

Unique Graph Homomorphisms onto Odd Cycles, II

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A natural generalization of graph colorings is graph homomorphisms. Let G and H be simple graphs. A map $\theta: V(G) \rightarrow V(H)$ is called a *homomorphism* if θ preserves adjacency. The set of all homomorphism from G to H is denoted by $\text{Hom}(G, H)$. A graph G is *uniquely H -colorable* if $\text{Hom}(G, H) \neq \emptyset$, and if for $\theta_1, \theta_2 \in \text{Hom}(G, H)$, there is an automorphism φ of H such that $\varphi\theta_1 = \theta_2$. In this paper, we investigate some necessary conditions of unique C^{2k+1} -colorings and prove a best possible sufficient condition involving $\delta(G)$ for G to be uniquely C^{2k+1} -colorable under some necessary conditions. This generalizes a result of Bollobás on unique C^3 -colorings [*J. Combin. Theory Ser. B* **25** (1978), 55–61]. We also find best possible conditions on the connectedness of the subgraphs of G induced by the preimages of θ , for any $\theta \in \text{Hom}(G, C^{2k+1})$. © 1989 Academic Press, Inc.

We shall use the notation of Bollobás [1]. Let G be a graph. We use $\delta(G)$ to denote the minimum degree of G . For simple graphs G and H , a map $\theta: V(G) \rightarrow V(H)$ is called a *homomorphism* if θ preserves adjacency. The set of all homomorphisms from G to H is denoted by $\text{Hom}(G, H)$. If $\varphi \in \text{Hom}(G, G)$ is bijective, then φ is called an *automorphism* of G . Cycles of length m are denoted by C^m and occasionally by \mathbb{Z}_m , the set of integers modulo m , where i and j are adjacent if and only if $i - j \equiv \pm 1 \pmod{m}$.

A graph G is *H -colorable* if $\text{Hom}(G, H) \neq \emptyset$, and G is said to be *uniquely H -colorable* if $\text{Hom}(G, H) \neq \emptyset$ and if $\forall \theta_1, \theta_2 \in \text{Hom}(G, H)$, there exists an automorphism φ of H such that $\varphi\theta_1 = \theta_2$.

Graph homomorphisms are regarded as a generalization of graph colorings. If a graph G is K^k -colorable, then G is k -colorable in the usual meaning. For homomorphisms into odd cycles, see [4] and [5].

In 1978, Bollobás proved two theorems on unique graph colorings.

THEOREM 1 (Bollobás [2]). *If G is a graph of order n , $\text{Hom}(G, K^k) \neq \emptyset$ and $\delta(G) > (3k - 5)/(3k - 2)$, then G is uniquely K^k -colorable.*

It is shown in [3] that if $k = \chi(G)$ and if G is uniquely K^k -colorable, then for any $\theta \in \text{Hom}(G, K^k)$, the following condition holds:

$$G[\theta^{-1}(i) \cup \theta^{-1}(j)] \text{ is connected, for any } ij \in E(K^k). \quad (1)$$

For graphs satisfying (1) for some $\theta \in \text{Hom}(G, K^k)$, the bound in Theorem 1 can be improved.

THEOREM 2 (Bollobás [2]). *Let G be a graph of order n such that $\exists \theta \in \text{Hom}(G, K^k)$ satisfying (1). If*

$$\delta(G) > n \frac{k-2}{k-1},$$

then G is uniquely K^k -colorable.

Bollobás gave two classes of graphs to show that both bounds are best possible.

In a recent paper [6], we proved

THEOREM 3. *Let G be a graph of order n . If, for $k > 1$,*

- (i) $\delta(G) \geq n/(k+1)$, and
- (ii) $\exists \theta \in \text{Hom}(G, C^{2k+1})$ such that $\theta(G) = C^{2k+1}$,

then G is uniquely C^{2k+1} -colorable.

This result is also best possible.

It is clear that Theorem 3 is analogous to Theorem 1. In this note we obtain the corresponding analogue to Theorem 2 for C^{2k+1} -colorings with $k \geq 2$ such that a condition analogous to (1) holds.

We note first that if G is a bipartite graph and $G \notin \{K^1, K^2\}$, then G is not uniquely \mathbb{Z}_{2k+1} -colorable.

Let G be a graph with $G \neq K^2$. For any vertex $v \in V(G)$, let $\Gamma(v)$ denote the neighborhood of v in G . Suppose $\theta \in \text{Hom}(G, \mathbb{Z}_{2k+1})$, where $\theta(G) = \mathbb{Z}_{2k+1}$. If $|\theta(\Gamma(x))| = 1$, for some $x \in V(G)$, then x can be recolored, and so G is not uniquely C^{2k+1} -colorable.

From the above observations, we conclude that if $G \neq K^1$, $G \neq K^2$, and G is uniquely \mathbb{Z}^{2k+1} -colorable, then for any $\theta \in \text{Hom}(G, \mathbb{Z}_{2k+1})$, the following holds:

$$\theta(G) = \mathbb{Z}_{2k+1} \quad \text{and} \quad |\theta(\Gamma(x))| = 2, \quad \text{for all } x \in V(G). \quad (2)$$

THEOREM 4. *Let G be a graph of order n and let $k \geq 2$ be an integer. If*

- (i) $\exists \theta \in \text{Hom}(G, \mathbb{Z}_{2k+1})$ such that (2) is satisfied, and
- (ii) $\delta(G) > 3n/(4k+2)$,

then G is uniquely \mathbb{Z}_{2k+1} -colorable.

We start with some lemmas.

LEMMA 1. Let G be a graph of order n . If $\exists \theta \in \text{Hom}(G, \mathbb{Z}_{2k+1})$ such that $\theta(G) = \mathbb{Z}_{2k+1}$, then

$$\theta_M \leq n - k(\delta(G)), \tag{3}$$

where $\theta_M = \max\{|\theta^{-1}(i)| : i \in \mathbb{Z}_{2k+1}\}$.

Proof. Let $n_i = |\theta^{-1}(i)|$, $i \in \mathbb{Z}_{2k+1}$. We may assume that $|\theta^{-1}(0)| = \theta_M$. Suppose that k is even. Consider the following k inequalities:

$$\begin{aligned} n_1 + n_3 &\geq \delta(G), \\ n_2 + n_4 &\geq \delta(G), \\ n_5 + n_7 &\geq \delta(G), \\ n_6 + n_8 &\geq \delta(G), \\ &\dots \\ n_{2k-3} + n_{2k-1} &\geq \delta(G), \\ n_{2k-2} + n_{2k} &\geq \delta(G). \end{aligned}$$

Adding them all together with n_0 , we get (3).

The proof when k is odd uses $n_1 + n_{2k} \geq \delta(G)$ as one of the k inequalities, and it is similar. ■

LEMMA 2. Suppose that G satisfies the hypotheses of Theorem 4. For any $x \in V(G)$, there exists an $m(2k+1)$ -cycle of G containing x , where m equals 1 or 2.

Proof. Denote, for subsets V, V' of $V(G)$.

$$(V, V') = \{v \in V' : \exists w \in V \text{ such that } vw \in E(G)\}.$$

Pick $x \in \theta^{-1}(0)$. Let

$$\begin{aligned} V_1 &= (\{x\}, \theta^{-1}(1)), & W_{2k} &= (\{x\}, \theta^{-1}(2k)), \\ V_2 &= (V_1, \theta^{-1}(2)), & W_{2k-1} &= (W_{2k}, \theta^{-1}(2k-1)), \\ &\dots & & \\ V_i &= (V_{i-1}, \theta^{-1}(i)), & W_i &= (W_{i+1}, \theta^{-1}(i)), \\ &\dots & & \\ V_{2k} &= (V_{2k-1}, \theta^{-1}(2k)), & W_1 &= (W_2, \theta^{-1}(1)), \\ V_0 &= (V_{2k}, \theta^{-1}(0)), & W_0 &= (W_1, \theta^{-1}(0)). \end{aligned}$$

Then we have, for all $i \in \mathbb{Z}_{2k+1}$,

$$\begin{aligned} |\theta^{-1}(i)| + |V_{i+2}| &\geq \delta(G), \\ |\theta^{-1}(i+4)| + |W_{i+2}| &\geq \delta(G). \end{aligned}$$

Notice that condition (2) implies that $V_i \neq \emptyset$ and $W_i \neq \emptyset$ for all $i \in \mathbb{Z}_{2k+1}$. If $V_i \cap W_i = \emptyset$, for all $i \in \mathbb{Z}_{2k+1}$, then we would have

$$\begin{aligned} 3n &\geq \sum_{i \in \mathbb{Z}_{2k+1}} \{|\theta^{-1}(i)| + |\theta^{-1}(i+4)| + (|V_{i+2}| + |W_{i+2}|)\} \\ &\geq 2(2k+1)(\delta(G)) \\ &> 3n, \end{aligned}$$

a contradiction.

Hence for some $i \in \mathbb{Z}_{2k+1}$, $V_i \cap W_i \neq \emptyset$. If $i=0$, then x lies on a $2(2k+1)$ -cycle of G . In other cases, x is in a $(2k+1)$ -cycle of G . ■

LEMMA 3. *If G satisfies the hypotheses of Theorem 4, then G does not have an odd cycle of length less than $2k+1$.*

Proof. This follows from the fact that $\text{Hom}(G, \mathbb{Z}_{2k+1}) \neq \emptyset$. ■

LEMMA 4. *Suppose that G satisfies the hypotheses of Theorem 4. Then G has a $(2k+1)$ -cycle.*

Proof. By Lemma 2, G has a cycle whose length is $2k+1$ or $2(2k+1)$. Let C be a cycle of G of length $m(2k+1)$ such that m is minimized. We shall show $m=1$.

By contradiction, suppose $m > 1$. By Lemma 2, $m=2$. Let

$$C = v_1 v_2 \cdots v_{m(2k+1)-1} v_{m(2k+1)} v_1.$$

By (ii) of Theorem 4, we have

$$\sum_{j=1}^{m(2k+1)} d(v_j) > 3nm \frac{2k+1}{4k+2} = \frac{3}{2} nm > nm. \quad (4)$$

Denote $\partial C = \{e \in E(G) - E(C) : e \text{ is incident with } V(C)\}$. We claim that there is no edge in ∂C that is incident with two vertices of $V(C)$. Suppose not, we can find an edge $e = v_i v_j \in \partial C$. Without loss of generality, we assume $1 \leq i \leq 2k+1$. Hence $j = (2k+1+i) \pm 1$.

If $j = 2k+i$, then $v_1 v_2 \cdots v_i v_{2k+i+1} \cdots v_{2(2k+1)} v_1$ is a $(2k+1)$ -cycle, a contradiction.

If $j = 2k + i + 2$, then the cycle $v_j v_{j+1} \cdots v_{2(2k+1)} v_1 \cdots v_{i-1} v_i$ has length $2k - 1$, contrary to Lemma 3. Hence the claim.

Thus no edge $e \in \partial C$ can be incident with two vertices of $V(C)$ and so

$$|\partial C| = \sum_{j=1}^{m(2k+1)} (d(v_j) - 2), \tag{5}$$

since exactly two edges incident with $v_j \in V(C)$ are in $E(C)$, and the $d(v_j) - 2$ other edges are in ∂C . Denote

$$Y = \{y \in V(G) - V(C) : \Gamma(y) \cap V(C) \neq \emptyset\}.$$

We claim that for all $y \in Y$,

$$|\Gamma(y) \cap V(C)| \leq 2. \tag{6}$$

Since $\theta \in \text{Hom}(G, \mathbb{Z}_{2k+1})$,

$$\theta[\Gamma(y) \cap V(C)] \subseteq \{\theta(y) - 1, \theta(y) + 1\}. \tag{7}$$

Suppose that (7) holds with equality. Then by the minimality of m , $|\Gamma(y) \cap V(C)| = 2$. Suppose that (7) is strict. Then $|\Gamma(y) \cap V(C)| \leq m$. By Lemma 2, $m \leq 2$. Hence in either case, (6) holds, as claimed.

By (6) and the definitions of Y and C ,

$$|\partial C| = \sum_{y \in Y} |\Gamma(y) \cap V(C)| \leq 2|Y|, \tag{8}$$

and

$$n \geq |Y| + |V(C)| = |Y| + m(2k + 1). \tag{9}$$

We combine (9), (8), (5), and (4) to get

$$\begin{aligned} 2n &\geq 2|Y| + 2m(2k + 1) \geq |\partial C| + 2m(2k + 1) \\ &= \sum_{j=1}^{m(2k+1)} (d(v_j) - 2) + 2m(2k + 1) \\ &\geq \sum_{j=1}^{m(2k+1)} d(v_j) \geq m(2k + 1)(\delta(G)) > \frac{3}{2} mn > mn. \end{aligned}$$

It follows that $m < 2$ and we are done. ■

For a graph H that has a $(2k + 1)$ -cycle, we define a new graph $C^{2k+1}(H)$ whose vertex set is the set of all $(2k + 1)$ -cycles of H , where two vertices of $C^{2k+1}(H)$ are adjacent if and only if the corresponding $(2k + 1)$ -cycles have at least one edge in common.

Let X be the subset of $V(G)$ that consists of all vertices lying on $(2k+1)$ -cycles of G . By Lemma 4, $X \neq \emptyset$. Let $W = V(G) - X$.

Let C_1 and C_2 be two $(2k+1)$ -cycles of G . For any $\theta_1, \theta_2 \in \text{Hom}(G, \mathbb{Z}_{2k+1})$, consider the following condition on the restrictions of θ_1 and θ_2 :

$$\theta_1|_{C_1} = \theta_2|_{C_1} \Leftrightarrow \theta_1|_{C_2} = \theta_2|_{C_2}. \quad (10)$$

We say that C_1 is *related to* C_2 if and only if (10) holds for any $\theta_1, \theta_2 \in \text{Hom}(G, \mathbb{Z}_{2k+1})$. It is clear that this defines an equivalence relation on $V(C^{2k+1}(G))$. We shall first show that $V(C^{2k+1}(G))$ has only one equivalence class. This will mean for $\theta_1 \in \text{Hom}(G, \mathbb{Z}_{2k+1})$, if $\theta_1(v) = \theta(v)$ for all vertices v in a $(2k+1)$ -cycle of G , then $\theta_1(x) = \theta(x)$ for all $x \in X$. Then we shall show that $\theta_1(w) = \theta(w)$ for all $w \in W$ also.

Let C_1 and C_2 be two $(2k+1)$ -cycles of G that have an edge in common. If θ_1 is a homomorphism from C_1 onto \mathbb{Z}_{2k+1} , then there is a unique way to extend θ_1 to a homomorphism from $C_1 \cup C_2$ onto \mathbb{Z}_{2k+1} . Hence C_1 is related to C_2 . As a consequence, if there is a (C_1, C_2) -path in $C^{2k+1}(G)$ for two $(2k+1)$ -cycles C_1 and C_2 in G , then C_1 is related to C_2 .

LEMMA 5. *If the hypotheses of Theorem 4 hold, then $V(C^{2k+1}(G))$ has only one equivalence class.*

Proof. Let C_1 and C_2 be two $(2k+1)$ -cycles of G . We shall also use C_1 and C_2 to denote the corresponding vertices in $C^{2k+1}(G)$. It suffices to show that C_1 is related to C_2 .

Case 1. $V(C_1) \cap V(C_2) \neq \emptyset$. We assume that $C_1 = x_0 x_1 \cdots x_{2k} x_0$ and $C_2 = y_0 y_1 \cdots y_{2k} y_0$ and that $x_0 = y_0$.

Suppose that C_1 is not related to C_2 . Thus there is no (C_1, C_2) -path in $C^{2k+1}(G)$.

First we claim that $|V(C_1) \cap V(C_2)| = 1$. Suppose not. We have $x_i = y_j$ for some $0 < i, j < 2k+1$.

If $i = j$, then $x_0 x_1 \cdots x_i y_{i+1} \cdots y_{2k} x_0$ is a $(2k+1)$ -cycle in G that is adjacent to both C_1 and C_2 in $C^{2k+1}(G)$, a contradiction.

If $i = 2k+1-j$, then $x_0 x_1 \cdots x_i y_{2k+2-j} \cdots y_{2k} x_0$ is a $(2k+1)$ -cycle in G that is adjacent to both C_1 and C_2 in $C^{2k+1}(G)$, a contradiction.

If $i \not\equiv \pm j \pmod{2k+1}$, then one of the following cycles,

$$x_0 x_1 \cdots x_i y_{i+1} \cdots y_{2k-1} y_{2k} x_0,$$

$$x_0 x_1 \cdots x_i y_{i-1} \cdots y_2 y_1 x_0,$$

$$x_0 x_{2k} \cdots x_i y_{i+1} \cdots y_{2k-1} y_{2k} x_0,$$

$$x_0 x_{2k} \cdots x_i y_{i-1} \cdots y_2 y_1 x_0,$$

is an odd cycle of length less than $2k + 1$, contrary to Lemma 3. Hence the claim holds and so $|V(C_1 \cup C_2)| = 4k + 1$.

Subcase I.1. Suppose $k \geq 3$. Define $S_i = \{x_{i-1}, y_{i-1}, x_{i+1}, y_{i+1}\}$ for all $i \in \mathbb{Z}_{2k+1}$, where S_i has fewer than four elements if $i \in \{1, -1\}$.

Let m_0 be the number of incidences of edges of G with $V(C_1 \cup C_2)$. We shall estimate m_0 in two ways.

We claim that $\forall i \in \mathbb{Z}^{2k+1} - \{0\}, |\Gamma(z) \cap S_i| \leq 2$, for all $z \in \theta^{-1}(i)$. Suppose not. Since $x_0 = y_0$, if $i \in \{-1, 1\}$, then $\{y_0, x_{2i}, y_{2i}\} \subseteq \Gamma(z)$ and so y_0z lies on a $(2k + 1)$ -cycle C_3 adjacent to C_1 and on a $(2k + 1)$ -cycle C_4 adjacent to C_2 . Thus $C_1C_3C_4C_2$ is a (C_1, C_2) -path in $C^{2k+1}(G)$, a contradiction. If $i \notin \{-1, 0, 1\}$, then without loss of generality, we may assume $\{y_{i-1}, x_{i+1}\} \subseteq \Gamma(z)$ with $z \in \theta^{-1}(i)$. Then $y_0 \cdots y_{i-1}zx_{i+1} \cdots x_{2k}y_0$ is a $(2k + 1)$ -cycle adjacent to both C_1 and C_2 in $C^{2k+1}(G)$, a contradiction. Hence the claim.

Note that $|\Gamma(z) \cap S_0| \leq 4, \forall z \in \theta^{-1}(0)$. Therefore, by the above claim, we sum over all $z \in V(G)$ to get

$$\begin{aligned} m_0 &= \sum_{z \in V(G)} |\Gamma(z) \cap V(C_1 \cup C_2)| = \sum_{i \in \mathbb{Z}_{2k+1}} \sum_{z \in \theta^{-1}(i)} |\Gamma(z) \cap S_i| \\ &\leq 2n + 2|\theta^{-1}(0)|. \end{aligned}$$

By (ii) of Theorem 4, and since $|V(C_1 \cup C_2)| = 4k + 1$,

$$m_0 \geq |V(C_1 \cup C_2)|\delta(G) > (4k + 1) \frac{3n}{4k + 2} = 3n - \frac{3n}{4k + 2}.$$

Combine these bounds to get

$$2|\theta^{-1}(0)| > n - \frac{3n}{4k + 2}.$$

But (3) and (ii) of Theorem 4 give

$$|\theta^{-1}(0)| < n - \frac{3kn}{4k + 2}.$$

Hence we must have

$$2\left(n - \frac{3kn}{4k + 2}\right) > n - \frac{3n}{4k + 2}.$$

It follows that $k < \frac{5}{2}$, contrary to $k \geq 3$.

Subcase I.2. Suppose $k = 2$. Then $\delta(G) > 3n/10$. Now we have

$$C_1 = x_0 x_1 x_2 x_3 x_4 x_0 \quad \text{and} \quad C_2 = y_0 y_1 y_2 y_3 y_4 y_0 \quad \text{with } x_0 = y_0.$$

CLAIM. $\Gamma(x_1), \Gamma(y_2), \Gamma(x_2), \Gamma(y_3)$ are pairwise disjoint.

Proof of the Claim. By Lemma 3, we have $\Gamma(x_1) \cap \Gamma(x_2) = \emptyset$ and $\Gamma(y_2) \cap \Gamma(y_3) = \emptyset$.

If $\Gamma(x_1) \cap \Gamma(y_2) \neq \emptyset$, then picking $w \in \Gamma(x_1) \cap \Gamma(y_2)$, we can see that $x_0 x_1 w y_2 y_1 x_0$ is a 5-cycle in G that is adjacent to both C_1 and C_2 in $C^5(G)$, a contradiction.

If $\Gamma(x_1) \cap \Gamma(y_3) \neq \emptyset$, then picking $w \in \Gamma(x_1) \cap \Gamma(y_3)$, we can see that $x_0 x_1 w y_3 y_4 x_0$ is a 5-cycle in G that is adjacent to both C_1 and C_2 in $C^5(G)$, a contradiction.

Suppose $\Gamma(x_2) \cap \Gamma(y_2) \neq \emptyset$. Let $\theta_1, \theta_2 \in \text{Hom}(G, \mathbb{Z}_{2k+1})$ and let $w \in \Gamma(x_2) \cap \Gamma(y_2)$. Without loss of generality, we assume that $\theta_1(x_i) = \theta_2(x_i) = i, \forall i \in \mathbb{Z}_{2k+1}$. Since $w \in \Gamma(x_2)$, $\theta_1(w) \in \{1, 3\}$. Since $\theta_1(x_0) = \theta_1(y_0)$ and since $C_2 = y_0 y_1 y_2 y_3 y_4 y_0$ is a 5-cycle, $\theta_1(y_2) \in \{2, 3\}$ and so we must have $\theta_1(y_2) = 2$, by $w \in \Gamma(y_2)$, $\theta_1(w) \in \{1, 3\}$, and $\theta_1(y_2) \in \{2, 3\}$. It follows that $\theta_1(y_i) = i, \forall i \in \mathbb{Z}_{2k+1}$. Similarly, we can see that $\theta_2(y_i) = i, \forall i \in \mathbb{Z}_{k+1}$. Hence C_1 is related to C_2 , a contradiction.

Similarly we can see that $\Gamma(x_2) \cap \Gamma(y_3) = \emptyset$. Hence the claim. \blacksquare

By the claim, we have

$$n \geq |\Gamma(x_1) \cup \Gamma(x_2) \cup \Gamma(y_2) \cup \Gamma(y_3)| \geq 4(\delta(G)) > \frac{12}{10}n > n,$$

a contradiction.

Case 2. $V(C_1) \cap V(C_2) = \emptyset$. Fix a $(2k+1)$ -cycle C of G . Define $C = v_0 v_1 \cdots v_{2k} v_0$ such that $\theta(v_i) = i, \forall i \in \mathbb{Z}_{2k+1}$. Let

$$N(C) = \bigcup_{i \in \mathbb{Z}_{2k+1}} [\Gamma(v_{i-1}) \cap \Gamma(v_{i+1})].$$

Note that

$$\begin{aligned} |N(C)| &= \sum_{i \in \mathbb{Z}_{2k+1}} |\Gamma(v_{i-1}) \cap \Gamma(v_{i+1})| \\ &\geq \sum_{i \in \mathbb{Z}_{2k+1}} \{|\Gamma(v_{i-1}) \cap \theta^{-1}(i)| + |\Gamma(v_{i+1}) \cap \theta^{-1}(i)| - |\theta^{-1}(i)|\} \\ &= \sum_{i \in \mathbb{Z}_{2k+1}} \deg(v_i) - n \\ &> (2k+1) \frac{3n}{4k+2} - n \\ &= \frac{n}{2}. \end{aligned}$$

Hence

$$|N(C_1) \cap N(C_2)| > \frac{n}{2} + \frac{n}{2} - n = 0. \tag{11}$$

By (11), there is a vertex $u \in N(C_1) \cup N(C_2)$. Thus there are i and j , $0 \leq i, j \leq 2k$, such that

$$u \in \Gamma(x_{i-1}) \cap \Gamma(x_{i+1}) \cap \Gamma(y_{j-1}) \cap \Gamma(y_{j+1}).$$

Let

$$C_3 = x_0 x_1 \cdots x_{i-1} u x_{i+1} \cdots x_{2k-1} x_{2k},$$

$$C_4 = y_0 y_1 \cdots y_{j-1} u y_{j+1} \cdots y_{2k-1} y_{2k}.$$

Then C_3 has an edge in common with C_1 and C_4 has an edge in common with C_2 and C_3 and C_4 have a vertex, namely u , in common. Thus C_1 is related to C_3 and C_2 is related to C_4 . By the result of Case 1, C_3 is related to C_4 and so by transitivity of the equivalence relation, C_1 is related to C_2 .

Thus C_1 is related to C_2 and we are done. ■

Proof of Theorem 4. Suppose that $\exists \theta, \theta_1 \in \text{Hom}(G, \mathbb{Z}_{2k+1})$ such that θ and θ_1 agree in X , the set of vertices lying in some $(2k+1)$ -cycle of G . Let $w \in W$, where $W = V(G) - X$. By Lemma 2, there is a $2(2k+1)$ -cycle C_5 that contains w . We may assume

$$C_5 = w_0 w_1 \cdots w_{2k+1} w_{2k+2} \cdots w_{4k+1} w_{4k+2}$$

such that $w = w_0 = w_{4k+2}$ and $\theta(w_i) \equiv i \pmod{2k+1}$.

For each $i \in \mathbb{Z}_{2k+1}$, define

$$T_i = \{w_{i-1}, w_{i+1}, w_{2k+i}, w_{2k+i+2}\}.$$

We claim that

$$\forall i \in \mathbb{Z}_{2k+1} - \{0\} \quad \text{and} \quad \forall z \in \theta^{-1}(i), \quad |\Gamma(z) \cap T_i| \leq 3. \tag{12}$$

If, instead, we have $T_i \subseteq \Gamma(z)$ for some $i \in \mathbb{Z}_{2k+1} - \{0\}$ and $z \in \theta^{-1}(i)$, then $w_0 w_i \cdots w_{i-1} z w_{2k+i+2} \cdots w_{4k+1} w_0$ is a $(2k+1)$ -cycle containing w , a contradiction. Hence the claim.

Let m_1 be the number of incidences of edges of G with $V(C_5)$.

Suppose $|\Gamma(z) \cap T_0| \leq 3, \forall z \in \theta^{-1}(0)$. Then by (12),

$$m_1 = \sum_{i \in \mathbb{Z}_{2k+1}} \sum_{z \in \theta^{-1}(i)} |\Gamma(z) \cap T_i| \leq 3n.$$

On the other hand,

$$m_1 = \sum_{i=1}^{4k+2} \deg(w_i) \geq (4k+2)(\delta(G)) > 3n,$$

a contradiction. Hence we must have

$$|\Gamma(z) \cap T_0| = 4, \quad \text{for some } z \in \theta^{-1}(0). \tag{13}$$

By (13), $zw_1w_2 \cdots w_{2k}z$ and $zw_{2k+2}w_{2k+3} \cdots w_{4k+1}z$ are $(2k+1)$ -cycles of G . Hence $w_1, w_{4k+1} \in X$. Thus $\theta(w_1) = \theta_1(w_1) = 1$ and $\theta(w_{4k+1}) = \theta_1(w_{4k+1}) = 2k$. Since $w_0w_1, w_0w_{4k+1} \in E(G)$, we must have $\theta(w_0) = \theta_1(w_0) = 0$.

Thus G is uniquely \mathbb{Z}_{2k+1} -colorable. ■

Theorem 4 is best possible in some sense. Let $k \geq 2$ and $m \geq 1$ be integers. Let $C_1 = v_0v_1 \cdots v_{2k}v_0$ and $C_2 = u_0u_1 \cdots u_{2k}u_0$ be two $(2k+1)$ -cycles with $V(C_1) \cap V(C_2) = \emptyset$. Let H be the graph such that $V(H) = V(C_1) \cup V(C_2)$ and $E(H) = E(C_1) \cup E(C_2) \cup \{u_iv_i : i = 0, 1, 2 \cdots 2k\}$. Let H_m be the graph obtained from H by replacing each vertex v_i by a set V_i of m vertices and replacing each vertex u_i by a set U_i of m vertices, where two vertices of H_m are adjacent in H_m if and only if they correspond to different vertices that are adjacent in H .

Hence $|V(H_m)| = (4k+2)m$ and $\delta(H_m) = 3m$. There are two homomorphisms $\theta_i \in \text{Hom}(H_m, \mathbb{Z}_{2k+1})$, $i = 1, 2$, where $\theta_1^{-1}(j) = U_j \cup V_{j-1}$ and $\theta_2^{-1}(j) = U_j \cup V_{j+1}$, $j \in \mathbb{Z}_{2k+1}$. Clearly $\theta_i(H_m) = \mathbb{Z}_{2k+1}$, $i = 1, 2$. Also, θ_1, θ_2 satisfy (2). But there is no automorphism φ of \mathbb{Z}_{2k+1} satisfying $\varphi\theta_1 = \theta_2$.

Examples of Bollobás [2] showing that Theorem 1 is best possible also show that Theorem 4 fails for $k = 1$.

Let $\theta \in \text{Hom}(G, \mathbb{Z}_{2k+1})$. One may consider the following analogue to (1):

$$G[\theta^{-1}(i) \cup \theta^{-1}(i+1)] \text{ is connected, } \forall i \in \mathbb{Z}_{2k+1}. \tag{14}$$

One may ask if (14) is a necessary condition for unique \mathbb{Z}_{2k+1} -colorings, and when (14) holds, if it is not such a necessary condition.

THEOREM 5. *Let G be a graph of order n and let $k \geq 1$ be an integer. Suppose that $\exists \theta \in \text{Hom}(G, \mathbb{Z}_{2k+1})$ such that $\theta(G) = \mathbb{Z}_{k+1}$. If one of the following holds:*

- (i) k is even and $\delta(G) > n/(k+1)$,
- (ii) k is odd and $\delta(G) > 4n/(4k+3)$,

then (14) holds.

Proof. By contradiction, we may assume that $G[\theta^{-1}(0) \cup \theta^{-1}(1)]$ is the disjoint union of two nontrivial subgraphs G_1 and G_2 such that there are no edges of G joining a vertex of G_1 and a vertex of G_2 . Let

$$V_0 = V(G_1) \cap \theta^{-1}(0),$$

$$W_0 = V(G_2) \cap \theta^{-1}(0),$$

$$V_1 = V(G_1) \cap \theta^{-1}(1),$$

$$W_1 = V(G_2) \cap \theta^{-1}(1).$$

Assume (i) of Theorem 5 first. We claim that $V_0, V_1, W_0,$ and W_1 are nonempty. Suppose, to the contrary, that $V_0 = \emptyset$. Since $\theta(G) = \mathbb{Z}_{2k+1}$ and $G[\theta^{-1}(0) \cup \theta^{-1}(1)]$ is disconnected, we may assume that $V_1 \neq \emptyset$ and there is no edge in G joining one vertex in V_1 and one vertex in W_0 . Let $x \in V_1$. Then $\Gamma(x) \subseteq \theta^{-1}(2)$. Thus

$$|\theta^{-1}(2)| \geq \delta(G) > \frac{n}{k+1}. \tag{15}$$

By (3), $|\theta^{-1}(2)| \leq n - k\delta(G) \leq n/(k+1)$, a contradiction. Hence V_0, V_1, W_0 and W_1 are nonempty. Without loss of generality, suppose

$$|V_0| + |W_0| \leq |V_1| + |W_1|. \tag{16}$$

Considering the degrees, we have

$$|W_0| + |\theta^{-1}(2)| \geq \delta(G), \tag{17}$$

$$|V_1| + |\theta^{-1}(2k)| \geq \delta(G), \tag{18}$$

$$|\theta^{-1}(2k-1)| + |\theta^{-1}(0)| \geq \delta(G).$$

If $k > 2$, we have the following besides the above three inequalities,

$$|\theta^{-1}(4j+3)| + |\theta^{-1}(4j+5)| \geq \delta(G),$$

$$|\theta^{-1}(4j+4)| + |\theta^{-1}(4j+6)| \geq \delta(G),$$

where $j = 0, 1, \dots, (k/2 - 2)$.

Adding all these inequalities, we get

$$\begin{aligned} n + (|W_0| - |W_1|) &\geq |W_0| + |V_1| + \sum_{j \in \mathbb{Z}_{2k+1} - \{1\}} |\theta^{-1}(j)| \\ &\geq \left[2 \left(\frac{k}{2} - 1 \right) + 3 \right] \delta(G) \\ &= (k+1) \delta(G) > n. \end{aligned}$$

This implies $|W_0| > |W_1|$. Replacing W_0 by V_0 in (17) and V_1 by W_1 in (18), we get similarly $|V_0| > |V_1|$, contrary to (16).

Now we assume that (ii) of Theorem 5 holds. A similar argument shows that V_0, V_1, W_0 , and W_1 are nonempty. Without loss of generality, suppose

$$|V_0| \leq |V_1|. \quad (19)$$

Considering the degrees, we get

$$\begin{aligned} |W_0| + |\theta^{-1}(2)| &\geq \delta(G), \\ |\theta^{-1}(2k-1)| + |\theta^{-1}(0)| &\geq \delta(G), \end{aligned}$$

and

$$\begin{aligned} |\theta^{-1}(2k)| + |W_1| &\geq \delta(G), \\ |\theta^{-1}(2i-1)| + |\theta^{-1}(2i+1)| &\geq \delta(G), \\ |\theta^{-1}(2i)| + |\theta^{-1}(2i+2)| &\geq \delta(G), \end{aligned}$$

where $i = 1, 2, \dots, k-1$.

Adding all these inequalities, we get

$$\begin{aligned} 2n - (|V_0| + |V_1|) &\geq \sum_{i \in \mathbb{Z}_{2k+1} - \{0\}} |\theta^{-1}(i)| + |W_0| \\ &\quad + \sum_{i \in \mathbb{Z}_{2k+1} - \{1\}} |\theta^{-1}(j)| + |W_1| \\ &\geq (2k+1)\delta(G) \\ &> (2k+1) \frac{4n}{4k+3} \\ &= 2n - \frac{2n}{4k+3}. \end{aligned}$$

Thus $|V_0| + |V_1| < 2n/(4k+3)$, and so (19) implies that $|V_0| < n/(4k+3)$. Considering the degrees of vertices in V_1 , we have

$$|V_0| + |\theta^{-1}(2)| \geq \delta(G) > 4n/(4k+3).$$

It follows that $|\theta^{-1}(2)| > 3n/(4k+3) > n - k\delta(G)$, contrary to (3).

This completes the proof of Theorem 5. ■

Theorem 5 is best possible in some sense also. Let H be a theta graph obtained from \mathbb{Z}_{2k+1} by adding two vertices α, β to \mathbb{Z}_{2k+1} in the following way: α is adjacent to β and 0, and β is adjacent to α and β .

Let $k > 1$ and $s > 1$ be integers. Let $G_j(k, s)$, $j = 1, 2$, be the graphs obtained from H by replacing each vertex $j \in \{\alpha, \beta\} \cup \mathbb{Z}_{2k+1}$ by a set $V_{i,j}$ of $n_{i,j}$ vertices, where the $n_{i,j}$'s are defined as follows:

When $j = 1$, we let

$$\begin{aligned} n_{\alpha,1} &= n_{1,1} = n_{\beta,1} = n_{2,1} = s, \\ n_{0,1} &= n_{3,1} = 3s, \\ n_{i,1} &= 2s, \quad \text{for all other } n_{i,1}'\text{s}; \end{aligned}$$

and when $j = 2$ and k is odd, we let

$$\begin{aligned} n_{\alpha,2} &= n_{1,2} = n_{\beta,2} = n_{2,2} = s, \\ n_{3,2} &= 3s, \end{aligned}$$

and for $p = 1, 2, \dots, (k-1)/2$,

$$\begin{aligned} n_{4p,2} &= n_{4p+2,2} = 2s, \\ n_{4p+1,2} &= s, \\ n_{4p+3,2} &= 3s. \end{aligned}$$

Two vertices in $H_j(k, s)$ are adjacent if and only if they correspond to different vertices that are adjacent in H .

Let H_j denote $H_j(k, s)$. It is easy to see that $\text{Hom}(H_j, \mathbb{Z}_{2k+1}) \neq \emptyset$, $j = 1, 2$. Note that

$$\delta(H_1) = 4s = |V(H_1)|/(k+1) \quad \text{and} \quad \delta(H_2) = 4s = |V(H_2)|/(4k+3).$$

By Theorem 3, both H_1 and H_2 are uniquely \mathbb{Z}_{2k+1} -colorable. Hence for $j \in \{1, 2\}$, $\forall \theta_j \in \text{Hom}(H_j, \mathbb{Z}_{2k+1})$, we have

$$\theta_j(V_{\alpha,j}) = \theta_j(V_{i,j}) \quad \text{and} \quad \theta_j(V_{\beta,j}) = \theta_j(V_{2,j}).$$

But $H_j[V_{\alpha,j} \cup V_{i,j} \cup V_{\beta,j} \cup V_{2,j}]$ is disconnected. These extremal graphs also show that (13) is not a necessary condition for unique \mathbb{Z}_{2k+1} -colorings.

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