Eulerian subgraphs in a class of graphs

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Abstract. Let G be a graph and let $D_1(G)$ denote the set of vertices of degree one in G. In [1], Behocine, Clark, Köhler and Veldman conjectured that for a connected simple graph G of n vertices, if $G - D_1(G)$ is 2-edged-connected, and if for any edge $xy \in E(G), d(x) + d(y) > \frac{2n}{5} - 2$, then $L(G)$ is hamiltonian.

In this note, we shall show that the conjecture above holds for a class of graphs that includes the $K_{1,3}$-free graphs, and we shall also characterize the extremal graphs.

I Introduction

We shall use the notation of Bondy and Murty [2] except for contraction and edge graphs. We assume that graphs have no loops, but multiple edges are allowed. Let G be a graph. We shall speak of the *line graph* of G, denoted by $L(G)$, instead of the edge graph of G. An *eulerian subgraph* H of G is a connected subgraph of G, each of whose vertices has even degree in H. Thus the graph K_1 is regarded as being an eulerian graph. Let K be a graph. A graph G is said to be *K-free* if G does not have induced subgraphs isomorphic to K. Let $\mathbf{N}$ denote the set of positive integers. For $n \in \mathbf{N}$, the n-cycle is denoted by C_n.

For a graph G, we denote

$$D_1(G) = \{v \in V(G) : deg_G(v) = 1\}.$$

For any graph G and any edge $e \in E(G)$, we denote by G/e the graph obtained from G by contracting e and by deleting any resulting loops. If H is a connected subgraph of G, then G/H denotes the graph obtained by contracting all edges of H and by deleting any resulting loops.

A family of graphs will be called a *family*. A family S is said to be *closed under contraction* if

$$G \in S, e \in E(G) \Rightarrow G/e \in S.$$

For any graph family S closed under contraction, define the *kernal* of S to be

$$S^0 = \{H : \text{For all supergraphs } G \text{ of } H, G \in S \Leftrightarrow G/H \in S\}. \qquad (1)$$

Following Catlin [4] , we let $\mathcal{SL}$ denote the family of all supereulerian graphs, that is, graphs with a spanning eulerian subgraph. Catlin proves ([3],[4])

$$K_2 \notin \mathcal{SL}^0, K_3 \in \mathcal{SL}^0. \qquad (2)$$

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If a graph G contains no nontrivial subgraphs in $S\mathcal{L}^0$, then G is called a *reduced* graph.

For a graph G, let $E'(G)$ be defined as follows:

$$E'(G) = \{e \in E(G) : e \text{ is in no subgraph } H \text{ of } G \text{ with } H \in S\mathcal{L}^0\}. \quad (3)$$

Catlin showed ([3],[4]) that each component of $G - E'(G)$ is a maximal subgraph of G that is in $S\mathcal{L}^0$. Let G' denote the graph obtained from G by contracting each edge of $E(G) - E'(G)$. Then G' is called the *reduction* of G. Note that each vertex of G' is the image of some maximal subgraph G that is in $S\mathcal{L}^0$ under the contraction. A vertex of G' is called *nontrivial* if it is the image of a nontrivial subgraph of G. A vertex is a *trivial* vertex of G' if it is not nontrivial. Catlin proved:

Theorem A. *(Catlin [3],[4]) Let G be a graph. Then*

(i) *G' is unique;*
(ii) *$G \in S\mathcal{L} \Leftrightarrow G' \in S\mathcal{L}$;*
(iii) *If H is a subgraph of G and $H \in S\mathcal{L}^0$, then $G \in S\mathcal{L} \Leftrightarrow G/H \in S\mathcal{L}$;*
(iv) *G' is reduced;*
(v) *If G is reduced, then G is simple and K_3-free;*
(vi) *If G has 2 edge-disjoint spanning trees, then $G \in S\mathcal{L}^0$.* ∎

Theorem A is a special case of a more general reduction method of Catlin [4]. We shall also need the following theorem.

Theorem B. *(Harary and Nash-Williams [6]) Let G be a connected graph with at least three edges. Then $L(G)$ is hamiltonian if and only if G has an eulerian subgraph Γ such that every edge of G has at least one end in $V(\Gamma)$.* ∎

There are several prior results on eulerian subgraphs in $K_{1,3}$-free graphs.

Theorem C. *(Paulraja [7]) Let G be a connected graph having no induced $K_{1,3}$ as a subgraph. If each edge of G is in a cycle of length at most 5, then $G \in S\mathcal{L}$.* ∎

The following theorem generalizes Theorem C.

Theorem D. *(Catlin and Lai [5]) Let G be a connected graph containing no induced $K_{1,3}$ as a subgraph, and define $E'(G)$ by (3). If each edge of $E'(G)$ is contained in a cycle of G of length at most 5, then exactly one of the following holds:*

(a) *$G \in S\mathcal{L}^0$;*
(b) *$G \in \{C_4, C_5\}$;*
(c) *G has a nontivial subgraph $H \in S\mathcal{L}^0$ such that G/H is the union of 4-cycles whose only common vertex is v_H, the vertex of G/H corresponding to H.* ∎

By (ii) and (iii) of Theorem A, it is easy to see that Theorem C follows from Theorem D.

Theorem E. *(Catlin and Lai [5]) Let G be a connected graph containing no induced $K_{1,3}$ as a subgraph, and define $E'(G)$ by (3). If each edge of $E'(G)$ is in a cycle of G of length at most 7, then $L(G)$ is hamiltonian.* ∎

Theorem F. *(Catlin and Lai [5]) Let G be a connected graph containing no induced $K_{1,3}$ as a subgraph, and define $E'(G)$ by (3). If each edge of $E'(G)$ is in a cycle of length at most 7 and $\delta(G) \geq 3$, then $G \in \mathcal{SL}$.* ∎

Theorem D, Theorem E and Theorem F are all best possible.

Let G be a graph and let G' be the reduction of G. Throughout this note we shall use $d(v)$ and $d'(v)$ to denote the degree of a vertex v in G and in G', respectively.

In [1], Benhocine, Clark, Köhler and Veldman conjectured that if G is a simple graph of n vertices, n large, and if

(i) $G - D_1(G)$ is 2-edge-connected,

(ii) for any edge $xy \in E(G)$,

$$d(x) + d(y) > \frac{2n}{5} - 2, \tag{4}$$

then $L(G)$ is hamiltonian.

In this note, we shall show that the conjecture above holds for a class of graphs that includes $K_{1,3}$-free graphs and we shall also characterize the extremal graphs.

II Main results

If G has no induced $K_{1,3}$ subgraphs, then for any vertex $v \in V(G)$, with $d(v) \geq 3$, all but at most one of the edges incident with v lie in triangles of G. Note that by (2), $K_3 \in \mathcal{PL}^0$. We thus let $\mathcal{L}$ be the collection of graphs having the following property:

> For every $v \in V(G)$ with $d(v) \geq 3$, there is a subgraph of G in $\mathcal{SL}^0$ that contains all but at most one of the edges incident with v.

Clearly $G \in \mathcal{L}$ if G has no induced $K_{1,3}$ subgraphs. The complete bipartite graph $K_{n,m}(n > 2, m > 2)$, containing induced $K_{1,3}$ subgraphs, are all in $\mathcal{L}$.

Let $c = v_1 v_2 v_3 v_4 v_1$ be a 4-cycle. Let X and Y be disjoint sets of vertices such that $X \cup Y \neq \phi$ and $(X \cup Y) \cap V(C) = \phi$. Define a graph $(C; X, Y)$ to be the graph with vertex set $V(C) \cup X \cup Y$ and edge set $E(C) \cup \{v_2 x : x \in X\} \cup \{v_4 y : y \in Y\}$.

Theorem 1. *Let $G \in \mathcal{L}$ be a connected simple graph of order $n \geq 46$ and let G' denote the reduction of G. If for any edge $xy \in E(G)$,*

$$d(x) + d(y) \geq \frac{2n}{5} - 2, \tag{5}$$

then exactly one of the following holds:

(a) *G has an edge e such that each component of $G - e$ has an edge.*
(b) *$G' \in \{K_1, C_4, C_5\}$.*
(c) *$G' = K_{1,m}$, for some $m \in \mathbf{N}$ such that if $m > 1$, then all the vertices of degree one are trivial, and such that if $m = 1$, then exactly one vertex of G' is trivial.*
(d) *$G' = (C; X, Y)$ such that all the vertices in $X \cup Y$ are trivial.*
(e) *$G' - D_1(G') = K_{2,m}$, for some $m \geq 3$, such that at least one of the divalent vertices of $G' - D_1(G')$ is a trivial vertex of G', and such that if $D_1(G') \neq \phi$, then every $v \in D_1(G')$ is incident with a vertex of degree m in G'.*
(f) *$G' = K_{2,3}$ and $n = 5s$, for some integer $s \geq 10$, such that the preimage of each vertex of G' is a K_s or a $K_s - e$, for some $e \in E(K_s)$.*

Proof: It is easy to check that the conclusions of Theorem 1 are mutually exclusive.

Let G satisfy the hypothesis of Theorem 1 and let G' be the reduction of G. Recall that we obtain G' from G by contracting all maximal subgraphs of G in $S\mathcal{L}^0$. Since vertices in $D_1(G)$ are maximal subgraphs of G in $S\mathcal{L}^0$, we can regard

$$D_1(G) = D_1(G'). \tag{6}$$

Suppose that (a) of Theorem 1 fails. Then for any cut-edge e of G, one of the components of $G - e$ is a K_1, namely, a vertex in $D_1(G)$. Thus we may assume that

$$\kappa'(G - D_1(G)) \geq 2,$$

since otherwise (a) of Theorem 1 holds.

Now $G - D_1(G)$ is 2-edge-connected, and so $G' - D_1(G')$ is either a K_1 or 2-edge-connected. If $G' - D_1(G') = K_1$, then (b) or (c) of Theorem 1 holds. By the assumption that $n \geq 46$, if $G' = K_2$, then exactly one vertex of G' is trivial. Thus we assume that

$$G' - D_1(G') \text{ is 2-edged-connected.} \tag{7}$$

Let $H_1, H_2, \ldots, H_c$ be all the maximal subgraphs of G in $S\mathcal{L}^0$. We shall use v_i, $1 \leq i \leq c$, to denote the vertex in G' onto which H_i is contracted.

From the way we obtain G', $E(G') = E'(G)$ can be regarded as a subset of $E(G)$, where $E'(G)$ is defined by (3). By the facts that $G \in \mathcal{L}$ and G' is reduced, we conclude that, for $1 \leq i \leq c$,

$$\text{each vertex in } V(H_i)\text{, is incident with at most one edge of } E(G'), \text{ unless } v_i \text{ is trivial and the degree of } v_i \text{ in } G' \text{ is 2.} \tag{8}$$

Suppose for some i, $|V(H_i)| > 1$. Since H_i is in SL^0, H_i contains an edge $xy \in E(H_i)$. By (8), we get

$$|N_G(x) \cup N_g(y) - V(H_i)| \leq 2. \tag{9}$$

By (5) and (9), if $|V(H_i)| > 1$, then

$$|V(H_i)| \geq \max\{d(x) - 1, d(y) - 1\} + 1 \geq \frac{1}{2}(\frac{2n}{5} - 2) = \frac{n}{5} - 1. \tag{10}$$

If $v_i v_j \in E(G')$ with $|V(H_i)| = 1$, then by (5) and (8), we get

$$|V(H_j)| \geq \frac{2n}{5} - 2 - d(v_i), \tag{11}$$

where we also use v_i to denote the unique vertex in $V(H_i)$.

We are now in a position to begin our proof. We divide the proof into several cases.

Case 1 $D_1(G) = \phi$ and $|V(H_i)| > 1$, for all $i \in \{1, 2, \ldots, c\}$.

Suppose $c \geq 6$. By (10),

$$n \geq 6(\frac{n}{5} - 1) = \frac{6n}{5} - 6.$$

It follows that $n \leq 30$, contrary to the hypothesis that $n \geq 46$. Hence $c < 6$.

Since G' is 2-edge-connected and has no triangles, G' has a cycle of length at least four. Thus G' must be either C_4, the 4-cycle, or C_5, the 5-cycle, or $K_{2,3}$.

If $G' \in \{C_4, C_5\}$, then (b) of Theorem 1 holds. Hence we suppose that $G' = K_{2,3}$.

Without loss of generality, we may assume that

$$|V(H_5)| \geq |V(H_4)| \geq |V(H_3)| \geq |V(H_2)| \geq |V(H_1)|. \tag{12}$$

Since $G' = K_{2,3}$, we have $d'(v_1) \leq 3$. Since $n \geq 46$, by (10), $|V(H_1)| \geq 9$. Thus we can find an edge $xy \in E(H_1)$ such that x is incident with no edges of $E(G')$.

If $d(x) \leq 3$, then by (5),

$$3 + d(y) \geq d(x) + d(y) \geq \frac{2n}{5} - 2.$$

It follows that

$$d(y) \geq \frac{2n}{5} - 5, \tag{13}$$

and so by (8),

$$|V(H_1)| \geq d(y) \geq \frac{2n}{5} - 5. \tag{14}$$

This, together with (12), implies

$$n \geq 5(\frac{2n}{5} - 5) = 2n - 25.$$

Thus $n \leq 25$. By assumption, $n \geq 46$, a contradiction.

Hence we must have $d(x) \geq 4$. Since $d'(v_1) \leq 3$, we can find an edge $xz \in E(H_1)$ such that neither x nor z is incident with any edge in $E(G')$.

By (5),

$$|V(H_1)| \geq \max\{d(x), d(z)\} + 1 \geq \frac{n}{5}.$$

By (12), we must have

$$|V(H_5)| \geq |V(H_4)| \geq |V(H_3)| \geq |V(H_2)| \geq |V(H_1)| \geq \frac{n}{5}. \tag{15}$$

Thus equalities hold everywhere in (15) and so $n = 5s$ for some $s \in \mathbf{N}$.

If, for some $i \in \{1,2,3,4,5\}$, $H_i \neq K_s$, then by (5), $H_i = K_s - e$, where each of the two ends of e is incident with an edge in E'. Hence (f) of Theorem 1 holds.

Case 2 $D_1(G) = \phi$ and $|V(H_i)| = 1$ for some $i \in \{1,2,\ldots,c\}$.

We may assume that $|V(H_1)| = 1$. By $D_1(G) = \phi$, by (6), (7) and (8), v_1 is adjacent to exactly two vertices in G', say v_2 and v_3. By (11),

$$|V(H_i)| \geq \frac{2n}{5} - 4, i = 2,3. \tag{16}$$

Note that (16) can be interpreted as follows:

If $v_i \in V(G')$ with $|V(H_i)| < \frac{2n}{5} - 4$, then v_i cannot be adjacent to some $v_j \in V(G')$ with $|V(H_j)| = 1$. (17)

By (10) and (16),

G' has at most one vertex other than v_2, v_3 with nontrivial preimages. (18)

For if $|V(H_i)| > 1, i \in \{4, 5\}$, then by (10) and (16),

$$n \geq \sum_{i=1}^{5} |V(H_i)| \geq 1 + 2(\frac{2n}{5} - 4) + 2(\frac{n}{5} - 1) = \frac{6n}{5} - 9.$$

It follows that $n \leq 45$, contrary to the assumption that $n \geq 46$. This contradiction yields (18).

By (18), we let $v_4 \in V(G') - \{v_1, v_2, v_3\}$ be the possible nontrivial vertex of G'.

Subcase 2.1 $|V(H_4)| \geq \frac{2n}{5} - 4$.

Then since $n \geq 46$, $c \leq 5$. For if $c > 5$, then

$$n \geq \sum_{i=1}^{6} |V(H_i)| \geq 1 + 3(\frac{2n}{5} - 4) + 2 = \frac{6n}{5} - 9.$$

It follows that $n \leq 45$, a contradiction.

With a similar argument, we can see that

$$|V(H_i)| = 1, i \notin \{2, 3, 4\}. \tag{19}$$

By $D_1(G) = \phi$ and (7),

every v_i is in a cycle. (20)

By (8), (17), (19) and by the fact that $G \in \mathcal{L}$,

each v_i, $i \notin \{2, 3, 4\}$, must be adjacent to exactly two vertices of $\{v_2, v_3, v_4\}$. (21)

By (20), (21), by the fact that $c \leq 5$ and by (iv) and (v) of Theorem A, $G' \in \{C_4, C_5, K_{2,3}\}$. Recall that v_1 is of degree 2 in G' and is a trivial vertex of G'. Hence (b) or (e) of Theorem 1 holds.

Subcase 2.2 Each $v_1 \in V(G')$, $i \notin \{2, 3\}$, satisfies

$$1 \leq |V(H_i)| < \frac{2n}{5} - 4. \tag{22}$$

By (18), G' has at most one nontrivial vertex other than v_2 and v_3. Thus by (17) and (22), $G' = K_{2,m}$ for some $m > 1$.

If $m = 2$, then (b) of Theorem 1 holds. If m>2, (e) of Theorem 1 holds since v_1 is a trivial vertex.

Case 3 $D_1(G') \neq \phi$.

Note that (17) is still valid.

By (6), $D_1(G) = D_1(G')$. We may assume that $v_1 \in D_1(G')$ and so $|V(H_1)| = 1$. Let $v_2 \in V(G')$ with $v_1 v_2 \in E(G')$. Then by (11),

$$|V(H_2)| \geq \frac{2n}{5} - 3. \tag{24}$$

Claim 1 $V(G')$ has at most 4 nontrivial vertices.

Suppose, to the contrary, that there are $v_i \in V(G') - D_1(G')$ with $|V(H_i)| > 1, i = 3, 4, 5, 6$. By (10) and (24),

$$n \geq \sum_{i=1}^{6} |V(H_i)| \geq 1 + \frac{2n}{5} - 3 + 4(\frac{n}{5} - 1) = \frac{6n}{5} - 6. \tag{25}$$

It folows that $n \leq 30$, a contradiction. Hence the claim.

Claim 2 G' has at most two nontrivial vertices with preimages of order at least $\frac{2n}{5} - 4$.

Suppose, by contradiction, that for $i \in \{3, 4\}$,

$$|V(H_i)| \geq \frac{2n}{5} - 4. \tag{26}$$

Then by (24) and (26), we have

$$n \geq \sum_{i=1}^{4} |V(H_i)| \geq 1 + (\frac{2n}{5} - 3) + 2(\frac{2n}{5} - 4) = \frac{6n}{5} - 10, \tag{27}$$

It follows that $46 \leq n \leq 50$. Let $k \in \mathbf{N}$ be such that $1 \leq k \leq 5$ and $n = 45 + k$. Then the right-hand-side of (27) is equal to $44 + k + \frac{k}{5}$. Since $|\bigcup_{i=1}^{4} V(H_i)|$ is an integer, by (27), $V(G) = \bigcup_{i=1}^{4} V(H_i)$, and so $G' - d_1(G')$ has exactly three vertices. By (7), $G' - D_1(G')$ must be nontrivial and collapsible, contrary to (iv) of Theorem A. Hence Claim 2.

If v_2 is the only nontrivial vertex satisfying (26) in G', then by (iv) and (v) of Theorem A, by (17) and by Claim 1, $G' - D_1(G')$ is a 4-cycle and all the trivial vertices are adjacent to v_2. Thus by (vi) and (v) of Theorem A, (d) of Theorem 1 holds with $Y \neq \phi$.

If G' has exactly two nontrivial vertices satisfying (26), say v_2 and v_4, then G' has at most three nontrivial vertices. For otherwise, we may assume that v_3 and v_5 are also nontivial, and so by (10) and (24),

$$n \geq \sum_{i=1}^{5} |V(H_i)| \geq 1 + (\frac{2n}{5} - 3) + (\frac{2n}{5} - 4) + 2(\frac{n}{5} - 1) = \frac{6n}{5} - 8.$$

It follows that $n \leq 40$, contrary to the assumption that $n \geq 46$.

Hence by Claim 2 and (17), every vertex of $G' - D_1(G')$ is adjacent to both v_2 and v_4, and so by (v) of Theorem A, $G' - D_1(G') = K_{2,m}$, for some $m \geq 2$. Thus either (d) (when $m = 2$) or (e) (when $m > 2$) of Theorem 1 holds. ∎

The bound $n \geq 46$ is best possible. Let G_1, G_2 and G_3 be three copies of K_{14} and let v_1, v_2 and v_3 be three vertices disjoint from $V(G_1) \cup V(G_2) \cup V(G_3)$. For $i \in \{1,2,3\}$, let $x_i, y_i \in V(G_i)$ be two distinct vertices. Obtain a graph G such that

$$V(G) = V(G_1) \cup V(G_2) \cup V(G_3) \cup \{v_1, v_2, v_3\}$$

and

$$E(G) = E(G_1) \cup E(G_2) \cup E(G_3) \cup \{x_1 v_1, v_1 y_2, x_2 v_2, v_2 y_3, x_3 v_3, v_3 y_1\}.$$

Then $|V(G)| = 45$ and for any edge $xy \in E(G)$

$$d(x) + d(y) \geq 16 = \frac{2}{5}|V(G)| - 2.$$

But G does not satisfy any conclusion of Theorem 1.

Corollary 2. *Let $G \in \mathcal{L}$ be a connected simple graph of $n \geq 46$ vertices and let G' denote the reduction of G. If for any edge $xy \in E(G)$,*

$$d(x) + d(y) \geq \frac{2n}{5} - 2, \tag{5}$$

then exactly one of the following holds:

(a) *G has a cut-edge e such that each component of $G - e$ has an edge.*
(b) *$L(G)$ is hamiltonian.*
(c) *$G' = K_{2,3}$ and $n = 5s$, for some integer $s \geq 10$, such that the preimage of each vertex of G' is a K_s or a $K_s - e$, for some $e \in E(K_s)$.*

Proof: Clearly (a) of Theorem 1 implies (a) of Corollary 2, and (f) of Theorem 1 implies (c) of Corollary 2. By (ii) of Theorem A, each of (b), (c), (d), and (e) of Theorem 1 implies that G has an eulerian subgraph Γ such that every edge of G has at least one end in $V(\Gamma)$. Hence by Theorem B, (b) of Corollary 2 follows from Theorem 1. ∎

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