace  $x_2x_4$  by  $x_2x_6x_4$  in any cycle in

$C - \{C_1, C_2\}$  containing  $x_2 x_4$  and still denote the resulting collection by  $C - \{C_1, C_2\}$ , for convenience. Thus  $(C - \{C_1, C_2\}) \cup \{C'_1, C'_2, F\}$  is a CC of  $G$  and so (9) must hold.

If  $C_2 \neq C_3$ , then let  $C'_2 = C_2 - x_2 x_4 + x_2 x_6 x_4$  and let  $C'_3 = C_3 - x_2 x_3 + x_2 x_5 x_6 x_3$ , and let  $F' = x_1 x_5 x_2 x_3 x_6 x_1$ . Thus  $(C - \{C_2, C_3\}) \cup \{C'_2, C'_3\}$  is a CC of  $G$ . Hence (9) must hold again.

When  $H = L_6^+$ , we can replace  $e'$  in the cycle containing  $e'$  by a path in  $H$  containing the multiple edge and so (9) holds again.

To show (iii), we let  $C$  be a CC of  $G_3$  and let  $C_e, C_0, C_3, C_4 \in C$  be cycles that contain  $e', x_2 x_4, x_3 x_4$ , and  $x_4 x_1$  respectively.

Assume that  $H = L_8''$  or  $L_8''^+$ . Since  $[e'] = \{e', x_2 x_4\}$ , we may assume that  $C_4 \neq C_0$  and  $C_3 \neq C_e$ . Let  $F_1 = x_1 x_7 x_8 x_2 x_5 x_6 x_3 x_4 x_1$  and let  $C'_0 = C_0 - x_2 x_4 + x_2 x_5 x_6 x_4$ ,  $C'_3 = C_3 - x_3 x_4 + x_3 x_5 x_4$ ,  $C'_e = C_e - e' + x_2 x_7 x_8 x_4$  and  $C'_4 = C_4 - x_4 x_1 + x_4 x_7 x_1$ , and let  $F_2$  be a cycle containing  $x_2 x_4$  and the multiple edge (if it exists). Thus  $(C - \{C_0, C_3, C_4, C_e\}) \cup \{C'_0, C'_3, C'_4, C'_e, F_1, F_2\}$  is a CC of  $G$  and so (10) holds also.

The proof for the case when  $H \in \{L_8', L_8'^+, L_8''', L_8'''+\}$  is similar to that for  $H \in \{L_8'', L_8''^+\}$  and the proof for (iv) is similar to that for (iii). Thus they are omitted.

We shall show (v) for  $H \in \{L_6'', L_6''^+\}$ . The proof for  $H \in \{L_6', L_6'^+\}$  is similar. Let  $v$  denote the vertex in  $G_5$  to which  $x_2 x_4$  is contracted. Let  $C$  be a CC of  $G'$  and let  $C_e, C_3$  be cycles in  $C$  containing  $e'$  and  $v x_3$ , respectively. (If  $H = L_6''$ , then just take  $C_3$ ).

If  $C_e = C_3$ , then let  $F' = x_1 x_2 x_3 x_6 x_5 x_4 x_1$ ,  $F'' = x_2 x_5 x_3 x_6 x_4 x_2$  and let  $F'''$  be a cycle that contains the multiple edge  $e$ . Thus  $(C - \{C_3\}) \cup \{F', F'', F'''\}$  is a CC of  $G$  and so (12) holds.

Thus we assume that  $C_e \neq C_3$ . Let  $C'_3 = G[E(C_3) - v x_3]$ . Thus either  $C'_3 + x_2 x_3$  or  $C'_3 + x_3 x_4$  is a cycle in  $G$ . Note that any cycle in  $C - \{C_e, C_3\}$  can easily be adjusted to cycles in  $G$  (still denoted by  $C - \{C_e, C_3\}$ , for convenience).

Let  $F_1 = x_1 x_2 x_5 x_6 x_3 x_4 x_1$ ,  $F_2 = x_2 x_3 x_6 x_4 x_2$ , and let  $C'_e$  be obtained from  $C_e$  by replacing  $e'$  by an  $(x_2 x_3)$ -path containing  $e$ . If  $C'_3 + x_2 x_3$  is a cycle in  $G$ , then let  $C''_3 = C'_3 + x_2 x_6 x_5 x_4 x_3$ ; and if  $C'_3 + x_3 x_4$  is a cycle in  $G$ , then let  $C''_3 = C'_3 + x_4 x_2 x_6 x_5 x_3$ . Thus in any case,  $(C - \{C_3, C_e\}) \cup \{C''_3, C'_e, F_1, F_2\}$  is a CC of  $G$ , and so (12) holds. ■

Let  $C$  be a cycle of a plane graph  $G$ . Define  $IntC$  to be the vertices of  $G$  inside (exclusively)  $C$ . Define  $ExtC$  similarly. The cycle  $C$  is *trivial* if  $IntC = \emptyset$ ; and is *acyclic* if the underlying simple graph of  $G[IntC]$  is acyclic.

A  $k$ -*face* of a plane graph  $G$  is a face of degree  $k$ . Define  $L(n)$  as the graph in Figure 5.

**Lemma 2.7.** *Let  $G$  be a plane triangulation with  $\mu(G) = 1$  and with  $n = |V(G)| \geq 3$ . If the exterior face of  $G$  is a 2-cycle  $C$ , and if  $C$  is acyclic, then*

$G \cong L(n)$ .

Proof: Let  $v_1, v_2$  be the two vertices in  $V(C)$  and let  $e_1, e_2$  be the two edges in  $E(C)$ . Since  $\mu(G) \leq 1$ , and since  $G$  is a plane triangulation,  $e_1$  must lie in a 3-face  $C_1$  inside  $C$ . Let  $v_3$  be the vertex in  $V(C_1) - \{v_1, v_2\}$ . If  $v_3$  has degree at least 4, then since  $G$  is a triangulation,  $v_3$  and two of its neighbors other than  $v_1, v_2$  would form a 3-cycle inside  $C$ , contrary to the assumption that  $C$  is acyclic. If  $v_3$  has degree 2, then we have  $n = 3$  and  $G = L(3)$ . Hence  $v_3$  has degree 3, and so  $G - v_3$  is also a plane triangulation with  $C$  as an acyclic exterior face. Thus by induction,  $G - v_3 = L(n-1)$  and so  $G = L(n)$ . ■

**Lemma 2.8.** *Let  $G$  be a simple plane triangulation. If the exterior face of  $G$  is a 3-cycle  $C$  and if  $C$  is acyclic, then either  $G$  contains a subgraph  $H \in \{L_6, \Gamma_6\}$  (using the notation in Figure 4) with  $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$  or  $G = \Gamma(n)$ , where  $n = |V(G)|$ .*

Proof: Let  $C = v_1 v_2 v_3 v_1$ . Since  $G$  is a plane triangulation,  $v_2 v_3$  lies in a 3-face  $C_1 = v_2 v_3 v v_2$  with  $v \in \text{Int}C$ . Let  $v_2 = u_1, u_2, \dots, u_m = v_3$  be the neighbors of  $v$  in  $G$  such that they are ordered clockwise by the planar imbedding of  $G$ .

Since  $G$  is a simple plane triangulation,  $vv_2 u_2 v, vu_2 u_3 v, \dots$ , must be 3-faces. Since  $C$  is acyclic, either  $m = 3$ , or  $4 \leq m \leq 5$  and  $u_3 = v_1$ .

If  $m = 5$  and  $u_3 = v_1$ , then  $G$  contains a subgraph  $H = \Gamma_6$  with  $x_1 = v_1, x_2 = v_2, x_3 = v, x_4 = v_3$ . If  $m = 4$  and  $u_3 = v_1$ , then, since  $vv_1 v_3 v$  and  $v_2 vv_3 v_2$  are now 3-faces,  $G = \Gamma(5)$ . Hence we may assume that  $m = 3$ . If  $x_2 = v_1$ , then  $G = \Gamma(4)$ . Thus we assume that  $x_2 \neq v_1$ , and so  $G - v$  is also a plane triangulation with  $C$  as the exterior face. By induction, Lemma 2.8 holds. ■

**Lemma 2.9.** *Suppose that  $G$  is a plane graph, and that  $G$  has a nontrivial 2-cycle  $C$  with  $V(C) = \{v_1, v_2\}$  and  $E(C) = \{e_1, e_2\}$ . Let  $H = G - \text{Ext}C$ .*

- (i) *If  $H = L(3)^+$  such that the extra edge  $e$  is parallel to  $v_1 v_3$ , then letting  $G' = G/v_3 v_2$ , we have  $cc(G) \leq cc(G')$ .*
- (ii) *If  $H = L(4)$ , then letting  $G' = G - \text{Int}C$ , we have  $cc(G) \leq cc(G') + 1$ .*
- (iii) *If  $H = L(4)^+$  such that the extra edge  $e$  is not parallel to any of  $\{e_1, e_2\}$ , then letting  $e'$  be an extra edge parallel to  $e_1$  and let  $G' = G - \text{Int}C + e'$ , we have  $cc(G) \leq cc(G') + 1$ .*
- (iv) *If  $H$  is isomorphic to  $\Gamma(5)^+$ , such that the exterior face of  $H$  is  $C$ , then letting  $G' = G - \text{Int}C$ , we have  $cc(G) \leq cc(G') + 2$ .*

Proof: (i) of Lemma 2.9 is trivial. We now show (ii). Let  $\mathcal{C}$  be a CC of  $G'$  and let  $C_1$  be a cycle in  $\mathcal{C}$  containing  $e_1$ . Define  $C'_1 = C_1 - e_1 + \{v_1 v_3, v_3 v_4, v_4 v_2\}$  and  $F = G[\{e_1, v_1 v_4, v_4 v_3, v_3 v_2\}]$ . Thus  $\mathcal{C} - \{C_1\} \cup \{C'_1, F\}$  is a CC of  $G$  and so (ii) of Lemma 2.9 holds.

Now we show (iii). Let  $\mathcal{C}$  be a CC of  $G'$ . Note that  $[e_1] = \{e_1, e_2, e'\}$  in  $G'$  this time. We may assume that  $e_1$  and  $e'$  are in distinct cycles  $C_1$  and  $C_e$ .



respectively. Define  $C'_1$  and  $F$  as above. Since  $e$  is not parallel to  $e_1$ , there is a  $(v_1, v_2)$ -path  $P$  in  $H - \{e_1, e_2\}$  containing  $e$ . Define  $C'_e = C_e - e' + P$ . Thus  $C - \{C_1, C_e\} \cup \{C'_1, C'_e, F\}$  is a CC of  $G$ .

Now we show (iv). Let  $\mathcal{C}$  be a CC of  $G'$  and let  $C_i$  be a cycle in  $\mathcal{C}$  containing  $e_i$ , ( $1 \leq i \leq 2$ ). Note that no matter where  $e_1, e_2$  lie in  $H$ ,  $H - \{e_1, e_2\}$  always has a spanning cycle and so  $H - \{e_1, e_2\}$  has two internally disjoint  $(v_1, v_2)$ -paths  $P_1$  and  $P_2$ . Let  $C'_i = C_i - e_i + P_i$ , ( $1 \leq i \leq 2$ ). (When  $C_1 = C_2 = C$ , let  $C'_1 = P_1 \cup P_2$ .) Thus it is easy to see that the edges in  $H - E(P_1) \cup E(P_2)$  can be covered by two cycles in  $H$  and so (iv) follows. ■

Define plane graphs  $\Gamma^4, \Gamma^5, \Gamma^6$  as the graphs in Figure 6.

**Lemma 2.10.** *Let  $G$  be a plane triangulation with  $\mu(G) \leq 1$  and with  $4 \leq |V(G)| \leq 5$ . If the exterior face of  $G$  is a 3-face, then  $G$  is isomorphic to one graph in  $\{\Gamma(4), \Gamma(5), \Gamma(4)^+, \Gamma(5)^+, \Gamma^4, \Gamma^5\}$ .*

**Proof:** The proof is straightforward. ■

For each  $i$ , ( $1 \leq i \leq 5$ ), define  $\Gamma'_i$  to be the simple plane triangulation obtained from  $\Gamma_i$  by adding a new vertex  $v_0$  in the exterior face of  $\Gamma_i$  and by joining  $v_0$  to each of  $v_1, v_2, v_3$  with a new edge, respectively.

**Lemma 2.11.** *If  $G$  is isomorphic to one of the graphs below,*

$$\{\Gamma_i, \Gamma'_i, \Gamma_i^+, (\Gamma'_i)^+, (1 \leq i \leq 5), \Gamma(6), \Gamma(6)^+, L(6), L(6)^+, \Gamma_6, \Gamma_6^+\},$$

then

$$cc(G) \leq \frac{2|V(G)| - 3}{3} + \frac{\mu(G)}{2}.$$

**Proof:** The proof is routine and so is omitted. ■

### 3. The Proof of Theorem 1.2

We argue by contradiction and assume that

$$G \text{ is a counterexample to Theorem 1.2} \quad (13)$$

such that

$$|V(G)| + \mu(G) \text{ is as small as possible,} \quad (14)$$

and subject to (14),

$$|E(G)| \text{ is minimized.} \quad (15)$$

If  $G$  has two 2-faces, then we pick two distinct edges,  $e, e'$  (say), from each of these 2-faces. Thus  $\mu(G) = \mu(G - \{e, e'\}) + 2$  and so by (14) and (15),  $G$  is not a counterexample, contrary to (13). Hence we assume that

$$G \text{ has at most one 2-face.} \quad (16)$$

Since  $G$  is a plane triangulation, and by (14), we have

$$\kappa(G) \geq 2 \text{ and } \delta(G) \geq 3. \quad (17)$$

**Lemma 3.1.** *If  $C$  is a nontrivial 2-face of  $G$ , then  $C$  is the exterior face of  $G$ .*

*Proof:* Suppose that  $G$  has a nontrivial 2-cycle  $C$  that is not the exterior face of  $G$ . Suppose also that  $C$  is so chosen that there is no nontrivial 2-cycle properly contained in the interior of  $C$ .

**Case 1:**  $|IntC| = 1$ .

By (16) and (17),  $G[IntC \cup V(C)] = L(3)^+$ . Define  $G'$  as in (i) of Lemma 2.9. If  $|V(G')| \geq 6$ , then by (14) we have

$$cc(G) \leq cc(G') \leq \frac{2(n-1) - 3}{3} + \frac{\mu(G) + 1}{2},$$

and so  $G$  is not a counterexample, a contradiction.

Hence by  $|V(G)| \geq 6$ ,  $|V(G')| = 5$ . It follows by Lemma 2.10 and by (17) that  $G'$  is either spanned by  $\Gamma(5)$  with  $\mu(G') = 3$  or spanned by  $\Gamma^5$  with  $\mu(G') = 4$ . Thus  $cc(G) \leq cc(G') \leq 4$ , by (i) of Lemma 2.9, a contradiction.

**Case 2:**  $|IntC| = 2$ .

By (16),  $G[IntC \cup V(C)] = L(4)$  or  $L(4)^+$ . Define  $G'$  as in (ii) or (iii) of Lemma 2.9. If  $|V(G')| \geq 6$ , then by arguing as above, one can derive a contradiction. Hence we assume that  $|V(G')| \leq 5$ , and so by Lemma 2.10 and by the fact that  $G'$  must have a 2-face,  $G'$  is isomorphic to one of  $\{\Gamma(4)^+, \Gamma(5)^+, \Gamma^{4+}, \Gamma^{5+}\}$ . Since  $cc(\Gamma(4)^+) = cc(\Gamma^{4+}) = 2$  and  $cc(\Gamma(5)^+) = cc(\Gamma^{5+}) = 3$ , and since  $\mu(G') \leq 1$ , it follows by (ii) and (iii) of Lemma 2.9 that  $G$  satisfies (1), contrary to (13).

**Case 3:**  $|IntC| \geq 3$ .

We may similarly assume that  $|ExtC| \geq 3$ . If both  $|IntC| = |ExtC| = 3$ , then  $G[IntC \cup V(C)] \cong \Gamma(5)^+$  and so by Lemma 2.10 and by (iv) of Lemma 2.9,  $cc(G) \leq 3 + 2 = 5$ , contrary to (13). Thus we assume that  $|ExtC| \geq 4$ . Let  $H = G[IntC \cup V(C)]$ . If  $|V(H)| \geq 6$ , then by Lemma 2.1, by (14), and noting that

$$\mu(H) + \mu(G - IntC) = \mu(G) + 1,$$

we have

$$cc(G) \leq cc(G - IntC) + cc(H) - 1 \leq \frac{2n-3}{3} + \frac{\mu(G)}{2}. \quad (18)$$

Hence we assume that  $|V(H)| \leq 5$ . Since  $H$  has an exterior 2-face, it is then easy to see by Lemma 2.9 that  $G$  satisfies (1), contrary to (13). ■

**Lemma 3.2.**  $\mu(G) \leq 1$ .

Proof: Suppose that  $\mu(G) \geq 2$ . Then by (16) and Lemma 3.1,  $G$  must have parallel edges  $e_1, e'_1$  and parallel edges  $e_2, e'_2$  with  $[e_1] \neq [e_2]$  such that, for each  $i$ ,  $G[\{e_i, e'_i\}]$  is a trivial 2-cycle of  $G$ . Note that  $G' = G - \{e'_1, e'_2\}$  is a plane triangulation. By (17) and since  $[e_1] \neq [e_2]$ ,  $G$  has a cycle containing both  $e'_1$  and  $e'_2$ . Note that  $\mu(G') = \mu(G) - 2$ . By (15),

$$cc(G) \leq cc(G') + 1 \leq \frac{2n-3}{3} + \frac{\mu(G) - 2}{2} + 1,$$

contrary to (13). ■

**Lemma 3.3.** *Each of the following subgraphs is forbidden in  $G$ :*

- (i)  $H \in \{L(3)^+, L(4), L(4)^+\}$  with  $A_G(H) \subseteq \{v_1, v_2\}$ .
- (ii)  $H = \Gamma_i$  or  $\Gamma_i^+$  with  $A_G(H) \subseteq \{v_1, v_2, v_3\}$ , ( $1 \leq i \leq 5$ ).
- (iii)  $H \in \{L, L^+\}$  where  $L \in \{\Gamma_6, L_6, L_7'', L_8', L_8'', L_8'''\}$  with  $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$ .
- (iv)  $H = \Gamma(6)$  or  $\Gamma(6)^+$  with  $A_G(H) \subseteq \{v_1, v_2, v_3\}$ .

Proof: Assume that  $H = \Gamma_1$  and let  $G'$  be defined as in Lemma 2.2. If  $|V(G')| = 3$ , then  $G = \Gamma_1$  or  $\Gamma_1^+$  and so (1) holds for  $G$ . If  $|V(G')| \geq 6$ , then by (14) and by Lemma 2.2,

$$cc(G) \leq cc(G') + 1 \leq \frac{2(n-3) - 3}{3} + \frac{\mu(G)}{2} + 1,$$

and so  $G$  is not a counterexample, a contradiction. Thus by  $|V(G)| \geq 6$ , we have  $4 \leq |V(G')| \leq 5$ . Thus by Lemma 2.10, one can easily check that  $G$  is not a counterexample, contrary to (13).

The proofs for the other cases are similar, by using reduction lemmas in section 2. ■

Proof of Theorem 1.2: Since  $G$  is a plane triangulation, the exterior face of  $G$  is either a 2-face or a 3-face. Since  $|V(G)| \geq 6$ ,  $G$  must have a nontrivial 3-cycle. If every nontrivial 3-cycle of  $G$  is acyclic, then in particular, the exterior 3-face or the 3-face obtained by deleting an edge from the exterior 2-face is also acyclic. Thus by Lemma 2.7 and Lemma 2.8,  $G$  must contain either  $\Gamma_6, L_6$  or  $\Gamma(6)$ , contrary to Lemma 3.3. Hence  $G$  has a cyclic 3-cycle. Let  $C_0$  be a cyclic 3-cycle of  $G$  such that

$$|IntC_0| \text{ is minimized.} \tag{19}$$

By (19), any 3-cycle contained in  $G[IntC_0]$  is either trivial or acyclic. By Lemmas 2.8 and 3.3, if  $Z$  is a nontrivial 3-cycle in  $G[IntC_0]$ , then

$$|IntZ| \leq 2. \tag{20}$$

Let  $M = G[\text{Int}C_0 \cup V(C_0)]$ . Let  $C = u_1 u_2 u_3 u_1$  be a trivial 3-cycle in  $M$  such that  $|V(C) \cap V(C_0)| = 0$ . Since  $G$  is a plane triangulation with  $\mu(G) \leq 1$ ,  $M$  has a 3-cycle containing  $u_i u_{i+1}$ , ( $i \equiv 1, 2, 3 \pmod{3}$ ). Let  $C_i$  be a 3-cycle in  $M$  containing  $u_i u_{i+1}$  such that  $E(C) \cap E(G[\text{Int}C_i \cup V(C_i)]) = \{u_i u_{i+1}\}$  and such that

$$|\text{Int}C_i| \text{ is maximized.} \quad (21)$$

Case 1: For  $i \neq j$ ,  $E(C_i) \cap E(C_j) = \emptyset$ .

Let  $C_1 = u_1 u_2 u_4 u_1$ ,  $C_2 = u_2 u_3 u_5 u_2$ , and  $C_3 = u_1 u_3 u_6 u_1$ . Thus  $u_4, u_5, u_6$  are distinct.

Suppose first that  $|\text{Int}C_i| = 0$ , ( $1 \leq i \leq 3$ ). Define

$$G_a = (G - \{u_1 u_6, u_3 u_5, u_2 u_4\}) / E(C), \quad (22)$$

and let  $u$  denote the vertex in  $G_a$  to which  $C$  is contracted. By (21), no new multiple edge will be produced by the contraction, and so

$$\mu(G_a) \leq \mu(G). \quad (23)$$

Since  $G$  is a plane triangulation and since the boundaries of other faces not incident with  $V(C)$  are unchanged,  $G_a$  is also a plane triangulation. We shall show

$$cc(G) \leq cc(G_a) + 1. \quad (24)$$

Let  $\mathcal{C}$  be a CC of  $G_a$ , and let  $C_j \in \mathcal{C}$  such that  $uu_j \in V(C_j)$ , ( $4 \leq j \leq 6$ ). For any cycle  $L$  in  $G_a$ ,  $L$  can be extended to a cycle  $L'$  in  $G$ , by using edges in  $E(C)$ , if necessary.

It is easy to see that we can extend  $C_4, C_5, C_6$  to  $C'_4, C'_5, C'_6$  so that any specified edge in  $E(C)$  can be covered twice by  $C'_4, C'_5, C'_6$ .

In fact, without loss of generality, we may assume that  $|[u_1 u_2]| = 2$ . Define  $L_j = G[E(C_j) - \{uu_4, uu_5, uu_6\}]$ . When  $C_4, C_5, C_6$  are distinct,  $L_j$  is a path in  $G$  joining  $u_j$  to a vertex  $u'_j \in V(C)$ , ( $4 \leq j \leq 6$ ). If  $C_4 = C_5$ , then define  $C'_4, C'_6$  as follows:

if  $u'_6 = u_1$ , then  $C'_4 = L_4 + u_4 u_2 u_1 u_3 u_5$  and  $C'_6 = L_6 + u_6 u_3 u_2 u_1$ ;

if  $u'_6 = u_2$ , then  $C'_4 = L_4 + u_4 u_1 u_2 u_3 u_5$  and  $C'_6 = L_6 + u_6 u_3 u_1 u_2$ ;

if  $u'_6 = u_3$ , then  $C'_4 = L_4 + u_4 u_2 u_1 u_3 u_5$  and  $C'_6 = L_6 + u_6 u_1 u_2 u_3$ .

When  $C_4, C_5, C_6$  are all distinct, define  $C'_4, C'_5, C'_6$  as follows:

if  $u'_5 = u_1$ , then  $C'_5 = L_5 + u_5 u_3 u_2 u_1$ ;

if  $u'_5 = u_2$ , then  $C'_5 = L_5 + u_5 u_3 u_1 u_2$ ;

if  $u'_5 = u_3$ , then  $C'_5 = L_5 + u_5 u_2 u_1 u_3$ ;

if  $u'_6 = u_1$ , then  $C'_6 = L_6 + u_6 u_3 u_2 u_1$ ;

if  $u'_6 = u_2$ , then  $C'_6 = L_6 + u_6 u_1 u_2 u_3$ ;

if  $u'_6 = u_3$ , then  $C'_6 = L_6 + u_6 u_1 u_2 u_3$ ;

and choose  $C'_4$  so that the remaining edge in  $E(C)$ , if there is any, is covered by  $C'_4$ .

Let  $F_1 = u_1 u_4 u_2 u_5 u_3 u_6 u_1$ . Then  $\{L' : L \in C\} \cup \{F\}$  is a CC of  $G$  and so (23) holds.

If  $|V(G_a)| \geq 6$ , then by (14), (22) and (23),

$$cc(G) \leq cc(G_a) + 1 \leq \frac{2(n-2) - 3}{3} + \frac{\mu(G)}{2} + 1,$$

contrary to (13).

If  $|V(G_a)| \leq 5$ , then since  $|V(C) \cap V(C_0)| = 0$  and since  $u$  is a vertex of degree at least 3 in  $G_a$ , it follows by Lemma 2.10 that  $G \in \{\Gamma_1, \Gamma_1^+, \Gamma_1', (\Gamma_1')^+\}$  and so by Lemma 2.11,  $G$  is not a counterexample, either.

By (20), we need to consider the cases when exactly  $k$  of the  $C_i$ 's are nontrivial, where  $1 \leq k \leq 3$ . The proofs for these subcases are similar to that when  $k = 0$  and so are omitted.

Case 2: For some  $i \neq j$ ,  $E(C_i) \cap E(C_j) \neq \emptyset$ .

If  $E(C_i) \cap E(C_j) \neq \emptyset$ , for every  $i \neq j$ , then  $u_4 = u_5 = u_6$ , contrary to Lemma 3.1 or to the assumption that  $n \geq 6$ . Hence we assume that

$$E(C_3) \cap (E(C_1) \cup E(C_2)) = \emptyset \text{ and } u_4 = u_5. \quad (25)$$

(2A)  $IntC_1 \neq \emptyset$  and  $IntC_2 \neq \emptyset$  or  $|IntC_3| \geq 1$  and  $IntC_1 = IntC_2 = \emptyset$ . Then  $G[IntC_1 \cup IntC_2 \cup \{u_1, u_2, u_3, u_4\}]$  contains a subgraph isomorphic to one of  $\{\Gamma_6, \Gamma_6^+, L_7'', L_7''^+\}$ , contrary to Lemma 3.3. Thus we assume that

$$IntC_1 = \emptyset. \quad (26)$$

(2B)  $|IntC_1| = 0$  and  $|IntC_2| > 0$ . Then  $G$  has a forbidden subgraph  $H$  isomorphic to one of  $\{L_6', L_6''\}$ . This case can be excluded by applying (v) of Lemma 2.6.

(2C)  $IntC_1 = IntC_2 = IntC_3 = \emptyset$  and  $u_4 \in IntC_0$ .

Since  $G$  is a triangulation, there are  $u_7$  and  $u_8$  in  $V(G)$ , such that  $C_4 = u_1 u_7 u_4 u_1$  and  $C_5 = u_3 u_8 u_4 u_3$  are 3-cycles satisfying (21). Applying the previous argument to the 3-cycles  $C_4$  and  $C_5$ , we conclude that  $IntC_4 = IntC_5 = \emptyset$ . Let  $H = G[\{u_1, u_2, u_3, u_4\}]$  and let  $G_b = (G - \{u_1 u_6, u_7 u_4, u_8 u_3\})/E(H)$ . Imitating the proof for (24), we can similarly show first that

$$\mu(G_b) \leq \mu(G) \text{ and } cc(G) \leq cc(G_b) + 2, \quad (27)$$

and then that  $G$  is not a counterexample, contrary to (13).

(2D)  $IntC_1 = IntC_2 = IntC_3 = \emptyset$  and  $u_4 \notin IntC_0$ .

Thus  $u_4 \in V(C_0)$ . It follows from Case 1 and Cases (2A) - (2C) that for any trivial 3-cycle  $C' = z_1 z_2 z_3 z_1$  in  $G[Int C_0]$ , there must be  $z_4, z_5 \in V(C_0)$  such that  $z_4 z_2 z_1 z_4$ ,  $z_4 z_2 z_3 z_4$  and  $z_1 z_3 z_5 z_1$  are trivial 3-faces in  $M$ , with  $z_i = u_i$  ( $1 \leq i \leq 4$ ) and  $z_5 = u_6$ . Note that  $C'' = z_1 z_5 z_4 z_1$  must be a trivial 3-face since otherwise  $G$  contains a  $L_6$ , contrary to Lemma 3.3. Call  $G[\{z_1, z_2, z_3, z_4, z_5\}]$  an associated  $\Gamma(5)$  with edge  $z_4 z_5 \in E(C_0)$ . For each edge in  $E(C_0)$ , there is at most one associated  $\Gamma(5)$  with the given edge. Delete  $z_1, z_2$  from the associated  $\Gamma(5)$  with  $z_4 z_5$ , and do the same for other associated  $\Gamma(5)$ 's with other edges in  $E(C_0)$ , (if there are any). Then the resulting graph is again a triangulation in which  $C_0$  is an acyclic 3-cycle, and so by Lemmas 7 and 8, either  $M$  contains a trivial 3-cycle that satisfies Case 1 or one of Cases (2A) - (2C), or  $G$  contains  $L_6$  or  $\Gamma(6)$ , or  $M - E(C_0)$  is isomorphic to the graph  $L_{11}$  in Figure 8.

Thus we may assume that  $M - E(C_0) \cong L_{11}$ . Let  $G_c = G - \{z_8, z_9, z_{10}\}$ . Then it is easy to see that

$$cc(G) \leq cc(G_c) + 2. \quad (28)$$

Thus by (14) and since  $|V(G_c)| \geq 7$ ,  $G$  must satisfy (1), contrary to (13).

Since every case leads to a contradiction, Theorem 1.2 is proved. ■

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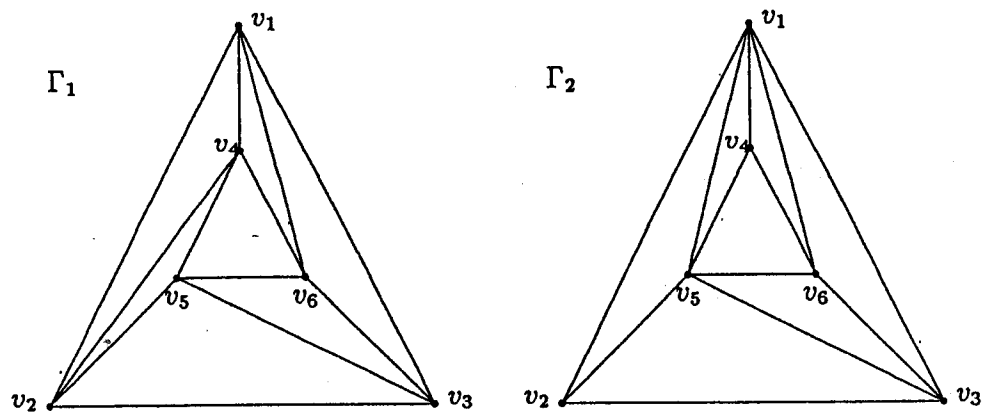


Figure 1 : The graphs  $\Gamma_1$  and  $\Gamma_2$

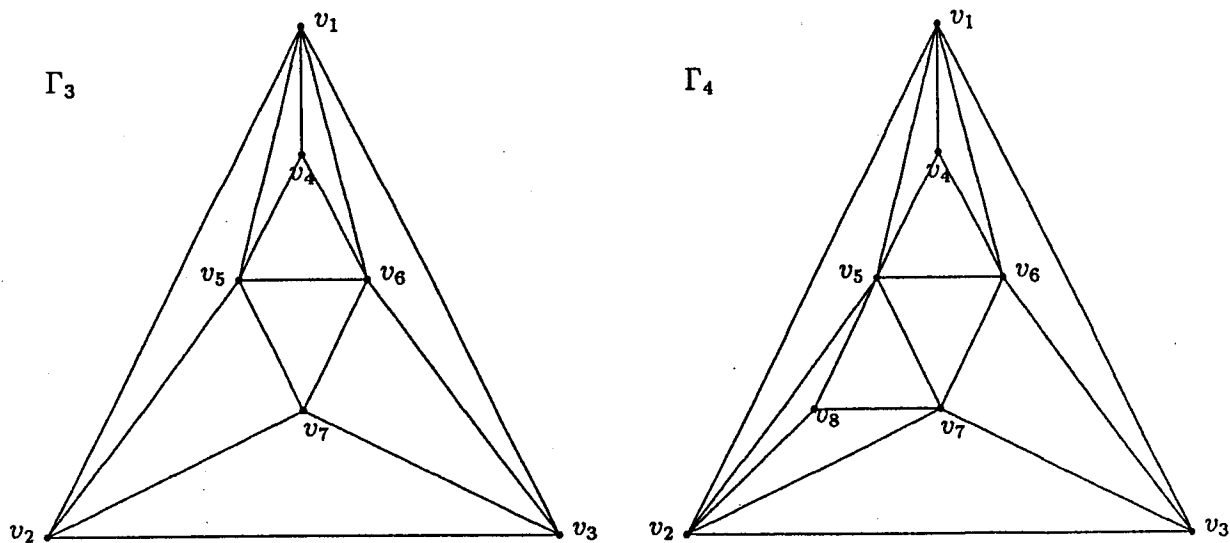


Figure 2 : The graphs  $\Gamma_3$  and  $\Gamma_4$

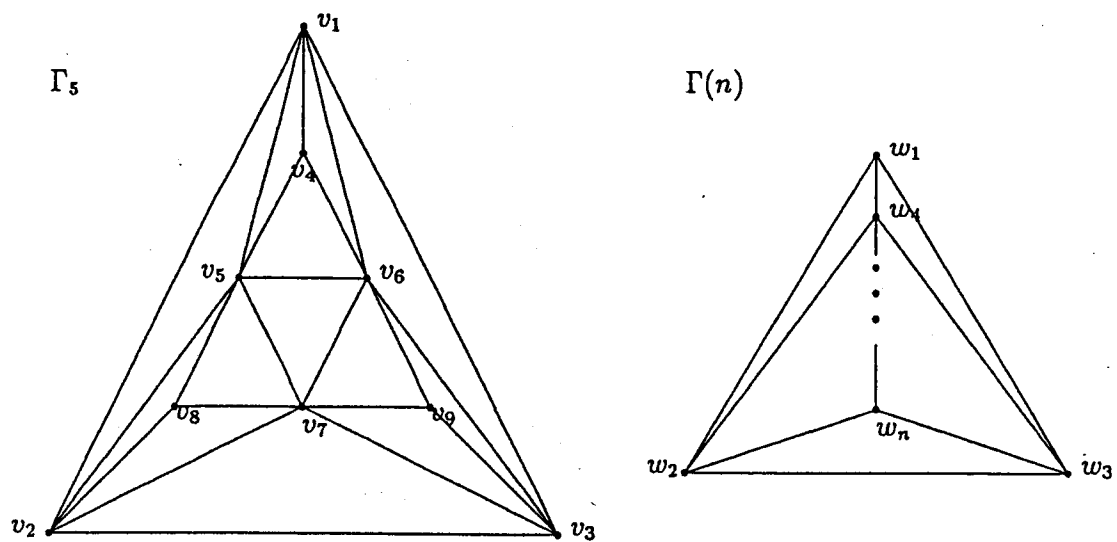


Figure 3 : The graphs  $\Gamma_5$  and  $\Gamma(n)$

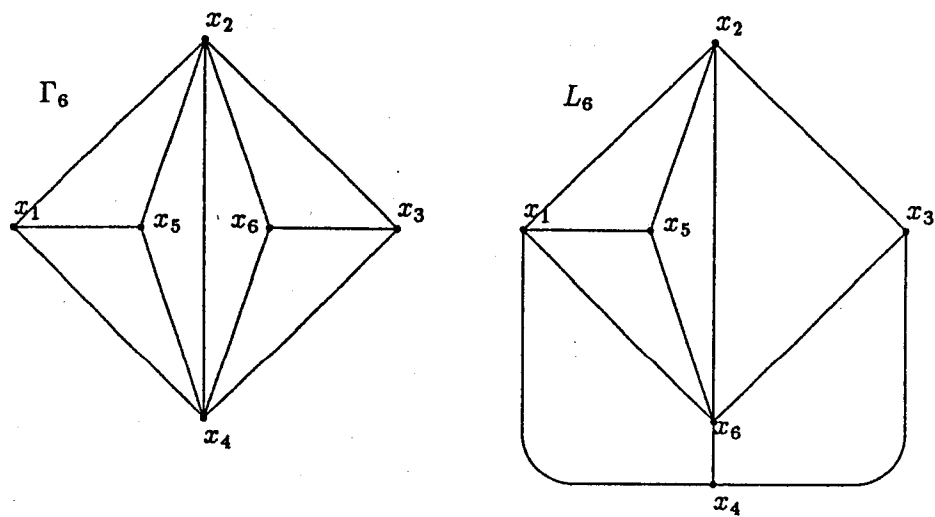
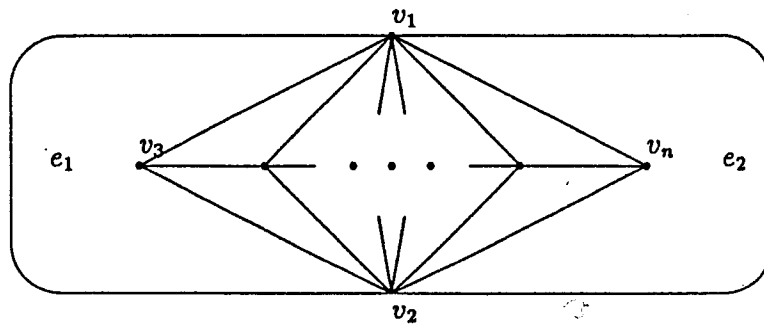


Figure 4 : The graphs  $\Gamma_6$  and  $L_6$





$L(n)$

Figure 5: The graph  $L(n)$

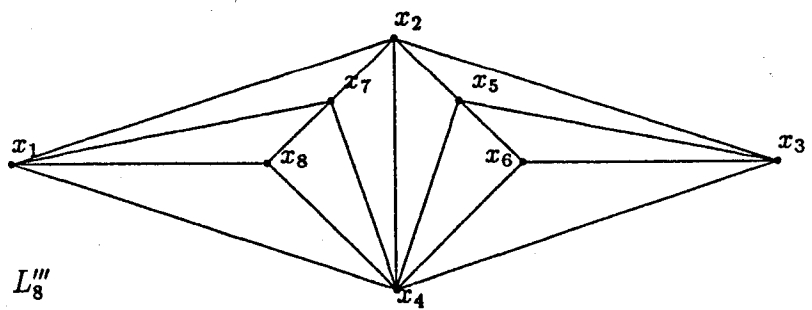
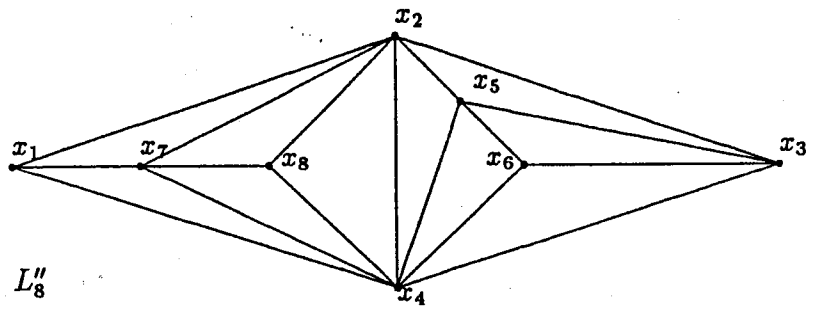
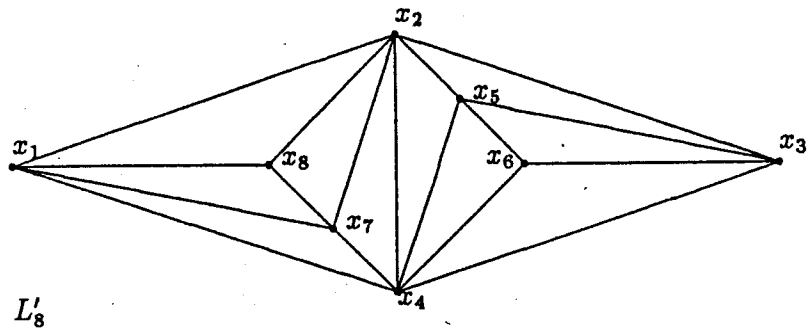


Figure 6 : Graphs  $L'_8$ ,  $L''_8$  and  $L'''_8$

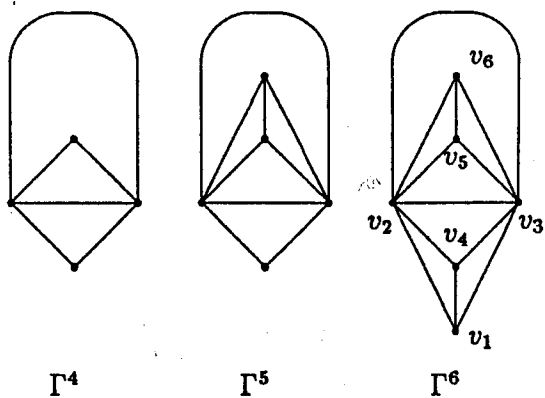


Figure 7 : Graphs  $\Gamma^4, \Gamma^5, \Gamma^6$

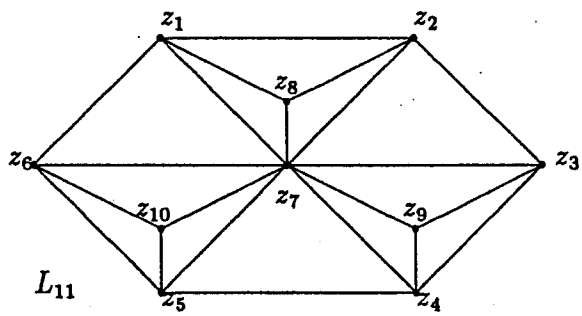


Figure 8 : Graph  $L_{11}$